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Some Approximations with Hurwitz Zeta Function

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Abstract. In this paper, we focus on some approximations with Hurwitz zeta function. By using these approximations, we present some asymptotic formulae related to Hurwitz zeta function. As an application, we give two corollaries related to Bernoulli polynomials.

1. Introduction, definitions and preliminaries

Throughout this article, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

Let $a, s \in \mathbb{C}$. Hurwitz zeta function and Riemann zeta function are respectively defined by (cf. [2], [8])

$$\zeta(s, w) = \sum_{n=0}^{\infty} \frac{1}{(n+w)^s} \quad, (Re(s) > 1, w \in \mathbb{C} \setminus \{0, -1, -2, -3, \cdots\})$$

and

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, (Re(s) > 1).$$

For $w = 1$, $\zeta(s, 1) = \zeta(s)$. Also, Hurwitz zeta function and Riemann zeta function are related to Bernoulli polynomials.

Bernoulli polynomials are defined by the following generating function:

$$\frac{t}{e^t - 1} e^{at} = \sum_{n=0}^{\infty} B_n(a) \frac{t^n}{n!} \quad \text{(cf. [8], [10])}$$

where $a \in \mathbb{C}$, $|t| < 2\pi$.

Hurwitz zeta function, Riemann zeta function and Bernoulli polynomials are the famous special functions for Analytic Number Theory. Also, it is possible to investigate the approximation of these functions. The special functions and their approximations were considered by Luke (cf. [9]).

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In Section 2, we use

$$\lim_{n\to\infty}\left(1+\frac{y}{n}\right)^{n}=e^{y}. \tag{1.1}$$

Also, (1.1) can be written by

$$\lim_{\lambda\to 0}(1+\lambda y)^{\frac{1}{\lambda}}=e^{y}.$$

(1.1) is a well-known result in Classical Analysis. Many authors have used this result in Analytic Number Theory. For instance, Carlitz introduced the degenerate Bernoulli polynomials given by the generating function:

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{a}{\lambda}}=\sum_{n=0}^{\infty}\beta_n(a\mid\lambda)\frac{t^n}{n!}\ \text{(cf. [6], [7] and [5]).} \tag{1.2}$$

For $a=0$, $\beta_n(0\mid\lambda)=\beta_n(\lambda)$ are called the degenerate Bernoulli numbers. From (1.2), we note that

$$\lim_{\lambda\to 0}\beta_n(a\mid\lambda)=B_n(a)\ \ (n\geq 0).$$

2. Main Results

In this section, we give a key lemma to give our main results related to approximation of the Hurwitz zeta function. Let $k\in\mathbb{N}$ and $0<y\in\mathbb{R}$.

We put $y\to ky$ in (1.1):

$$\lim_{n\to\infty}\left(1+\frac{ky}{n}\right)^{-n}=e^{-ky}$$

Then, we expand the series on k:

$$\sum_{k=0}^{\infty}\left\{\lim_{n\to\infty}\left(1+\frac{ky}{n}\right)^{-n}\right\}=\sum_{k=0}^{\infty}e^{-ky}$$

or

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\left(1+\frac{ky}{n}\right)^{-n}=\sum_{k=0}^{\infty}e^{-ky}$$

We set

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=0}^{\infty}\left(1+\frac{ky}{n}\right)^{-n} &= \lim_{n\to\infty}\left(\frac{y}{n}\right)^{-n}\sum_{k=0}^{\infty}\left(\frac{n}{y}+k\right)^{-n}\\ &= \lim_{n\to\infty}\left(\frac{y}{n}\right)^{-n}\zeta\left(n,\frac{n}{y}\right).\end{aligned}$$

From the property of geometric sum, we know

$$\frac{e^{y}}{e^{y}-1}=\sum_{k=0}^{\infty}e^{-ky}.$$

Then, we arrive at the following Lemma:

Lemma 2.1.

$$\lim_{n\to\infty}\left(\frac{y}{n}\right)^{-n}\zeta\left(n,\frac{n}{y}\right)=\frac{e^y}{e^y-1}. \tag{2.1}$$

Theorem 2.2.

$$\lim_{n\to\infty}\frac{(-1)^n\,\zeta\left(n,\frac{n}{y}\right)}{\zeta\left(n,-\frac{n}{y}\right)}=-e^y. \tag{2.2}$$

Proof. We put $y\to -y$ in (2.1):

$$\lim_{n\to\infty}\left(-\frac{y}{n}\right)^{-n}\zeta\left(n,-\frac{n}{y}\right)=-\frac{1}{e^y-1}. \tag{2.3}$$

From (2.1) and (2.3), we obtain the desired result. □

Theorem 2.3. *Let $\lambda\in\mathbb{R}$. Then, we have*

$$\lim_{n\to\infty}\frac{(-1)^{n+1}\,\zeta\left(n,\frac{n}{\lambda y}\right)}{\zeta\left(n,-\frac{n}{\lambda y}\right)}=\lim_{n\to\infty}\left(\frac{(-1)^{n+1}\,\zeta\left(n,\frac{n}{y}\right)}{\zeta\left(n,-\frac{n}{y}\right)}\right)^{\lambda}.$$

Proof. It is immediately seen from Theorem 2.2 for $y\to\lambda y$. □

We note that

$$\begin{aligned}\frac{d}{dy}\zeta\left(n,\frac{n}{y}\right) &= \frac{d}{dy}\sum_{k=0}^{\infty}\frac{1}{\left(\frac{n}{y}+k\right)^n}\\ &= \sum_{k=0}^{\infty}\frac{d}{dy}\left\{\frac{1}{\left(\frac{n}{y}+k\right)^n}\right\}\\ &= \frac{n^2}{y^2}\sum_{k=0}^{\infty}\frac{1}{\left(\frac{n}{y}+k\right)^{1+n}}\\ &= \frac{n^2}{y^2}\zeta\left(1+n,\frac{n}{y}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dy}\zeta\left(n,-\frac{n}{y}\right) &= \frac{d}{dy}\sum_{k=0}^{\infty}\frac{1}{\left(-\frac{n}{y}+k\right)^n}\\ &= \sum_{k=0}^{\infty}\frac{d}{dy}\left\{\frac{1}{\left(-\frac{n}{y}+k\right)^n}\right\}\\ &= -\frac{n^2}{y^2}\sum_{k=0}^{\infty}\frac{1}{\left(-\frac{n}{y}+k\right)^{1+n}}\\ &= -\frac{n^2}{y^2}\zeta\left(1+n,-\frac{n}{y}\right).\end{aligned}$$

Theorem 2.4.

$$\lim_{n\to\infty} \frac{(-1)^n \zeta\left(1+n, \frac{n}{y}\right)}{\zeta\left(1+n, -\frac{n}{y}\right)} = e^y. \tag{2.4}$$

Proof. We take the first derivative respect to y into (2.2):

$$\begin{aligned}
&\lim_{n\to\infty} \frac{(-1)^n \left\{\frac{n^2}{y^2}\zeta\left(1+n, \frac{n}{y}\right)\zeta\left(n, -\frac{n}{y}\right) + \frac{n^2}{y^2}\zeta\left(1+n, -\frac{n}{y}\right)\zeta\left(n, \frac{n}{y}\right)\right\}}{\left(\zeta\left(n, -\frac{n}{y}\right)\right)^2} \\
= &\left\{\lim_{n\to\infty} \frac{n^2(-1)^n}{y^2} \frac{\zeta\left(1+n, \frac{n}{y}\right)}{\zeta\left(n, -\frac{n}{y}\right)}\right\} - e^y \left\{\lim_{n\to\infty} \frac{n^2}{y^2} \frac{\zeta\left(1+n, -\frac{n}{y}\right)}{\zeta\left(n, -\frac{n}{y}\right)}\right\} \\
= &-e^y
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2(-1)^n}{y^2} \frac{\zeta\left(1+n, \frac{n}{y}\right)}{\zeta\left(n, -\frac{n}{y}\right)} &= e^y \left\{-1 + \lim_{n\to\infty} \frac{n^2}{y^2} \frac{\zeta\left(1+n, -\frac{n}{y}\right)}{\zeta\left(n, -\frac{n}{y}\right)}\right\} \\
&= e^y \left\{\lim_{n\to\infty} \frac{n^2}{y^2} \left(-\frac{y^2}{n^2} + \frac{\zeta\left(1+n, -\frac{n}{y}\right)}{\zeta\left(n, -\frac{n}{y}\right)}\right)\right\}.
\end{aligned}$$

Then, we obtain the desired result. □

Theorem 2.5. *Let $\lambda \in \mathbb{R}$. Then, we have*

$$\lim_{n\to\infty} \frac{(-1)^n \zeta\left(1+n, \frac{n}{\lambda y}\right)}{\zeta\left(1+n, -\frac{n}{\lambda y}\right)} = \lim_{n\to\infty} \left(\frac{(-1)^n \zeta\left(1+n, \frac{n}{y}\right)}{\zeta\left(1+n, -\frac{n}{y}\right)}\right)^\lambda.$$

Proof. It is immediately seen from Theorem 2.4 for $y \to \lambda y$. □

Theorem 2.6.

$$\lim_{n\to\infty} \left(\frac{y}{n}\right)^{-n} \left\{\zeta\left(n, \frac{n}{y}\right) + (-1)^n \zeta\left(n, -\frac{n}{y}\right)\right\} = 1 \tag{2.5}$$

Proof. It is easily seen from (2.1) and (2.3). □

Theorem 2.7. *Let $\lambda \in \mathbb{R}$. Then, we have*

$$\begin{aligned}
&\lim_{n\to\infty} \left(\frac{\lambda y}{n}\right)^{-n} \left\{\zeta\left(n, \frac{n}{\lambda y}\right) + (-1)^n \zeta\left(n, -\frac{n}{\lambda y}\right)\right\} \\
= &\lim_{n\to\infty} \left(\frac{y}{n}\right)^{-\lambda n} \left\{\zeta\left(n, \frac{n}{y}\right) + (-1)^n \zeta\left(n, -\frac{n}{y}\right)\right\}^\lambda \\
= &\lim_{n\to\infty} \left(\frac{y}{n}\right)^{-n} \left\{\zeta\left(n, \frac{n}{y}\right) + (-1)^n \zeta\left(n, -\frac{n}{y}\right)\right\}.
\end{aligned}$$

Proof. It is easily seen from (2.5). □

Theorem 2.8. *Let $y > 0$. Then, we have*

$$\lim_{n\to\infty} n^2 \left(\frac{y}{n}\right)^{-n} \left\{\zeta\left(1+n, \frac{n}{y}\right) - (-1)^n \zeta\left(1+n, -\frac{n}{y}\right)\right\} = \infty. \tag{2.6}$$

Proof. We take the first derivative respect to y into (2.5):

$$\begin{aligned} & -\lim_{n\to\infty} \left(\frac{y}{n}\right)^{-1-n} \left\{\zeta\left(n, \frac{n}{y}\right) + (-1)^n \zeta\left(n, -\frac{n}{y}\right)\right\} \\ & +\lim_{n\to\infty} \left(\frac{y}{n}\right)^{-n} \left\{\frac{n^2}{y^2}\zeta\left(1+n, \frac{n}{y}\right) - (-1)^n \frac{n^2}{y^2}\zeta\left(1+n, -\frac{n}{y}\right)\right\} \\ = \ & 0. \end{aligned}$$

By using (2.5) into the above equation, we obtain the desired result. □

Theorem 2.9. *Let $\lambda, y > 0$. Then, we have*

$$\begin{aligned} & \lim_{n\to\infty} n^2 \left(\frac{\lambda y}{n}\right)^{-n} \left\{\zeta\left(1+n, \frac{n}{\lambda y}\right) - (-1)^n \zeta\left(1+n, -\frac{n}{\lambda y}\right)\right\} \\ = \ & \lim_{n\to\infty} n^{2\lambda} \left(\frac{y}{n}\right)^{-\lambda n} \left\{\zeta\left(1+n, \frac{n}{y}\right) - (-1)^n \zeta\left(1+n, -\frac{n}{y}\right)\right\}^{\lambda} \\ = \ & \lim_{n\to\infty} n^2 \left(\frac{y}{n}\right)^{-n} \left\{\zeta\left(1+n, \frac{n}{y}\right) - (-1)^n \zeta\left(1+n, -\frac{n}{y}\right)\right\}. \end{aligned}$$

Proof. It is easily seen from (2.6). □

We note that

$$\begin{aligned} \frac{d}{dy}\zeta\left(1+n, \frac{n}{y}\right) &= \frac{d}{dy}\sum_{k=0}^{\infty} \frac{1}{\left(\frac{n}{y}+k\right)^{n+1}} \\ &= \sum_{k=0}^{\infty} \frac{d}{dy}\left\{\frac{1}{\left(\frac{n}{y}+k\right)^{n+1}}\right\} \\ &= \frac{n(n+1)}{y^2}\sum_{k=0}^{\infty} \frac{1}{\left(\frac{n}{y}+k\right)^{2+n}} \\ &= \frac{n(n+1)}{y^2}\zeta\left(2+n, \frac{n}{y}\right) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dy}\zeta\left(1+n, -\frac{n}{y}\right) &= \frac{d}{dy}\sum_{k=0}^{\infty} \frac{1}{\left(-\frac{n}{y}+k\right)^{n+1}} \\ &= \sum_{k=0}^{\infty} \frac{d}{dy}\left\{\frac{1}{\left(-\frac{n}{y}+k\right)^{n+1}}\right\} \\ &= -\frac{n(n+1)}{y^2}\sum_{k=0}^{\infty} \frac{1}{\left(-\frac{n}{y}+k\right)^{2+n}} \\ &= -\frac{n(n+1)}{y^2}\zeta\left(2+n, -\frac{n}{y}\right). \end{aligned}$$

Theorem 2.10.

$$\lim_{n\to\infty}\frac{(-1)^n\,\zeta\left(2+n,\frac{n}{y}\right)}{\zeta\left(2+n,-\frac{n}{y}\right)}=-e^y. \tag{2.7}$$

Proof. We take the first derivative respect to y into (2.4):

$$\begin{aligned}
&\lim_{n\to\infty}\frac{(-1)^n\left\{\frac{n(n+1)}{y^2}\zeta\left(2+n,\frac{n}{y}\right)\zeta\left(1+n,-\frac{n}{y}\right)+\frac{n(n+1)}{y^2}\zeta\left(2+n,-\frac{n}{y}\right)\zeta\left(1+n,\frac{n}{y}\right)\right\}}{\left(\zeta\left(1+n,-\frac{n}{y}\right)\right)^2}\\
&= \left\{\lim_{n\to\infty}\frac{n(n+1)(-1)^n}{y^2}\frac{\zeta\left(2+n,\frac{n}{y}\right)}{\zeta\left(1+n,-\frac{n}{y}\right)}\right\}+e^y\left\{\lim_{n\to\infty}\frac{n(n+1)}{y^2}\frac{\zeta\left(2+n,-\frac{n}{y}\right)}{\zeta\left(1+n,-\frac{n}{y}\right)}\right\}\\
&= -e^y
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\lim_{n\to\infty}\frac{n(n+1)(-1)^n}{y^2}\frac{\zeta\left(2+n,\frac{n}{y}\right)}{\zeta\left(1+n,-\frac{n}{y}\right)} &= e^y\left\{1-\lim_{n\to\infty}\frac{n(n+1)}{y^2}\frac{\zeta\left(2+n,-\frac{n}{y}\right)}{\zeta\left(1+n,-\frac{n}{y}\right)}\right\}\\
&= e^y\left\{\lim_{n\to\infty}\frac{n(n+1)}{y^2}\left(\frac{y^2}{n(n+1)}-\frac{\zeta\left(2+n,-\frac{n}{y}\right)}{\zeta\left(1+n,-\frac{n}{y}\right)}\right)\right\}.
\end{aligned}$$

Then, we obtain the desired result. □

Theorem 2.11. *Let $\lambda\in\mathbb{R}$. Then, we have*

$$\lim_{n\to\infty}\frac{(-1)^{n+1}\,\zeta\left(2+n,\frac{n}{\lambda y}\right)}{\zeta\left(2+n,-\frac{n}{\lambda y}\right)}=\lim_{n\to\infty}\left(\frac{(-1)^{n+1}\,\zeta\left(2+n,\frac{n}{y}\right)}{\zeta\left(2+n,-\frac{n}{y}\right)}\right)^{\lambda}.$$

Proof. It is immediately seen from Theorem 2.10 for $y\to\lambda y$. □

Theorem 2.12. *Let $y>0$. Then, we have*

$$\lim_{n\to\infty}n^3(n+1)\left(\frac{y}{n}\right)^{-n}\left\{\zeta\left(2+n,\frac{n}{y}\right)+(-1)^n\,\zeta\left(2+n,-\frac{n}{y}\right)\right\}=\infty. \tag{2.8}$$

Proof. We take the first derivative respect to y into (2.6):

$$\begin{aligned}
&\lim_{n\to\infty}n^2\left(\frac{y}{n}\right)^{-1-n}\left\{-\zeta\left(1+n,\frac{n}{y}\right)+(-1)^n\,\zeta\left(1+n,-\frac{n}{y}\right)\right\}\\
&+\lim_{n\to\infty}n^2\left(\frac{y}{n}\right)^{-n}\left\{\frac{n(n+1)}{y^2}\zeta\left(2+n,\frac{n}{y}\right)+(-1)^n\,\frac{n(n+1)}{y^2}\zeta\left(2+n,-\frac{n}{y}\right)\right\}=\infty.
\end{aligned}$$

By using (2.6) into the above equation, we obtain the desired result. □

Theorem 2.13. *Let $\lambda, y>0$. Then, we have*

$$\begin{aligned}
&\lim_{n\to\infty}n^3(n+1)\left(\frac{\lambda y}{n}\right)^{-n}\left\{\zeta\left(2+n,\frac{n}{\lambda y}\right)+(-1)^n\,\zeta\left(2+n,-\frac{n}{\lambda y}\right)\right\}\\
&= \lim_{n\to\infty}(n^3(n+1))^{\lambda}\left(\frac{y}{n}\right)^{-\lambda n}\left\{\zeta\left(2+n,\frac{n}{y}\right)+(-1)^n\,\zeta\left(2+n,-\frac{n}{y}\right)\right\}^{\lambda}\\
&= \lim_{n\to\infty}n^3(n+1)\left(\frac{y}{n}\right)^{-n}\left\{\zeta\left(2+n,\frac{n}{y}\right)+(-1)^n\,\zeta\left(2+n,-\frac{n}{y}\right)\right\}.
\end{aligned}$$

Proof. It is easily seen from (2.8). □

3. Applications

Finally, we give some approximations related to Bernoulli polynomials, by using the following property:

$$\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1} \text{ (cf. [1], [3], [4], [8]).} \tag{3.1}$$

From (2.1), we set

$$\frac{e^y}{e^y-1} = \lim_{n\to-\infty}\left(-\frac{y}{n}\right)^n \zeta\left(-n,-\frac{n}{y}\right). \tag{3.2}$$

Then, we choose $a=-n/y$ into (3.1):

$$\zeta\left(-n,-\frac{n}{y}\right) = -\frac{B_{n+1}\left(-\frac{n}{y}\right)}{n+1}. \tag{3.3}$$

By using (3.2) and (3.3), we arrive at the following corollary:

Corollary 3.1.

$$\lim_{n\to-\infty}\left(-\frac{y}{n}\right)^n \frac{B_{n+1}\left(-\frac{n}{y}\right)}{n+1} = \frac{e^y}{1-e^y}. \tag{3.4}$$

Putting $y \to -y$ in (3.4), we have

$$\lim_{n\to-\infty}\left(\frac{y}{n}\right)^n \frac{B_{n+1}\left(\frac{n}{y}\right)}{n+1} = \frac{1}{e^y-1}. \tag{3.5}$$

From (3.4) and (3.5), we arrive at the following corollary:

Corollary 3.2.

$$\lim_{n\to-\infty}\frac{(-1)^n B_{n+1}\left(-\frac{n}{y}\right)}{B_{n+1}\left(\frac{n}{y}\right)} = -e^y.$$

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