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On Fully Degenerate Bell Numbers and Polynomials

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Abstract. Recently, the partially degenerate Bell numbers and polynomials were introduced as a degenerate version of Bell numbers and polynomials. In this paper, as a further degeneration of them, we study fully degenerate Bell numbers and polynomials. Among other things, we derive various expressions for the fully degenerate Bell numbers and polynomials.

1. Introduction

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad (\text{see } [4, 9, 11-14]). \tag{1}$$

Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$. For brevity, we also write

$$e_\lambda(t) = e_\lambda^1(t). \tag{2}$$

It is well known that the degenerate Stirling numbers of the second kind are given by

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (\text{see } [9]). \tag{3}$$

Note that $\lim_{\lambda\to 0} S_{2,\lambda}(n,k) = S_2(n,k)$, where $S_2(n,k)$ are the ordinary Stirling numbers of the second kind.

The Bell polynomials (also called Tochard or exponential polynomials and denoted by $\phi_n(x)$) are defined by the generating function

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad (\text{see } [1-3, 5-8, 10]). \tag{4}$$

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From (4), we note that

$$B_n(x) = e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k, \quad \text{(see [8,15])}, \tag{5}$$

which are known as Dobinski's formula.

It is not difficult to show that

$$B_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k, (n \geq 0), \quad \text{(see [7,8,15,16])}. \tag{6}$$

In [10], the partially degenerate Bell polynomials are introduced as

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!}. \tag{7}$$

When $x = 1$, $b_{n,\lambda} = b_{n,\lambda}(1)$ are called the partially degenerate Bell numbers.

From (7), we note that

$$b_{n,\lambda}(x) = e^{-x}\sum_{k=0}^{\infty}\frac{(k)_{n,\lambda}}{k!}x^k, \quad \text{(see [12])}, \tag{8}$$

where $(k)_{0,\lambda} = 1, (k)_{n,\lambda} = k(k-\lambda)(k-2\lambda)\cdots(k-(n-1)\lambda), (n \geq 1)$.

Recently, the partially degenerate Bell numbers and polynomials were introduced as a degenerate version of Bell numbers and polynomials. In this paper, as a further degeneration of them, we study fully degenerate Bell numbers and polynomials. Among other things, we derive various expressions for the fully degenerate Bell numbers and polynomials.

2. Fully degenerate Bell numbers and polynomials

Motivated by (4), we consider the fully degenerate Bell polynomials, $B_{n,\lambda}(n \geq 0)$, which are given by

$$e_\lambda(x(e_\lambda(t)-1)) = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}, \quad (\lambda \in \mathbb{R}). \tag{9}$$

When $x = 1$, $B_{n,\lambda} = B_{n,\lambda}(1)$ are called the fully degenerate Bell numbers.

Note that

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{\lambda\to 0} B_{n,\lambda}(x)\frac{t^n}{n!} &= \lim_{\lambda\to 0} e_\lambda(x(e_\lambda(t)-1)) \\ &= \lim_{\lambda\to 0}(1+\lambda x((1+\lambda t)^{\frac{1}{\lambda}}-1))^{\frac{1}{\lambda}} \\ &= e^{x(e^t-1)} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.\end{aligned} \tag{10}$$

By comparing the coefficients on both sides, we get

$$\lim_{\lambda\to 0} B_{n,\lambda}(x) = B_n(x), \quad (n \geq 0).$$

From (9), we have

$$\begin{aligned} e_\lambda(x(e_\lambda(t)-1)) &= (1+\lambda x(e_\lambda(t)-1))^{\frac{1}{\lambda}} \\ &= \sum_{k=0}^{\infty}(1)_{k,\lambda}x^k\frac{1}{k!}(e_\lambda(t)-1)^k \\ &= \sum_{k=0}^{\infty}(1)_{k,\lambda}x^k\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}(1)_{k,\lambda}x^kS_{2,\lambda}(n,k)\Big)\frac{t^n}{n!}. \end{aligned} \tag{11}$$

Therefore, by (9) and (11), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$B_{n,\lambda}(x) = \sum_{k=0}^{n}(1)_{k,\lambda}x^kS_{2,\lambda}(n,k).$$

In particular,

$$B_{n,\lambda} = \sum_{k=0}^{n}(1)_{k,\lambda}S_{2,\lambda}(n,k).$$

By (9), we get

$$\begin{aligned} e_\lambda(x(e_\lambda(t)-1)) &= e^{\frac{1}{\lambda}\log(1+\lambda x(e_\lambda(t)-1))} \\ &= \sum_{k=0}^{\infty}\lambda^{-k}\frac{1}{k!}\Big(\log(1+\lambda x(e_\lambda(t)-1))\Big)^k \\ &= \sum_{k=0}^{\infty}\lambda^{-k}\sum_{l=k}^{\infty}S_1(l,k)\lambda^lx^l\frac{1}{l!}(e_\lambda(t)-1)^l \\ &= \sum_{k=0}^{\infty}\lambda^{-k}\sum_{l=k}^{\infty}S_1(l,k)\lambda^lx^l\sum_{n=l}^{\infty}S_{2,\lambda}(n,l)\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\Big(\sum_{l=k}^{n}S_1(l,k)S_{2,\lambda}(n,l)\lambda^{l-k}x^l\Big)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\sum_{l=k}^{n}S_1(l,k)S_{2,\lambda}(n,l)\lambda^{l-k}x^l\Big)\frac{t^n}{n!}, \end{aligned} \tag{12}$$

where $S_1(n,k)$ are the Stirling numbers of the first kind.

Therefore, by (9) and (12), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$B_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{l=k}^{n}S_1(l,k)S_{2,\lambda}(n,l)\lambda^{l-k}x^l.$$

From (9), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!} &= e_\lambda(x(e_\lambda(t)-1)) \\
&= (1+\lambda x((1+\lambda t)^{\frac{1}{\lambda}}-1))^{\frac{1}{\lambda}} \\
&= \sum_{l=0}^{\infty}(1)_{l,\lambda}x^l\frac{1}{l!}((1+\lambda t)^{\frac{1}{\lambda}}-1))^l \\
&= \sum_{l=0}^{\infty}(1)_{l,\lambda}x^l\frac{1}{l!}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1+\lambda t)^{\frac{m}{\lambda}} \\
&= \sum_{l=0}^{\infty}(1)_{l,\lambda}x^l\frac{1}{l!}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}\sum_{n=0}^{\infty}(m)_{n,\lambda}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\Big(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1)_{l,\lambda}(m)_{n,\lambda}x^l\frac{1}{l!}\Big)\frac{t^n}{n!}.
\end{aligned} \tag{13}$$

Therefore, by comparing the coefficients on both sides of (13), we obtain the following theorem.

Theorem 2.3. *(Dobinski-like formula) For $n \geq 0$, we have*

$$B_{n,\lambda}(x) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1)_{l,\lambda}(m)_{n,\lambda}x^l\frac{1}{l!}.$$

In particular,

$$B_{n,\lambda} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1)_{l,\lambda}(m)_{n,\lambda}\frac{1}{l!}.$$

Remark. By (5), we get

$$\begin{aligned}
B_n(x) &= e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k \\
&= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}x^l\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k \\
&= \sum_{m=0}^{\infty}\Big(\sum_{k=0}^{m}\frac{k^n}{k!}\frac{(-1)^{m-k}m!}{(m-k)!}\Big)\frac{x^m}{m!} \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}k^n\frac{1}{m!}x^m.
\end{aligned} \tag{14}$$

From Theorem 2.3, we note that

$$\lim_{\lambda\to 0}B_{n,\lambda}(x) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}k^n\frac{1}{m!}x^m = B_n(x).$$

Now, we observe that

$$\begin{aligned}
&\sum_{n=1}^{\infty} B_{n,\lambda}(x)\frac{t^{n-1}}{(n-1)!}\\
&= \frac{\partial}{\partial t} e_\lambda(x(e_\lambda(t)-1))\\
&= \frac{\partial}{\partial t}(1+\lambda x((1+\lambda t)^{\frac{1}{\lambda}}-1))^{\frac{1}{\lambda}}\\
&= x(1+\lambda x((1+\lambda t)^{\frac{1}{\lambda}}-1))^{\frac{1}{\lambda}(1-\lambda)}(1+\lambda t)^{\frac{1}{\lambda}(1-\lambda)}\\
&= x e_\lambda^{1-\lambda}(x(e_\lambda(t)-1))e_\lambda^{1-\lambda}(t)\\
&= x\sum_{l=0}^{\infty}(1-\lambda)_{l,\lambda}\frac{x^l}{l!}(e_\lambda(t)-1)^l\sum_{m=0}^{\infty}(1-\lambda)_{m,\lambda}\frac{t^m}{m!}\\
&= x\sum_{l=0}^{\infty}(1-\lambda)_{l,\lambda}x^l\sum_{k=l}^{\infty}S_{2,\lambda}(k,l)\frac{t^k}{k!}\sum_{m=0}^{\infty}(1-\lambda)_{m,\lambda}\frac{t^m}{m!}\\
&= x\sum_{k=0}^{\infty}\sum_{l=0}^{k}(1-\lambda)_{l,\lambda}x^l S_{2,\lambda}(k,l)\frac{t^k}{k!}\sum_{m=0}^{\infty}(1-\lambda)_{m,\lambda}\frac{t^m}{m!}\\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(1-\lambda)_{l,\lambda}x^{l+1}S_{2,\lambda}(k,l)(1-\lambda)_{n-k,\lambda}\Big)\frac{t^n}{n!}.
\end{aligned} \tag{15}$$

By (15), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_{n+1,\lambda}(x)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(1-\lambda)_{l,\lambda}x^{l+1}S_{2,\lambda}(k,l)(1-\lambda)_{n-k,\lambda}\Big)\frac{t^n}{n!}.
\end{aligned} \tag{16}$$

Therefore, by comparing the coefficients on both sides of (16), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$B_{n+1,\lambda}(x) = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(1-\lambda)_{l,\lambda}x^{l+1}S_{2,\lambda}(k,l)(1-\lambda)_{n-k,\lambda}.$$

In particular,

$$B_{n+1,\lambda} = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(1-\lambda)_{l,\lambda}S_{2,\lambda}(k,l)(1-\lambda)_{n-k,\lambda}.$$

Note that

$$\begin{aligned}
\lim_{\lambda\to 0} B_{n+1,\lambda}(x) &= \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}x^{l+1}S_2(k,l)\\
&= x\sum_{k=0}^{n}\binom{n}{k}B_k(x)\\
&= B_{n+1}(x).
\end{aligned}$$

For $n \in \mathbb{N}$, by Theorem 2.3, we get

$$\begin{aligned}
B_{n,\lambda}(x) &= \sum_{l=1}^{\infty}\sum_{m=1}^{l}\binom{l}{m}(-1)^{l-m}(1)_{l,\lambda}(m)_{n,\lambda}\frac{1}{l!}x^l \\
&= \sum_{l=1}^{\infty}\sum_{m=0}^{l-1}\binom{l}{m+1}(-1)^{l-m-1}(1)_{l,\lambda}(m+1)_{n,\lambda}\frac{1}{l!}x^l \\
&= \sum_{l=1}^{\infty}\sum_{m=0}^{l-1}\frac{l!}{(m+1)!(l-m-1)!}(-1)^{l-m-1}(1)_{l,\lambda}(m+1)_{n,\lambda}\frac{1}{l!}x^l \\
&= x\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{(-1)^{l-m}}{(l-m)!}\frac{1}{m!}(1)_{l+1,\lambda}\left(\sum_{k=0}^{n}S_1(n,k)\lambda^{n-k}(m+1)^{k-1}\right)x^l
\end{aligned}$$

$$\begin{aligned}
&= x\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1)_{l+1,\lambda}\sum_{k=0}^{n}S_1(n,k)\lambda^{n-k}(m+1)^{k-1}\frac{x^l}{l!} \\
&= x\sum_{k=0}^{n}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1)_{l+1,\lambda}S_1(n,k)\frac{x^l}{l!}\lambda^{n-k}(m+1)^{k-1} \\
&= x\sum_{k=0}^{n}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1)_{l+1,\lambda}S_1(n,k)\frac{x^l}{l!}\lambda^{n-k}\sum_{j=0}^{k-1}\binom{k-1}{j}m^j \qquad (17)\\
&= x\sum_{k=0}^{n}\lambda^{n-k}S_1(n,k)\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1)_{l+1,\lambda}\frac{x^l}{l!}\sum_{j=1}^{k}\binom{k-1}{j-1}m^{j-1} \\
&= x\sum_{k=1}^{n}\sum_{j=1}^{k}\lambda^{n-k}S_1(n,k)\binom{k-1}{j-1}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1)_{l+1,\lambda}m^{j-1}\frac{x^l}{l!}.
\end{aligned}$$

By comparing the coefficients on both sides of (17), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}$, we have*

$$B_{n,\lambda}(x) = x\sum_{k=1}^{n}\sum_{j=1}^{k}\lambda^{n-k}S_1(n,k)\binom{k-1}{j-1}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1)_{l+1,\lambda}m^{j-1}\frac{x^l}{l!}.$$

In particular,

$$B_{n,\lambda} = \sum_{k=1}^{n}\sum_{j=1}^{k}\lambda^{n-k}S_1(n,k)\binom{k-1}{j-1}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1)_{l+1,\lambda}m^{j-1}\frac{1}{l!}.$$

From (9), we can derive the following equation.

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{d}{dx}B_{n,\lambda}(x)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\frac{d}{dx}B_{n,\lambda}(x)\frac{t^n}{n!}\\
&=\frac{\partial}{\partial x}e_\lambda(x(e_\lambda(t)-1))\\
&=(e_\lambda(t)-1)\frac{e_\lambda(x(e_\lambda(t)-1))}{1+\lambda x((1+\lambda t)^{\frac{1}{\lambda}}-1)}\\
&=\frac{e_\lambda(t)-1}{1+\lambda x((1+\lambda t)^{\frac{1}{\lambda}}-1)}e_\lambda(x(e_\lambda(t)-1))\\
&=\frac{1}{\lambda}\frac{d}{dx}\log(1+\lambda x(e_\lambda(t)-1))e_\lambda(x(e_\lambda(t)-1))\\
&=\frac{1}{\lambda}\frac{d}{dx}\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l}\lambda^l x^l(e_\lambda(t)-1)^l\sum_{m=0}^{\infty}B_{m,\lambda}(x)\frac{t^m}{m!}\\
&=\sum_{l=1}^{\infty}(-1)^{l-1}\lambda^{l-1}x^{l-1}l!\frac{1}{l!}(e_\lambda(t)-1)^l\sum_{m=0}^{\infty}B_{m,\lambda}(x)\frac{t^m}{m!}\\
&=\sum_{l=1}^{\infty}(-1)^{l-1}\lambda^{l-1}x^{l-1}l!\sum_{k=l}^{\infty}S_{2,\lambda}(k,l)\frac{t^k}{k!}\sum_{m=0}^{\infty}B_{m,\lambda}(x)\frac{t^m}{m!}\\
&=\sum_{k=1}^{\infty}(\sum_{l=1}^{k}(-1)^{l-1}\lambda^{l-1}x^{l-1}l!S_{2,\lambda}(k,l))\frac{t^k}{k!}\sum_{m=0}^{\infty}B_{m,\lambda}(x)\frac{t^m}{m!}\\
&=\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}\sum_{l=1}^{k}\binom{n}{k}(-1)^{l-1}\lambda^{l-1}x^{l-1}l!S_{2,\lambda}(k,l))B_{n-k,\lambda}(x)\Big)\frac{t^n}{n!}.
\end{aligned}
\tag{18}
$$

Therefore, by comparing the coefficients on both sides of (18), we obtain the following theorem.

Theorem 2.6. *For $n \geq 1$, we have*

$$\frac{d}{dx}B_{n,\lambda}(x)=\sum_{k=1}^{n}\sum_{l=1}^{k}\binom{n}{k}(-1)^{l-1}\lambda^{l-1}x^{l-1}l!S_{2,\lambda}(k,l))B_{n-k,\lambda}(x).$$

Note that

$$
\begin{aligned}
\lim_{\lambda\to 0}\frac{d}{dx}B_{n,\lambda}(x)&=\sum_{k=1}^{n}\binom{n}{k}B_{n-k}(x)\\
&=\sum_{k=0}^{n-1}\binom{n}{k}B_k(x)\\
&=\frac{d}{dx}B_n(x),\ (n\in\mathbb{N}).
\end{aligned}
$$

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