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# A Decomposition of Arf Semigroups

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**Abstract.** The aim of this work is to exhibit a kind of primitive semigroup decomposition of Arf semigroups using combinatorial properties of partitions of a positive integer $n$.

## 1. Introduction

Numerical semigroups have several applications to many branches of mathematics. They have become important because of their applications in algebraic geometry, coding theory during the half of the last century, see [1, 2, 4, 7, 8, 12].

A numerical semigroup $S$ is a monoid of $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and has a finite complement $G(S) = \mathbb{N}_0 \backslash S$. The elements of $G(S)$ are called *gaps* of $S$. The largest element of $G(S)$ is called the *Frobenius number* of $S$ and denoted by $F(S)$. The *conductor* of $S$ is the number $c := F(S) + 1$. We say that $S$ is generated by $A \subseteq S$, if $S = \{\sum_{i=1}^{m} h_i a_i : m \in \mathbb{N}, h_i \in \mathbb{N}_0, a_i \in A,\ i = 1, \ldots, m\}$. In this case, $A$ is a *system of generators* of $S$ and we denote $S$ by $\langle A \rangle$. Note that a system of generators of a numerical semigroup is a *minimal system of generators* if none of its proper subsets generates the numerical semigroup. If $\{n_1 < n_2 < \cdots < n_e\}$ is the minimal system of generators of $S$, then $n_1$ is called the *multiplicity*, and $e$ is called the *embedding dimension* of $S$. We say that $S$ has *maximal embedding dimension* if $e = n_1$.

If $S$ is a numerical semigroup, then unless otherwise stated we assume $S = \{0 = s_0, s_1, \ldots, s_r = F(S)+1, \longrightarrow \}$, where "$\longrightarrow$" means that all subsequent natural numbers which are bigger then $s_r$ belong to $S$ and $r$ denotes the number of small elements of $S$.

Partitions occur in several branches of mathematics, including the study of symmetric polynomials, the symmetric groups in group representation theory, see [6]. A *partition* $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r]$ of a positive integer $n$ is a non increasing list of positive integers, $\lambda_r \leq \lambda_{r-1} \leq \ldots \leq \lambda_1$, whose sum is $n$ and length is $r$. We refer to the $\lambda_i$ as a *part* of partition $\lambda$. If $\lambda_i \neq \lambda_{i+1}$, $1 \leq i \leq r-1$, then we called $\lambda$ is a *strict dominant* partition.

The *Young diagram* of $\lambda$ consists of a left-justified shape of $r$ columns of boxes with lengths $\lambda_1, \lambda_2, \ldots, \lambda_r$. If there are $r$ columns in a Young diagram and there are $u_i$ rows of length $i$, for $i = 1, \ldots, r$, then we denote this diagram of the form $\mathbf{1}^{u_1}\mathbf{2}^{u_2}\cdots\mathbf{r}^{u_r}$ and $n = \sum_{j=1}^{r} j u_j$. If $u_j = 0$ for some $1 \leq j \leq r$, then we omit $j^0$ in the

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presentation of a Young diagram $Y$ and if $u_j = 1$, then we write $\mathbf{j}$. If $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r]$ is the partition corresponding to $Y = \mathbf{1}^{u_1}\mathbf{2}^{u_2}\cdots\mathbf{r}^{u_r}$, then $\lambda_j = \sum_{i=j}^{r} u_i$ , $1 \leq j \leq r$. Note that $\lambda_j - \lambda_{j+1} = u_j$ for each $j = 1, \ldots, r-1$ and $\lambda_r = u_r$. We will use the notation $m{\cdot}\lambda_{\mathbf{1}} := [\overbrace{\lambda_1, \ldots, \lambda_1}^{m \text{ times}}]$ to avoid misunderstanding. Flipping a diagram over its main diagonal (from upper left to lower right) gives the *conjugate* diagram, the conjugate of $\lambda$ will be denoted here by $c(\lambda)$. If $\lambda = c(\lambda)$, then we say $\lambda$ is a *symmetric* partition. Given a box of a Young diagram, the shape formed by the boxes directly to the right of it, the boxes directly below it and the box itself is called the *hook* of that box. The number of boxes in a column (or a row) is called the *length* of that column (or, respectively, that row). The boxes to the right form the arm and the boxes below form the leg of the hook. The hook of a box is a column if it has no arm, it is a row if it has no leg, and it consists of the box itself if it has no arm and no leg. The number of boxes in the hook of a box is called the *hook-length* of that box. The *Young tableau* (plural, "tableaux") of a Young diagram is obtained by placing the numbers $1, \ldots, m$ in the diagram which has $m$ boxes.

A connection among partitions, Young diagrams, numerical semigroups was given by [3, 5, 10, 11, 14]. We think of a path as lying in $\mathbb{N}^2$ with bottom left corner of Young diagram at the origin. Starting with $x = 0$. If $x \in S$, then we draw a line segment of unit length to the right. If $x \notin S$, then we draw a line segment of unit length up. Repeat for $x + 1$. For any $x$ greater than the Frobenius number of $S$ we draw a line to the right. The lattice lying above the path and below the horizontal line defines a *Young diagram of* $S$ (see [3, 5]). If $Y_S$ is the Young diagram of a numerical semigroup $S$, then we denote the $j$th column by $G_j$, for each $j \geq 0$. We know that 0th column, $G_0$, gives the gap set $G(S)$ of $S$.

$S = \{0, 4, 7, 8, 11, \rightarrow\}$ is a numerical semigroup and we have the following Young tableau for $S$:

$$Y_S = \begin{array}{|c|c|c|c|} \hline 10 & 6 & 3 & 2 \\ \hline 9 & 5 & 2 & 1 \\ \hline 6 & 2 & & \\ \cline{1-2} 5 & 1 & & \\ \cline{1-2} 3 & & & \\ \cline{1-1} 2 & & & \\ \cline{1-1} 1 & & & \\ \cline{1-1} \end{array}$$

This Young tableau consists of a Young diagram with the hook lengths of each box in the diagram.

Let $\check{S}$ be the set of numerical semigroups, $\hat{Y}$ be the set of Young diagrams and let $P$ be the set of partitions. Here we define the following maps :

$\sigma : \check{S} \rightarrow \hat{Y}$, $\sigma(S) = Y_S$, where $Y_S$ is the Young diagram of $S$.

$\tau : \hat{Y} \rightarrow P$, $\sigma(Y) = \lambda$, where $\lambda$ is the partition of the Young diagram $Y$.

The map $S \xrightarrow{\sigma} Y_S \xrightarrow{\tau} \lambda$ is an injection between the set of numerical semigroups and the set of partitions. For a numerical semigroup $S$, $\lambda = \tau\sigma(S)$ is called the *partition of* $S$.

For a given numerical semigroup $S$, we have several related semigroups. For each $i \geq 0$, the sets $S_i$ and $S(i)$ defined as follows:

$$S_i = \{s \in S : s \geq s_i\}, \quad S - s_i = \{s - s_i \in \mathbb{N}_0 : s \in S\}$$

$$S(i) = S - S_i = \{z \in \mathbb{N}_0 : z + S_i \subseteq S\}.$$

It is obvious that every $S(i)$ is itself a numerical semigroup, and we obtain the following chain:

$$\cdots \subset S_r \subset S_{r-1} \subset \cdots \subset S_1 \subset S \subset S(1) \subset \cdots \subset S(r) = \mathbb{N}_0.$$

For $i \geq 1$ we define $i$th *type set* $T(i) := S(i) \backslash S(i-1)$ and $t_i = |T(i)|$. We call $((t_i) : i \geq 1)$ the *type sequence* of $S$.

A numerical semigroup $S$ is called *Arf semigroup* if $x + y - z \in S$, for all $x, y, z \in S$ with $z \leq y \leq x$. This property is equivalent to, $2x - y \in S$, for all $x, y \in S$ with $y \leq x$. An Arf semigroup has maximal embedding dimension. There are several equivalent conditions on Arf semigroups, see [2, 8, 9, 12, 13].

The combinatorial properties of an Arf semigroup allow us to define an Arf partition of a positive integer $n$. In [14], the concept of Arf partition was firstly introduced. In [15], the authors analyzed the relation among an Arf partition, its Young dual diagram, and the corresponding rational Young diagram. Here, we continue these works. In Section 2, firstly, we recall the construction of Arf partitions with respect to a Young diagram, then we add some new properties to Arf partitions.

For a given hook, if $u$ denotes the number of boxes of the leg and $\mathbf{x}-1$ denotes the number of boxes of the arm, then this hook can be represented by $\mathbf{1}^u\mathbf{x}$. If $\mathbf{x}=0$, then the hook is a leg. Let $K=\{\mathbf{1}^u\mathbf{x} : u\geq 0,\ \mathbf{x}\geq 2 \text{ or } \mathbf{x}=0\}$ be the set of hooks. Then we define an operation $\odot$ on $K$ such that

$$\Gamma_1\odot\Gamma_2=\mathbf{1}^{u_1-(u_2+1)}\mathbf{2}^{u_2}(\mathbf{x}_2{+}1)\mathbf{x}_1$$

where $\Gamma_1=\mathbf{1}^{u_1}\mathbf{x}_1$, $\Gamma_2=\mathbf{1}^{u_2}\mathbf{x}_2$ and $u_1>u_2$, $\mathbf{x}_1-1\geq\mathbf{x}_2$. In Section 3, Lemma 3.3 states that any partition $\lambda$ can be written with respect to the operation $\odot$ on the set of hooks. Additionally, using Lemma 3.3, we exhibit a primitive semigroup decomposition of an Arf semigroup via the operation $\odot$ in Theorem 3.6.

## 2. Some properties of Arf Partitions

For the calculation of hook lengths of a partition, using the definition, one can prove Lemma 2.1.

**Lemma 2.1.** *Let $\lambda=[\lambda_1,\ldots,\lambda_r]$ be the partition of a numerical semigroup $S$, $\lambda_i\neq 0$, $1\leq i\leq r$. From the bottom, the hook lengths of the $(\lambda_1)$st row of the Young diagram of $\lambda$ form the partition $\lambda+\rho$, and the hook lengths of $(\lambda_1+1)$st row form $\lambda+\rho+r\cdot 1$, where $\rho=[r-1,r-2,\ldots,1,0]$. The small elements of $S$ are obtained by $r\cdot(\lambda_1+r)-(\lambda+\rho)-r\cdot 1$. For $j\leq\lambda_1$, there exist $k,t\in\mathbb{N}_0$ such that the hook lengths of the $j$th row form the partition*

$$[(\lambda_1-\lambda_t)+k+t-1,(\lambda_2-\lambda_t)+k+t-2,\ldots,(\lambda_{t-1}-\lambda_t)+k].$$

If $\lambda$ is a partition and $Y=\mathbf{1}^{u_1}\mathbf{2}^{u_2}\cdots\mathbf{r}^{u_r}$ is the Young diagram of $\lambda$, then the complement of the hook set of the first column of $Y$ is $\{0,u_1+1,u_1+u_2+2,u_1+u_2+u_3+3,\ldots,u_1+u_2+\cdots+u_r+r,\longrightarrow\}$.

Here, in order to ensure completeness, we recall the characterization of the numerical semigroup $S$ with respect to the corresponding Young diagram $Y_S$. Theorem 2.2 proved in [14].

**Theorem 2.2.** *Let $S=\{s_0,s_1,\ldots,s_{r-1},s_r,\longrightarrow\}$ be a numerical semigroup and $Y_S$ be the Young diagram of $S$. Let $G_i$ be the hook set of the $i$th column of $Y_S$, $i\geq 0$. Then the following statements hold:*

1. *For $0\leq i\leq r-1$, we have $G_i=\{s-s_i : s\in G_0,\ s\geq s_i\}$.*
2. *For $0\leq i\leq r-1$, the set $G_i$ does not contain any element of $S$.*
3. *For $1\leq i\leq r$, we have $u_i=s_i-s_{i-1}-1$, and $Y_S=\mathbf{1}^{u_1}\mathbf{2}^{u_2}\cdots\mathbf{r}^{u_r}$. In this case, the conductor of $S$ is $c=r+\sum_{i=1}^{r}u_i$. If $u_i=0$, $1\leq i\leq r$, then $S=\mathbb{N}_0$.*
4. *If $Y_S=\mathbf{1}^{u_1}\mathbf{2}^{u_2}\cdots\mathbf{r}^{u_r}$, then $|G_i|=s_r-s_i-(r-i)$, $0\leq i\leq r-1$.*
5. *The first hook length of $G_i$ is $\min\{b\in G(S) : b>s_i\}-s_i$, $i\geq 1$, the last hook length of $G_i$ is $F(S)-s_i$.*
6. *$S(i)=\bigcap_{j\geq i}(S-s_j)=\mathbb{N}_0\backslash\bigcup_{j=i}^{r-1}G_j$.*
7. *$x\in T(i)$ if and only if $x\in G_{i-1}$ and $x\notin G_j$, $i-1<j\leq r$.*

Recall that genus is the cardinality of $G(S)$.

**Proposition 2.3.** *Let $S$ be a numerical semigroup of genus $g$ and $\lambda=[\lambda_1,\ldots,\lambda_r]$ be the partition of $S$. Then the following statements hold:*

1. *$\lambda_1$ is the genus of $S$, $c=\lambda_1+r$ is the conductor of $S$.*
2. *If $S$ is an Arf semigroup and $s_e$ is the largest minimal generator, then $s_e=\lambda_1+r+t_1$, where $t_1$ is the first type of $S$.*

*Proof.* (1) is clear. (2) $S$ has maximal embedding dimension, we have $F(S)=s_e-s_1$. Using (1), we obtain $s_e=\lambda_1+r+t_1$. □

If $\lambda = [\lambda_1, \ldots, \lambda_r]$ is a partition of a positive integer $n$, then we say $\lambda_1 + r$ is the *conductor* of $\lambda$.
If we consider the Young diagram of an Arf semigroup $S$, we can add new properties to Theorem 2.2. For example, for any numerical semigroup $S$, the hook set of the $i$th column of $Y_S$ is a subset of the complement of the semigroup $S(i)$, $0 \leq i \leq r$. In particular, $S$ is an Arf semigroup if and only if $G_i$ is the complement of $S(i)$, and $S(i)$ is also Arf, see [14].

Let $\lambda$ be a partition of a positive integer $n$. If $\lambda$ is the partition of an Arf semigroup $S$, then $\lambda$ is called an *Arf partition* of $n$. Any positive integer $n$ has at least one Arf partition. For example, $\lambda = [n]$ is an Arf partition of $n$. Some of the Arf partitions of 13 are [13], [9, 4], [9, 3, 1], [10, 3], [10, 2, 1]. Let $S$ be an Arf semigroup. If $Y_S = \mathbf{1}^{u_1}\mathbf{2}^{u_2}\cdots\mathbf{r}^{u_r}$, then $u_i \neq 0$, for all $1 \leq i \leq r$. Equivalently, $\lambda = [\lambda_1, \ldots, \lambda_r] = \tau(Y_S)$, then $\lambda_i \neq \lambda_{i+1}$, $1 \leq i < r$, see [15].

The proof of Proposition 2.4 is obtained from [14].

**Proposition 2.4.** *$\lambda = [\lambda_1, \ldots, \lambda_r]$ is an Arf partition if and only if*

$$\lambda_j - \lambda_{j+1} + 1 \in \{\lambda_{j+1} - \lambda_{j+2} + 1, \lambda_{j+1} - \lambda_{j+3} + 2, \ldots, \lambda_{j+1} - \lambda_r + r - j - 1, \lambda_{j+1} + r - j, \rightarrow\}$$

*for all $j = 1, \ldots, r-1$.*

In [14], the authors gave an algorithm that uses Arf partitions to obtain the Arf closure (the smallest Arf semigroup containing $S$) of a numerical set $S$. Now, we can obtain some Arf partitions associated with an Arf partition $\lambda$. These are listed in Proposition 2.5.

**Proposition 2.5.** *Let $\lambda = [\lambda_1, \ldots, \lambda_r]$ be an Arf partition of a positive integer $n$.*

1. *For any $1 \leq i \leq r$, the partition $\beta = [\lambda_i, \ldots, \lambda_r]$ is an Arf partition.*
2. *For any $0 \leq i < \lambda_r$, the partition $\beta = [\lambda_1 - i, \ldots, \lambda_r - i]$ is an Arf partition of length $r$.*
3. *For any $0 \leq i \leq \lambda_1$, the partition $\beta = \lambda - r \cdot i$ (non-negative parts) is an Arf partition of length $s$, where $s \leq r$.*

*Proof.* If $\lambda$ is an Arf partition, there is an Arf numerical semigroup $S$ such that $G(S)$ is the hook set of the first column of $Y_S$.

(1) The hook set of the $i$th column of $Y_S$ is the complement of the semigroup $S(i)$, since $S$ is Arf, $S(i)$ is also Arf. Wiping the columns from left to right does not change any hook length in other columns.

(2) Deleting rows from top to bottom does not change hook lengths in other rows.

(3) The proof follows from (2). □

Recall the *trace of a partition* is defined by $tr(\lambda) = \max\{i : \lambda_i \geq i\}$.

**Corollary 2.6.** *Let $\lambda$ be an Arf partition of length $r$ and $tr(\lambda)$ be the trace of $\lambda$. Then $[\lambda_{i+1} - i, \ldots, \lambda_j - i]$ is also an Arf partition, where $j$ is the biggest number $j \leq r$ such that $\lambda_j - i \geq 0$ and $0 \leq i < tr(\lambda)$.*

*Proof.* The proof follows from Proposition 2.5 (1) and (3). □

**Corollary 2.7.** *Let $\lambda$ be an Arf partition of length $r$ and $t = tr(\lambda)$ be the trace of $\lambda$. Then $[\lambda_1, \ldots, \lambda_t] - t \cdot t$ and $[\lambda_{t+1}, \ldots, \lambda_r]$ are Arf partitions.*

*Proof.* The proof follows from Proposition 2.5. □

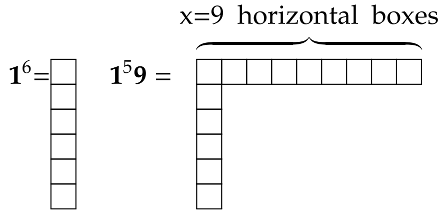
Let $P$ be the set of partitions obtained from the set of all numerical semigroups. The intersection of two numerical semigroups is again a numerical semigroup. The intersection of two semigroups induces a binary operation $\circledast$ on $P$ and $P$ is a semigroup.

**Theorem 2.8.** *If $A$ is the set of Arf partitions, then $A$ is a semigroup.*

*Proof.* Let $\alpha, \beta \in A$ and let $S, T$ be corresponding Arf semigroups. Then $S \cap T$ is also an Arf semigroup and we define $\circledast : AxA \longrightarrow A$, $(\alpha, \beta) \longrightarrow \alpha \circledast \beta = \gamma$, where $\gamma$ is the partition of $S \cap T$. Since the intersection of numerical semigroups has associative property, the set A becomes a semigroup with the operation $\circledast$. □

## 3. Decomposition of an Arf Semigroup

In this section, we give a decomposition of an Arf semigroup to the primitive semigroups. Now, we consider the special subset of the set of Young diagrams. Any element of this subset is a hook of some diagram. If $u$ denotes the number of boxes of the leg of that hook and $\mathbf{x}-1$ denotes the number of boxes of the arm of that hook, then we represent the hook by $\mathbf{1}^u\mathbf{x}$. If $\mathbf{x}=0$, then the hook is a leg. For $u=5$, $\mathbf{x}=9$, we have the following hook:



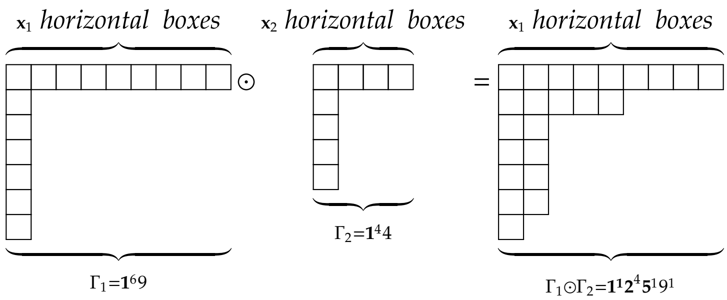

Determining the Arf partitions of positive integers is equivalent to determining Arf semigroups. We want to explain Arf semigroups with the help of partitions. Any partition consists of finitely many hooks. For this reason, we think of the separation of a partition into hooks. This is motivation for Definition 3.1 and Lemma 3.3.

**Definition 3.1.** *Let $K=\{\mathbf{1}^u\mathbf{x} : u\geq 0,\ \mathbf{x}\geq 2$ or $\mathbf{x}=0\}$ be the set of hooks. Then we define an operation $\odot$ on $K$ such that*

$$\Gamma_1\odot\Gamma_2=\mathbf{1}^{u_1-(u_2+1)}\mathbf{2}^{u_2}(\mathbf{x}_2+1)\mathbf{x}_1$$

*where $\Gamma_1=\mathbf{1}^{u_1}\mathbf{x}_1$, $\Gamma_2=\mathbf{1}^{u_2}\mathbf{x}_2$ and $u_1>u_2$, $\mathbf{x}_1-1\geq\mathbf{x}_2$.*

**Example 3.2.** *Let $\Gamma_1=\mathbf{1}^{u_1}\mathbf{x}_1=\mathbf{1}^6 9$ and $\Gamma_2=\mathbf{1}^{u_2}\mathbf{x}_2=\mathbf{1}^4 4$. Then we obtain $\Gamma_1\odot\Gamma_2=\mathbf{1}^{6-(4+1)}\mathbf{2}^4(\mathbf{4}+\mathbf{1})^1\mathbf{9}^1=\mathbf{1}^1\mathbf{2}^4\mathbf{5}^1\mathbf{9}^1$.*




Here, we define an ordering $\geq$ over the set of hooks as follows:

$$\Gamma_1\ \geq\Gamma_2\ \Leftrightarrow u_1>u_2,\ \mathbf{x}_1-1\geq\mathbf{x}_2$$

where $\Gamma_1=\mathbf{1}^{u_1}\mathbf{x}_1$ and $\Gamma_2=\mathbf{1}^{u_2}\mathbf{x}_2$. Definition 3.1 can be explained as the nesting of two hooks $\Gamma_1,\Gamma_2$ with $\Gamma_1\ \geq\Gamma_2$. The result $\Gamma_1\odot\Gamma_2$ is not a hook, but a partition of a positive integer $n$. In Example 3.2, we obtain $n=23$.

If $\Gamma_1$, $\Gamma_2$ are hooks with the same notation as in Definition 3.1, then we obtain the partition $\Gamma_1\odot\Gamma_2$. On the other hand, the hook $\Gamma$ can be added to the right-hand side or left-hand side of $\Gamma_1\odot\Gamma_2$, for a suitable hook $\Gamma$. Therefore, we observe that the operation $\odot$ has associative property: $(\Gamma_1\odot\Gamma_2)\odot\Gamma=\Gamma_1\odot(\Gamma_2\odot\Gamma)$, where $\Gamma_2=\mathbf{1}^{u_2}\mathbf{x}_2$, $\Gamma=\mathbf{1}^u\mathbf{x}$ and $u_2>u$, $\mathbf{x}_2-1\geq\mathbf{x}$.

**Lemma 3.3.** *If $\lambda$ is a partition and $t=tr(\lambda)$, then it can be written of the form*

$$\lambda=\Gamma_1\odot\Gamma_2\odot\cdots\odot\Gamma_t=\mathbf{1}^{v_1}\mathbf{2}^{v_2}\mathbf{3}^{v_3}\cdots(\mathbf{t}-\mathbf{1})^{v_{t-1}}\mathbf{t}^{v_t}\mathbf{y}_t\mathbf{y}_{t-1}\cdots\mathbf{y}_2\mathbf{y}_1$$

*where $\Gamma_i=\mathbf{1}^{u_i}\mathbf{x}_i$, $1\leq i\leq t$, $v_i=u_i-(u_{i+1}+1)$, $1\leq i\leq t-1$, $v_t=u_t$ and $\mathbf{y}_j=\mathbf{x}_j+(j-1)$, $1\leq j\leq t$.*

*Proof.* Let $\lambda = [\lambda_1, \dots, \lambda_r]$. The trace of a partition can be seen by its Young diagram; it is equal to the number of boxes of the Young diagram of shape $\lambda$ in the main diagonal. i.e., $tr(\lambda) = max\{i : \lambda_i \geq i\}$. Then $tr(\lambda)$ gives the number of components which can be seen on the decomposition. Given partition $\lambda$, we can calculate $\mathbf{x}_i$, $i \leq t$, by the following algorithm:

· $u_i = \lambda_i - i$, $i \leq t$.

· $\lambda - r{\cdot}t = [m_1, \dots, m_t, \bar{m}_{t+1}, \dots, \bar{m}_r]$, where $\bar{m}_j = -m_j$ is negative integer, $t + 1 \leq j \leq r$.

· If $m := [m_{t+1}, \dots, m_r]$, then $c(m)$ denotes the conjugate of $m$.

· $[\mathbf{x}_1, \dots, \mathbf{x}_t] = t{\cdot}r - c(m)^w - [0, 1, 2, \cdots, t - 1]$, where $c(m)^w$ means that the reverse ordering of $c(m)$.

Hence, $u_i > u_{i+1}$, $\mathbf{x}_i - 1 \geq \mathbf{x}_{i+1}$. Let $\Gamma_i = \mathbf{1}^{u_i}\mathbf{x}_i \in K$, $1 \leq i \leq t = tr(\lambda)$. Then we get,

$$\Gamma_1 \odot \Gamma_2 = \mathbf{1}^{u_1-(u_2+1)}\mathbf{2}^{u_2}(\mathbf{x}_2\mathbf{+1})\mathbf{x}_1$$

$$\Gamma_1 \odot \Gamma_2 \odot \Gamma_3 = \mathbf{1}^{u_1-(u_2+1)}\mathbf{2}^{u_2-(u_3+1)}\mathbf{3}^{u_3}(\mathbf{x}_3\mathbf{+2})(\mathbf{x}_2\mathbf{+1})\mathbf{x}_1$$

and by using induction we obtain

$$\Gamma_1 \odot \Gamma_2 \odot \cdots \odot \Gamma_t = \mathbf{1}^{v_1}\mathbf{2}^{v_2}\mathbf{3}^{v_3} \cdots (\mathbf{t-1})^{v_{t-1}}\mathbf{t}^{v_t}\mathbf{y}_t\mathbf{y}_{t-1}\cdots\mathbf{y}_2\mathbf{y}_1,$$

where $v_i = u_i - (u_{i+1} + 1)$, $1 \leq i \leq t - 1$, $v_t = u_t$ and $\mathbf{y}_j = \mathbf{x}_j + (j - 1)$, $1 \leq j \leq t$. □

**Example 3.4.** *For a partition* $\lambda = [9, 6, 3, 2, 1, 1]$, *we have the decomposition* $1^3 2^3 3^1 4^1 6^1 = 1^8 6 \odot 1^4 3 \odot 1^1$, *in other words,* $\lambda = [9, 1, 1, 1, 1, 1] \odot [5, 1, 1] \odot [1]$.

**Lemma 3.5.** *Let* $K = \{\mathbf{1}^u\mathbf{x} : u \geq 0,\ \mathbf{x} \geq 2 \text{ or } \mathbf{x} = 0\}$. *Then the following statements hold:*

1. *If* $K_1 = \{\mathbf{1}^u\mathbf{x} \in K : u \geq 1,\ 2 \leq \mathbf{x} \leq u + 1 \text{ or } \mathbf{x} = 0\}$, *then any* $\Gamma \in K_1$ *is a partition of a numerical semigroup.*
2. *If* $\lambda$ *is a partition of a numerical semigroup, then it has a decomposition via the set K, but components may not be a partition of some semigroup.*

*Proof.* (1) Let $\Gamma = \mathbf{1}^u\mathbf{x} \in K_1$. If $x = 0$, then the corresponding semigroup is $S = \{0, u + 1, \longrightarrow\}$. Otherwise, $S = \{0, u + 1, u + 2, \cdots, u + \mathbf{x} - 1, u + \mathbf{x} + 1, \longrightarrow\}$.

(2) The proof follows from the definition of the partition of a semigroup. □

Recall that if $F(S) < 2s_1$, then $S$ is called a *primitive* semigroup. Hence, if $\Gamma \in K_1$, then $\Gamma$ is a partition of a primitive semigroup by Lemma 3.5.

**Theorem 3.6.** *If S is an Arf semigroup, then S has a primitive semigroup decomposition and the length of the decomposition is the trace of the Arf partition of S. Additionally, the component semigroups do not commute.*

*Proof.* Let $S$ be an Arf semigroup and $\lambda$ be its partition. Then $\lambda$ is a strict dominant Arf partition. By Proposition 2.5, and Corollary 2.6, we see that $[\lambda_i - j, \dots, \lambda_r - j]$ is also an Arf partition, for $1 \leq i \leq r$ and $0 \leq j \leq \lambda_1$. In other words, if we separate the last row together with the first column which is left-aligned, then we obtain two partitions; one is an Arf partition, the other is an element of $K_1$. Both are partitions of the appropriate semigroups, since $\lambda$ is Arf. Using Lemma 3.3, we write $\lambda = \mathbf{1}^{v_1}\mathbf{2}^{v_2}\mathbf{3}^{v_3} \cdots (\mathbf{t-1})^{v_{t-1}}\mathbf{t}^{v_t}\mathbf{y}_t\mathbf{y}_{t-1}\cdots\mathbf{y}_2\mathbf{y}_1$, where $v_i = u_i - (u_{i+1} + 1)$, $1 \leq i \leq t - 1$, $v_t = u_t$ and $\mathbf{y}_j = \mathbf{x}_j + (j - 1)$, $1 \leq j \leq t = tr(\lambda)$. Let $\tilde{S}_i$ denote the semigroup of the partition $\mathbf{1}^{u_i}\mathbf{x}_i$. Hence, $S$ has a decomposition of the form $S = \tilde{S}_1 \odot \tilde{S}_2 \odot \cdots \odot \tilde{S}_t$ and each component $\tilde{S}_i$ is a primitive semigroup by Lemma 3.5. □

**Corollary 3.7.** *If* $\lambda$ *is the partition of an Arf semigroup S and* $S = \tilde{S}_1 \odot \tilde{S}_2 \odot \cdots \odot \tilde{S}_t$ *is the primitive semigroup decomposition of S where* $\tilde{S}_i$ *denotes the semigroup of the partition* $\mathbf{1}^{u_i}\mathbf{x}_i$, *then for any* $1 \leq j \leq t = tr(\lambda)$ *the following statements hold:*

1. *If* $\mathbf{x}_j = 1$, *then* $\tilde{S}_j = \{0, \lambda_j - j + 2, \longrightarrow\}$, *and if* $\mathbf{x}_j - 1 > 0$, *we have*

$$\tilde{S}_j = \{0, (\lambda_j - j) + 1, \dots, (\lambda_j - j) + (\mathbf{x}_j - 1), (\lambda_j - j) + (\mathbf{x}_j + 1), \longrightarrow\}.$$

2. $g(\tilde{S}_j) = \lambda_j - j + 1$ *and* $F(\tilde{S}_j) = \begin{cases} \lambda_j - j + \mathbf{x}_j, & \mathbf{x}_j > 1 \\ \lambda_j - j + 1, & \mathbf{x}_j = 1. \end{cases}$

*Proof.* For the case $j = t$, we may have two situations for $\tilde{S}_t$. If $\Gamma = \mathbf{1}^u$, then $\tilde{S}_t = \{0, \lambda_j - j + 2, \longrightarrow\}$, otherwise, any primitive component is

$$\tilde{S}_t = \{0, (\lambda_j - j) + 1, \ldots, (\lambda_j - j) + (\mathbf{x}_j - 1), (\lambda_j - j) + (\mathbf{x}_j + 1), \longrightarrow\}.$$

Direct calculation gives the Frobenius number $F(\tilde{S}_j)$ and the genus $g(\tilde{S}_j)$. □

**Example 3.8.** *For $S = \{0, 4, 8, 12, 15, \longrightarrow\}$, we have the following tableaux:*

$$Y_S = \begin{array}{|c|c|c|c|} \hline 14 & 10 & 6 & 2 \\ \hline 13 & 9 & 5 & 1 \\ \hline 11 & 7 & 3 \\ \cline{1-3} 10 & 6 & 2 \\ \cline{1-3} 9 & 5 & 1 \\ \cline{1-3} 7 & 3 \\ \cline{1-2} 6 & 2 \\ \cline{1-2} 5 & 1 \\ \cline{1-2} 3 \\ \cline{1-1} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|} \hline 14 & 3 & 2 & 1 \\ \hline 10 \\ \cline{1-1} 9 \\ \cline{1-1} 8 \\ \cline{1-1} 7 \\ \cline{1-1} 6 \\ \cline{1-1} 5 \\ \cline{1-1} 4 \\ \cline{1-1} 3 \\ \cline{1-1} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \odot \begin{array}{|c|c|c|} \hline 9 & 2 & 1 \\ \hline 6 \\ \cline{1-1} 5 \\ \cline{1-1} 4 \\ \cline{1-1} 3 \\ \cline{1-1} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \odot \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$$

*The partition of $S$ is $\lambda = [11, 8, 5, 2] = \mathbf{1}^3\mathbf{2}^3\mathbf{3}^3\mathbf{4}^2$, $r = 4$ and $tr(\lambda) = 3$, $\lambda - 4{\cdot}3 = [11, 8, 5, 2] - [3, 3, 3, 3] = [8, 5, 2, -1]$. Then $m = [1]$ and its conjugate is $c(m) = [1]$, we extend $c(m)$ to the partition of length $tr(\lambda) = 3$. Hence, $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = [4, 4, 4] - [0, 0, 1] - [0, 1, 2] = [4, 3, 1]$. Therefore, $\Gamma_1 = \mathbf{1}^{10}\mathbf{4}$, $\Gamma_2 = \mathbf{1}^7\mathbf{3}$ and $\Gamma_3 = \mathbf{1}^2\mathbf{1} = \mathbf{1}^3$. Then, we obtain primitive semigroups $\tilde{S}_1 = \{0, 11, 12, 13, 15, \longrightarrow\}$, $\tilde{S}_2 = \{0, 7, 8, 10, \longrightarrow\}$, and $\tilde{S}_3 = \{0, 4, \longrightarrow\}$. Thus $S$ has a decomposition of the form $S = \tilde{S}_1 \odot \tilde{S}_2 \odot \tilde{S}_3$ where*

$$\{0, 4, 8, 12, 15, \longrightarrow\} = \{0, 11, 12, 13, 15, \longrightarrow\} \odot \{0, 7, 8, 10, \longrightarrow\} \odot \{0, 4, \longrightarrow\}.$$

**Lemma 3.9.** *If $S$ is a symmetric numerical semigroup, then the partition of $S$ is a symmetric partition.*

*Proof.* The proof follows from the construction of the partition of a numerical semigroup. □

**Proposition 3.10.** *If $S$ is a symmetric Arf semigroup and $\lambda$ is the partition of $S$, then $\lambda$ is a symmetric Arf partition. Additionally, the primitive semigroup decomposition of $S$ can be written of the form $S = \tilde{S}_1 \odot \tilde{S}_2 \odot \cdots \odot \tilde{S}_t$ where $1 \leq j \leq t = tr(\lambda)$ and*

$$\tilde{S}_j = \begin{cases} \{0, \lambda_j - j + 2, \longrightarrow\}, & \lambda_j = j \\ \{0, \lambda_j - j + 1, \ldots, 2(\lambda_j - j),\ 2(\lambda_j - j) + 2, \longrightarrow\}, & \lambda_j \geq j + 1. \end{cases}$$

*Proof.* The first assertion follows from Lemma 3.9. Since $\lambda$ is a symmetric Arf partition of the semigroup $S$ and $t = tr(\lambda)$, we see that $u_{t-k} = u_{t+k+1}, 1 \leq k \leq t - 1$. On the other hand $\lambda_i - (i - 1) = \mathbf{x}_i$, $1 \leq i \leq t$. Using Corollary 3.7, if $\mathbf{x}_j = 1$ for some $j \leq t$, then $\tilde{S}_j = \{0, \lambda_j - j + 2, \longrightarrow\}$ and if $\mathbf{x}_j - 1 > 0$, we have

$$\begin{aligned} \tilde{S}_j &= \{0, (\lambda_j - j) + 1, \ldots, (\lambda_j - j) + ((\lambda_j - j + 1) - 1), (\lambda_j - j) + (\lambda_j - j + 1) + 1), \longrightarrow\} \\ &= \{0, \lambda_j - j + 1, \ldots, 2(\lambda_j - j),\ 2(\lambda_j - j) + 2, \longrightarrow\}. \end{aligned}$$

Therefore, we have $g(\tilde{S}_j) = \lambda_j - j + 1$ and $F(\tilde{S}_j) = \begin{cases} 2\lambda_j - 2j + 1, & \mathbf{x}_j > 1 \\ \lambda_j - j + 1, & \mathbf{x}_j = 1. \end{cases}$ □

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