Filomat 34:2 (2020), 475–482
https://doi.org/10.2298/FIL2002475K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Generalized q-Poly-Euler Polynomials of the Second Kind

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Abstract. In this work, we define the generalized q-poly-Euler numbers of the second kind of order α and the generalized q-poly-Euler polynomials of the second kind of order α. We investigate some basic properties for these polynomials and numbers. In addition, we obtain many identities, relations including the Roger-Szégo polynomials, the Al-Salam Carlitz polynomials, q-analogue Stirling numbers of the second kind and two variable Bernoulli polynomials.

1. Introduction, Definitions and Notations

The classical Bernoulli polynomials and the classical Euler polynomials are defined by the following generating functions, respectively;

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{xt},\ |t| < 2\pi \tag{1}$$

and

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t + 1}e^{xt},\ |t| < \pi. \tag{2}$$

Also, let

$B_n = B_n(0)$ and $E_n = E_n(0)$

where B_n and E_n are respectively, the Bernoulli numbers and the Euler numbers.

$k \in \mathbb{Z}$, $k > 1$, then k-th polylogarithm is defined by ([2], [12], [14], [22]) as

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}. \tag{3}$$

2010 *Mathematics Subject Classification*. 11B68; 11B73, 11S80.

Keywords. Bernoulli polynomials and numbers; Euler polynomials and numbers; Apostol-Bernoulli polynomials and numbers; Apostol-Euler polynomials and numbers; the Stirling numbers of second kind; Alternating sums; Unified Apostol-Bernoulli, Euler and Genocchi polynomials; modified Apostol-type polynomials.

Received: 08 January 2019; Revised: 19 March 2019; Accepted: 26 May 2019

Communicated by Yilmaz Simsek

This article is dedicated to Professor Gradimir V. Milovanovic on the Occasion of his 70th anniversary.

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This function is convergent for $|z| < 1$, when $k = 1$

$$Li_1(z) = -\log(1-z).$$

The q-numbers and q-factorial are defined by

$$[n]_q = \frac{1-q^n}{1-q}, q \neq 1, \ [n]_q! = [n]_q [n-1]_q \cdots [1]_q, \tag{4}$$

$n \in \mathbb{N}$, $q \in \mathbb{C}$, respectively where $[0]_q! = 1$.

The analogue of $(x-y)_q^n$ is defined by in [11]

$$(x-y)_q^n = \begin{cases} 1, & \text{if} \quad n = 0 \\ (x-y)(x-qy)\cdots\left(x-q^{n-1}y\right), & \text{if} \quad n > 1 \end{cases} . \tag{5}$$

From (5), we get

$$(x+y)_q^n := \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\binom{k}{2}} x^{n-k} y^k, n \in \mathbb{N}. \tag{6}$$

The q-exponential functions are given by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, 0 < |q| < 1, \ |z| < \frac{1}{|1-q|} \tag{7}$$

and

$$E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1 + (1-q)q^k z\right), 0 < |q| < 1, z \in \mathbb{C}. \tag{8}$$

From here, we easily see that $e_q(z)E_q(-z) = 1$ in [11].

The above q-notation can be found in [11]. Luo in [24], Liu in [23], Wei *et al.* [34] and Srivastava in [32] introduced and investigated Euler numbers and Euler polynomials. They gave several basic properties and recursion relations of these polynomials. Carlitz [5] extended the classical Bernoulli and Euler numbers and polynomials and introduced the q-Bernoulli and the q-Euler numbers and polynomials. Ozden et al. in [29], by using a p-adic q-Volkenborn integral gave a new extension of q-Euler numbers and polynomials. Kim *et al.* in [16] considered the poly-Bernoulli polynomials. Kim *et al.* in [17] and Kurt [18] gave some relations for the poly-Genocchi polynomials. Mahmudov ([25], [26]) considered two variables the q-Bernoulli polynomials, q-Euler polynomials and q-Genocchi polynomials. He gave some summation properties of these polynomials. Kim *et al.* [15], Kurt ([20], [21]) gave some identities and the analogues of the Srivastava-Pintér summation formulae for these polynomials. Ryoo et al. [30] introduced the q-poly-tangent polynomials and gave the distribution of their zeros. Agarwal *et al.* [1] introduced and investigated the q-extension of Euler polynomial of the second kind. Cieśliński in [6] improved q-exponential and q-trigonometric functions. Duran *et al.* in ([7], [8], [9]) investigated the (p,q)-Euler polynomials and the (p,q)-Hermite polynomials.

Sadjang [31] introduced and investigated to q-addition theorems for the q-Appell polynomials and the associated classes of q-polynomials expressions.

Mahmudov ([25], [26]) defined and investigated the q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ of order α, the q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ of order α and the q-Genocchi polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x,y)$ of order α respectively, the following generating functions

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t)-1}\right)^{(\alpha)} e_q(tx) E_q(ty), \ |t| < 2\pi, \tag{9}$$

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(\alpha)}(x,y)\,\frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t)+1}\right)^{(\alpha)} e_q(tx)\,E_q(ty)\,,\ |t|<\pi \tag{10}$$

and

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(\alpha)}(x,y)\,\frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t)+1}\right)^{(\alpha)} e_q(tx)\,E_q(ty)\,,\ |t|<\pi \tag{11}$$

where $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$ and $0 < \left|q\right| < 1$.

Hamahata *et al.* [10] defined poly-Euler polynomials by

$$\sum_{n=0}^{\infty} E_n^{(k)}(x)\,\frac{t^n}{n!} = \frac{2Li_k\left(1-e^{-t}\right)}{t\left(e^t+1\right)}e^{xt}.$$

For $k=1$, we get $E_n^{(1)}(x) = E_n(x)$.

The q-analogue of the Stirling numbers of the second kind $S_{2,q}(n,k)$ is defined [26] as

$$\sum_{n=0}^{\infty} S_{2,q}(n,k)\,\frac{t^n}{[n]_q!} = \frac{\left(e_q(t)-1\right)^k}{[k]_q!}. \tag{12}$$

The q-Hermite polynomials $H_{n,q}(x)$ is defined by Mahmudov in [27] as

$$e_q(tx)\,E_{q^2}\left(-\frac{t^2}{[2]_q}\right) = \sum_{n=0}^{\infty} H_{n,q}(x)\frac{t^n}{[n]_q!}. \tag{13}$$

It is clear that

$$\lim_{q\longrightarrow 1^-} H_{n,q}(x) = \exp\left(tx - \frac{t^2}{2}\right).$$

The Roger-Szégo polynomials $H_n(x:q)$ [see [3], Equ. (1)] and the Al-Salam Carlitz polynomials $U_n^{(a)}(x:q)$ [see [13], page 534] are defined by the generating functions

$$e_q(t)e_q(xt) = \sum_{n=0}^{\infty} H_n(x:q)\frac{t^n}{[n]_q!} \tag{14}$$

and

$$\frac{e_q(xt)}{e_q(t)e_q(at)} = \sum_{n=0}^{\infty} U_n^{(a)}(x:q)\frac{t^n}{[n]_q!}. \tag{15}$$

The classical Euler numbers of order α and the classical Euler polynomials of order α are defined [33] by the following generating functions, respectively

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^{\alpha},\ |t|<\pi$$

and

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt},\ |t|<\pi$$

where $\alpha \in \mathbb{R}$ and $x \in \mathbb{C}$.

The classical Euler numbers of the second kind $\breve{E}_n$ and the classical Euler polynomials of the second kind $\breve{E}_n(x)$ are defined in [1] by means of the following generating functions, respectively

$$\sum_{n=0}^{\infty} \breve{E}_n \frac{t^n}{n!} = \frac{2}{e^t + e^{-t}} \text{ and } \sum_{n=0}^{\infty} \breve{E}_n(x) \frac{t^n}{n!} = \frac{2}{e^t + e^{-t}} e^{xt}.$$

Agarwal *et al.* in [1] defined the q-Euler polynomials of second kind in two parameters as:

$$\sum_{n=0}^{\infty} \breve{\mathcal{E}}_{n,q}(x,y) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t) + e_q(-t)} e_q(xt) E_q(ty) \tag{16}$$

where $x, y \in \mathbb{C}$.

By this motivation, we define the generalized q-poly-Euler numbers $\breve{\mathcal{E}}_{n,q}^{[k,\alpha]}$ of the second kind of order α and the generalized q-poly-Euler polynomials $\breve{\mathcal{E}}_{n,q}^{[k,\alpha]}(x,y)$ of the second kind of order α as follows, respectively

$$\sum_{n=0}^{\infty} \breve{\mathcal{E}}_{n,q}^{[k,\alpha]} \frac{t^n}{[n]_q!} = \left(\frac{2Li_k(1-e^{-t})}{t\left(e_q(t) + e_q(-t)\right)} \right)^{\alpha} \tag{17}$$

and

$$\sum_{n=0}^{\infty} \breve{\mathcal{E}}_{n,q}^{[k,\alpha]}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2Li_k(1-e^{-t})}{t\left(e_q(t) + e_q(-t)\right)} \right)^{\alpha} e_q(xt) E_q(yt). \tag{18}$$

For $k = 1$, $Li_1(z) = -\log(1-z)$, from (17) and (18), we get

$$\lim_{q \longrightarrow 1^-} \breve{\mathcal{E}}_{n,q}^{[1,\alpha]} = E_n^{(\alpha)} \text{ and } \lim_{q \longrightarrow 1^-} \breve{\mathcal{E}}_{n,q}^{[1,\alpha]}(x,y) = E_n^{(\alpha)}(x+y).$$

2. Main Theorems

In this section, we give explicit relations for these polynomials. Also, we prove some relations between the generalized q-poly-Euler polynomials of the second kind, the q-Stirling numbers of the second kind, the two variable Bernoulli numbers and the Bernoulli polynomials.

Theorem 2.1. *The generalized q-poly-Euler polynomials of the second kind of order α satisfy the following relations:*

$$\breve{\mathcal{E}}_{n,q}^{[k,\alpha]}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (x+y)_q^l \, \breve{\mathcal{E}}_{n-l,q}^{[k,\alpha]}, \tag{i}$$

$$\breve{\mathcal{E}}_{n,q}^{[k,\alpha]}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \breve{\mathcal{E}}_{n-l,q}^{[k,\alpha]}(x,0) \, q^{\binom{l}{2}} y^l \tag{ii}$$

and

$$\breve{\mathcal{E}}_{n,q}^{[k,\alpha]}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \breve{\mathcal{E}}_{n-l,q}^{[k,\alpha]}(0,y) \, x^l. \tag{iii}$$

The proof of this Theorem is easily obtained by using (17) and (18).

Theorem 2.2. *The following relations hold true:*

$$(x+y)_q^n = \frac{1}{2}\sum_{l=0}^{n}\left[\begin{array}{c} n \\ l \end{array}\right]_q \left(1+(-1)^l\right)\breve{\mathcal{E}}_{n-l,q}(x,y) \tag{19}$$

and

$$x^n = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q H_{n-k}(a:q)\,U_k^{(a)}(x:q). \tag{20}$$

The proof of these relations are easily obtained by applying the Cauchy product to (14), (15) and (16) and comparing the coefficients. For $y=0$, Theorem 2.2 is reduced to Theorem 2.12-(ii) in [1, p.142].

Theorem 2.3. *We have the following relation*

$$(x+y)_q^n = \sum_{m=0}^{n}\left[\begin{array}{c} n \\ m \end{array}\right]_q \sum_{k=0}^{n-m}\left[\begin{array}{c} n-m \\ k \end{array}\right]_q q^{\binom{k}{2}}(-1)^k q^{\binom{n-m-k}{2}} y^{n-m-k} H_m(x:q). \tag{21}$$

The proof of this Theorem is depend on the equations (7), (8) and (14) and also the property of q-exponential functions such as $E_q(-t)\,e_q(t)=1$.

We get the following corollary from (19) and (21).

Corollary 2.4. *There is the following relation*

$$\begin{aligned}&\sum_{m=0}^{n}\left[\begin{array}{c} n \\ m \end{array}\right]_q \sum_{k=0}^{n-m}\left[\begin{array}{c} n-m \\ k \end{array}\right]_q q^{\binom{k}{2}}(-1)^k q^{\binom{n-m-k}{2}} y^{n-m-k} H_m(x:q) \\ = &\ \frac{1}{2}\sum_{l=0}^{n}\left[\begin{array}{c} n \\ l \end{array}\right]_q \left(1+(-1)^l\right)\breve{\mathcal{E}}_{n-l,q}(x,y).\end{aligned}$$

Theorem 2.5. *There is the following relation between the generalized q-poly-Euler polynomials of the second kind and q-Bernoulli polynomials $B_{n,q}^{(\alpha)}(x,y)$ of order α:*

$$\breve{\mathcal{E}}_{n,q}^{[k,\alpha]}(x,y) = \sum_{j=0}^{n}\left[\begin{array}{c} n \\ j \end{array}\right]_q \breve{\mathcal{E}}_{n-j,q}^{[k,\alpha]}(0,y)\sum_{r=0}^{j}\left[\begin{array}{c} j \\ r \end{array}\right]_q \frac{\mathcal{B}_{j-r,q}^{(1)}(mx,0)}{m^j\,[r+1]_q!}. \tag{22}$$

Proof. By (9) and (18), we write as

$$\begin{aligned}&\sum_{n=0}^{\infty}\breve{\mathcal{E}}_{n,q}^{[k,\alpha]}(x,y)\frac{t^n}{[n]_q!} \\ = &\left(\frac{2Li_k(1-e^{-t})}{t\left(e_q(t)+e_q(-t)\right)}\right)^{\alpha} E_q(yt)\frac{e_q(\frac{t}{m})-1}{\frac{t}{m}}\frac{\frac{t}{m}}{e_q(\frac{t}{m})-1}e_q(mx\frac{t}{m}) \\ = &\ \frac{m}{t}\sum_{n=0}^{\infty}\breve{\mathcal{E}}_{n,q}^{[k,\alpha]}(0,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^{n+1}}{m^{n+1}[n+1]_q!}\sum_{n=0}^{\infty}B_{n,q}(mx,0)\frac{t^n}{m^n\,[n]_q!}.\end{aligned}$$

By using Cauchy product and comparing the coefficeints of $\frac{t^n}{[n]_q!}$, we have (22). □

Theorem 2.6. *The following relation holds true:*

$$\begin{aligned}\breve{\mathcal{E}}_{n-1,q}^{[k,\alpha]}(x,y) &= \frac{1}{2[n]_q}\sum_{l=0}^{n}\left[\begin{array}{c} n \\ l \end{array}\right]_q \frac{1}{m^{l-1}}\left\{\breve{\mathcal{E}}_{n-l,q}^{[k,\alpha]}\left(\frac{1}{m},y\right)+\breve{\mathcal{E}}_{n-l,q}^{[k,\alpha]}(0,y)\right\} \\ &\quad \times \mathcal{G}_{l,q}^{(1)}(mx,0). \end{aligned} \tag{23}$$

Proof. By (11) and (18), we write as

$$\begin{aligned} & \sum_{n=0}^{\infty}\breve{\mathcal{E}}_{n,q}^{[k,\alpha]}(x,y)\frac{t^n}{[n]_q!} \\ = & \left(\frac{2Li_k(1-e^{-t})}{t\left(e_q(t)+e_q(-t)\right)}\right)^{\alpha} E_q(yt)\frac{e_q(\frac{t}{m})+1}{\frac{2t}{m}}\frac{\frac{2t}{m}}{e_q(\frac{t}{m})+1}e_q\left(mx\frac{t}{m}\right) \\ = & \frac{m}{2t}\left\{\left(\frac{2Li_k(1-e^{-t})}{t\left(e_q(t)+e_q(-t)\right)}\right)^{\alpha} E_q(yt)e_q\left(\frac{t}{m}\right)\frac{\frac{2t}{m}}{e_q(\frac{t}{m})+1}e_q\left(mx\frac{t}{m}\right)\right. \\ & \left. +\left(\frac{2Li_k(1-e^{-t})}{t\left(e_q(t)+e_q(-t)\right)}\right)^{\alpha} E_q(yt)\frac{\frac{2t}{m}}{e_q(\frac{t}{m})+1}e_q\left(mx\frac{t}{m}\right)\right\} \end{aligned}$$

$= \dfrac{m}{2t}\{A+B\}$, *where*

$$A=\sum_{m=0}^{\infty}\breve{\mathcal{E}}_{m,q}^{[k,\alpha]}\left(\frac{1}{m},y\right)\frac{t^m}{[m]_q!}\sum_{l=0}^{\infty}\mathcal{G}_{l,q}^{(1)}(mx,0)\frac{t^l}{m^l[l]_q!} \tag{24}$$

and

$$B=\sum_{m=0}^{\infty}\breve{\mathcal{E}}_{m,q}^{[k,\alpha]}(0,y)\frac{t^m}{[m]_q!}\sum_{l=0}^{\infty}\mathcal{G}_{l,q}^{(1)}(mx,0)\frac{t^l}{m^l[l]_q!}. \tag{25}$$

By using Cauchy product to (24) and (25), we get

$$\begin{aligned} & \sum_{n=0}^{\infty}[n+1]_q\breve{\mathcal{E}}_{n,q}^{[k,\alpha]}(x,y)\frac{t^{n+1}}{[n+1]_q!} \\ = & \frac{1}{2}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\left[\begin{array}{c} n \\ l \end{array}\right]_q\frac{1}{m^{l-1}}\left\{\breve{\mathcal{E}}_{n-l,q}^{[k,\alpha]}\left(\frac{1}{m},y\right)+\breve{\mathcal{E}}_{n-l,q}^{[k,\alpha]}(0,y)\right\}\mathcal{G}_{l,q}^{(1)}(mx,0)\frac{t^n}{[n]_q!}. \end{aligned}$$

From comparing the coefficients of the both side, we have (23).
□

Theorem 2.7. *There is the following relation between the generalized q-poly-Euler polynomials of the second kind and the q-Stirling numbers $S_{2,q}(n,k)$ of the second kind as*

$$\begin{aligned} & \sum_{s=0}^{n}\left[\begin{array}{c} n \\ s \end{array}\right]_q \breve{E}_{n-s,q}^{[1,1]}(x,y)\sum_{m=0}^{s}\left[\begin{array}{c} s \\ m \end{array}\right]_q S_{2,q}(m,l)\left(1+(-1)^{s-m}\right) \\ = & 2\sum_{m=0}^{n}\left[\begin{array}{c} n \\ m \end{array}\right]_q S_{2,q}(m,l)(x+y)_q^{n-m}. \end{aligned} \tag{26}$$

Proof. By (12) and (18) and for $\alpha = 1$, we write as

$$\sum_{n=0}^{\infty} \breve{\mathcal{E}}_{n,q}^{[k,1]}(x,y)\frac{t^n}{[n]_q!}$$
$$= \frac{2Li_k(1-e^{-t})}{t\left(e_q(t)+e_q(-t)\right)}\frac{\left(e_q(t)-1\right)^l}{[l]_q!}\frac{[l]_q!}{\left(e_q(t)-1\right)^l}e_q(xt)E_q(yt)$$

$$\sum_{n=0}^{\infty} \breve{\mathcal{E}}_{n,q}^{[k,1]}(x,y)\frac{t^n}{[n]_q!}t\left(e_q(t)+e_q(-t)\right)\frac{\left(e_q(t)-1\right)^l}{[l]_q!}$$
$$= 2Li_k\left(1-e^{-t}\right)\frac{\left(e_q(t)-1\right)^l}{[l]_q!}e_q(xt)E_q(yt). \tag{27}$$

The left hand side of the equation (27) is

$$t\sum_{n=0}^{\infty} \breve{\mathcal{E}}_{n,q}^{[k,1]}(x,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}(1+(-1)^n)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}S_{2,q}(n,l)\frac{t^n}{[n]_q!}. \tag{28}$$

The right hand side of the equation (27) is

$$2\sum_{n=0}^{\infty}S_{2,q}(n,l)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}(x+y)_q^n\frac{t^n}{[n]_q!}Li_k\left(1-e^{-t}\right). \tag{29}$$

For $k = 1$, using $Li_1(1-e^{-t}) = t$ in (29). By using the Cauchy product of the equation (28) and (29) and comparing the coefficients in (27). We have (26). □

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