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On Radical Formula in Modules over Noncommutative Rings

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Abstract. This paper examines the radical formula in noncommutative case and for this purpose, a generalization of prime submodule is defined. It is proved that there is a direct connection between one-sided prime ideals and one-sided prime submodules. Moreover the connections between the intersection of all one-sided prime submodules and strongly nilpotent elements of a module are studied.

1. Introduction

As it is well known, prime ideals and prime submodules form an important part to characterize rings and modules and have been studied for long time by many authors ([5], [9], [13], [17]). It is well known that the set of nilpotent elements of a commutative ring R with unity forms an ideal which is equal to the intersection of all the prime ideals. This notion has been generalized in [5] to modules.
Let N be a proper submodule of an R-module M. The radical of N in M, denoted by $rad_M(N)$, is defined to be the intersection of all prime submodules of M containing N. The envelope submodule $RE_M(N)$ of N in M is a submodule of M generated by the set

$$E_M(N) = \{rm : r \in R \text{ and } m \in M \text{ such that } r^n m \in N \text{ for some } n \in \mathbb{N}\}.$$

Then N is said to satisfy the radical formula in M if $rad_M(N) = RE_M(N)$. By using this concept, some useful characterizations for Dedekind domains and modules were proved but unfortunately, in noncommutative case, there are not enough useful results about the radical formula and radical submodule.

Let R be a ring. Let M be an R-module and let N be a submodule of M.

i) A set $\eta(a) = \{a, a_1, ...\}$ is said to be an sequence of an element a of R if for all $i \in \mathbb{N}$, $a_{i+1} \in a_i R a_i$ and $a_0 = a$.

ii) Let $a \in R$, $m \in M$. Then an element am of M is said to be a strongly nilpotent on N if for all subsets $K = \{a_i \in R : a_0 = a \text{ and } a_{i+1} \in a_i R a_i,\ i \in \mathbb{N}\}$ of R, $0 \in N \cap Km$. We use the notation $W_M(N)$ to denote the submodule generated by the strongly nilpotent elements on N. Now it is clear that $W_M(N) = RE_M(N)$ when R is a commutative ring.

In [14], to examine the radical formula in noncommutative case, a generalization of prime ideal was defined. Let P be a left ideal of R. Following [14], P is said to be a one-sided prime ideal (left O-prime ideal) if for any left ideals I, J such that $PJ \subseteq P$ and $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$ holds. It is clear that every maximal

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left ideal is one-sided prime ideal. In this sense, the class of one-sided prime ideals is different from other known classes.

In commutative ring theory, there are so much useful results about the radical formula. In particular, it plays important role in the characterization of Dedekind domain. Unfortunately, in noncommutative case, there are not enough useful results about the radical formula and radical submodule. In this paper, we examine the radical formula in noncommutative case and for this purpose, we firstly define a new submodule class which are not only the module version of one-sided prime ideal but also a generalization of prime submodules called one-sided prime submodule and so our main objective is to bring a new perspective for the prime radical and radical formula in noncommutative ring theory. Then we investigate the relations between the intersection of all one-sided prime submodules and strongly nilpotent elements of a module. On the other hand as is well-known, the property that when a submodule P becomes prime in case $(P : M)$ is prime ideal, which is known not to hold in general, is of central importance in the prime module theory, but it is proved that there is a direct connection between one-sided prime ideals and one-sided prime submodules.

Let P be a submodule of a left R-module M. Then P is said to be a one-sided prime submodule of M if for all $a, b \in R$ and $m \in M$ such that $aRbm \subseteq P$ and $(P : m)Rb \subseteq (P : m)$, either $am \in P$ or $bm \in P$ holds. To show the difference of two classes, we give an example of a one-sided prime submodule which is not a prime submodule of a module M. We also define the O-radical of a submodule N as the intersection of all one-sided prime submodules of M containing N, denoted by $O\text{-}rad_M(N)$. Let M be a finitely generated left R-module and let $K = \{a_i \in R : a_0 = a \text{ and } a_{i+1} \in a_iRa_i, i \in \mathbb{N}\}$ be a set of R. If there is a submodule N of M such that $N \cap Km = \emptyset$, we verify that there is a one-sided prime submodule P of left R-module M containing N but not am. (i.e. $P \cap Km = \emptyset$). By using this result, we characterize elements of $O\text{-}rad_M(N)$ under some conditions. Finally, we prove that any O-radical submodule N in M is the intersection of a finite number of one-sided prime submodules if M satisfies the ascending chain condition on O-radical submodules.

2. One-Sided Prime Submodule

Throughout this paper, all rings will be associative rings with identity and all modules will be unital left modules unless otherwise stated.

In this section, we start with the definition of one-sided prime submodule of an R-module M and examine some of its properties.

Definition 2.1. *Let M be an R-module. A submodule P of M is called a one-sided prime submodule if for all $a,b \in R$ and $m \in M$ such that $(P : m)Rb \subseteq (P : m)$ and $aRbm \subseteq P$, either $am \in P$ or $bm \in P$ holds.*

It is clear that every prime submodule of M is one-sided prime. If I is an ideal of R such that $I = (I : m)$ for all $m \in R - I$ then I is a one-sided prime submodule of $_RR$ if and only if I is a one-sided prime ideal of R. On the other hand, the set of one-sided prime submodule is different from the set of prime submodule.

We focus on the set $\Omega_m(P) = (P : m) = \{r \in R : rm \in P\}$ for $m \in M$. Clearly, it is a left ideal of R and $\Omega_m(P) = R$ if and only if $m \in P$. We give a basic property related to this set as follows:

Lemma 2.2. *Let M and M^*be R-modules and let P be a submodule of M. If $f : M \to M^*$ is an R-module homomorphism, then we have $\Omega_m(P) \subseteq \Omega_{f(m)}(f(P))$ for some $m \in M$. The converse of this inclusion is true when $Kerf \subseteq P$.*

Proof. Let r be in $\Omega_m(P)$. Then $rm \in P$ and so $f(rm) = rf(m) \in f(P)$. Thus we have $r \in \Omega_{f(m)}(f(P))$.

Let r be in $\Omega_{f(m)}(f(P))$. Then $rf(m) = f(rm) \in f(P)$. There exists an element $p \in P$ such that $f(rm) = f(p)$ and so $f(rm - p) = 0$. Thus we have $rm - p \in Kerf$. Since $Kerf \subseteq P$, it follows $rm \in P$. Consequently, $r \in \Omega_m(P)$. □

The following lemma is also an example for one-sided prime submodules, which is not prime submodule.

Lemma 2.3. *Let R be a domain, $M = R \oplus R$ be an R-module and $0 \neq P$ a prime ideal of R. Then $N = 0 \oplus P$ is a one-sided prime submodule of M but not prime submodule.*

Proof. Let r be in P and $m = (0, 1)$ be in M such that $rRm \subseteq N$. It is clear that $m \notin N$ and $rM \nsubseteq N$. Thus N is not a prime submodule of M.

Now we show that N is a one-sided prime submodule of M. Let x,y,a and b be in R such that $xRy(a, b) \subseteq N$ and $\Omega_{(a,b)}(N)Ry \subseteq \Omega_{(a,b)}(N)$. Then $xtya = 0$ and $xtyb \in P$ for all $t \in R$. We have the situations as follows.

i) If $x = 0$, then $x(a, b) \in N$.

ii) If $y = 0$, then $y(a, b) \in N$.

iii) If both x and y are not zero then $a = 0$ since R is a prime ring and $xRyb \subseteq P$. Since P is a prime ideal of R, $x \in P$ or $yb \in P$. If $x \in P$, then $x(0, b) \in N$ and if $yb \in P$, then $y(0, b) \in N$. Thus N is a one-sided prime submodule of M. □

Let M be a module over a commutative ring and P be a submodule of M. The theorem stating that P is a prime submodule of M if and only if M/P is a torsion-free $R/(P : M)$-module and $(P : M)$ is a prime ideal of R is very useful to characterize the module. It is also well known that P need not be a prime submodule of M while $(P : M)$ is a prime ideal of R. Hence some papers deal with the problem that when a submodule P has the property of being prime in case $(P : M)$ is a prime ideal.

The following theorem shows that a similar property holds for one-sided prime submodules.

Theorem 2.4. *Let M be an R-module and P a submodule of M. Then P is a one-sided prime submodule of M if and only if $\Omega_m(P)$ is a one-sided prime ideal of R for $m \in M - P$.*

Proof. Let P be a one-sided prime submodule. Let $aRb \subseteq \Omega_m(P)$ for $a \in R$ and $b \in R - \Omega_m(P)$ such that $\Omega_m(P)Rb \subseteq \Omega_m(P)$. Then $aRbm \subseteq P$ and so $a \in \Omega_m(P)$.

Let $\Omega_m(P)$ be a one-sided prime ideal. Let $aRbm \subseteq P$ for $a \in R$ and $b \in R - \Omega_m(P)$ such that $\Omega_m(P)Rb \subseteq \Omega_m(P)$. Then $aRb \subseteq \Omega_m(P)$ and $\Omega_m(P)Rb \subseteq \Omega_m(P)$. Since $\Omega_m(P)$ is a one-sided prime ideal, $am \in P$. □

Proposition 2.5. *Let M and M^* be R-modules, $\varphi : M \to M^*$ an R-epimorphism and $Ker\varphi \subseteq P$. Then P is a one-sided prime submodule of M if and only if $\varphi(P)$ is a one-sided prime submodule of M^*.*

Proof. Let $a, b \in R$ and $\varphi(m) \in M^*$ such that $aRb\varphi(m) \subseteq \varphi(P)$ and $\Omega_{\varphi(m)}(\varphi(P))Rb \subseteq \Omega_{\varphi(m)}(\varphi(P))$. Since $\varphi(aRbm) \subseteq \varphi(P)$ and $Ker\varphi \subseteq P$, it follows that $aRbm \subseteq P$ and also $\Omega_m(P)Rb \subseteq \Omega_m(P)$. By the hypothesis, we get that $a \in \Omega_m(P)$ or $b \in \Omega_m(P)$. Thus either $\varphi(am) \in \varphi(P)$ or $\varphi(bm) \in \varphi(P)$ and so $\varphi(P)$ is a one-sided prime submodule of M^*.

Conversely, let $\varphi(P)$ be a one-sided prime submodule of M^*. Let $a, b \in R$ and $m \in M$ such that $aRbm \subseteq P$ and $\Omega_m(P)Rb \subseteq \Omega_m(P)$. Thus $aRb\varphi(m) \subseteq \varphi(P)$ and $\Omega_{\varphi(m)}(\varphi(P))Rb \subseteq \Omega_{\varphi(m)}(\varphi(P))$. Because $\varphi(P)$ is a one-sided prime submodule, $\varphi(am) \in \varphi(P)$ or $\varphi(bm) \in \varphi(P)$. Since $Ker\varphi \subseteq P$, either $am \in P$ or $bm \in P$. □

Corollary 2.6. *Let M be an R-module. Then P is a one-sided prime submodule of M if and only if P/N is a one-sided prime submodule of an R-module M/N for all $N \subseteq P \subseteq M$.*

3. Radical Formula

In this section, we define the O-radical submodule of M and focus on the relationships between submodules generated by the strongly nilpotent elements and O-radical submodules.

Definition 3.1. *Let N be a submodule of an R-module M. Then O-radical of N is defined as intersection one-sided prime submodules of M containing N, denoted by $O\text{-}rad_M(N)$.*

In particular, if $O\text{-}rad_M(N) = N = W_M(N)$ then N is said to be an O-radical submodule of M.

Theorem 3.2. *Let M be a finitely generated R-module and let N, L be submodules of M. Then $O\text{-}rad_M(N) + O\text{-}rad_M(L) = M$ if and only if $N + L = M$.*

Proof. Suppose that $O\text{-}rad_M(N) + O\text{-}rad_M(L) = M$ and $N + L \neq M$. Thus, there exists a maximal submodule T of M such that $N + L \subseteq T$. Since T is a one-sided prime submodule of M, we have $O\text{-}rad_M(N) \subseteq T$ and $O\text{-}rad_M(L) \subseteq T$. Then

$$O\text{-}rad_M(N) + O\text{-}rad_M(L) \subseteq T.$$

This is a contradiction. Then $N + L = M$.

Since $N \subseteq O\text{-}rad_M(N)$, $L \subseteq O\text{-}rad_M(L)$ and $N + L = M$, it follows that

$$O\text{-}rad_M(N) + O\text{-}rad_M(L) = M.$$

□

Let P be a submodule of an R-module M and let K be a multiplicative set of R. The module M is said to satisfy the condition (∗) if $(P + Ran) \cap Km \neq \emptyset$ implies $((P : M) + Ra) \cap K \neq \emptyset$ for any $a \in R$ and $n, m \in M$.

For the rest of this paper, all modules will be assumed to satisfy the condition (∗).

Proposition 3.3. *Let N be a submodule of a finitely generated R-module M and let K be a multiplicative set of R such that $N \cap Km = \emptyset$ for any $m \in M$. Then there is a one-sided prime submodule P of M containing N such that $P \cap Km = \emptyset$.*

Proof. Consider the set

$$\Psi = \{L : L \cap Km = \emptyset\ ,\ N \subseteq L \leq M\}.$$

Then Ψ is a partially ordered set with the inclusion property of set and we also observe that $L \cap Km = \emptyset$ implies $(L : M) \cap K = \emptyset$ for some submodule L of M. Now take a chain Λ of Ψ. Then $A = \cup_{A_i \in \Lambda} A_i$ and so $A \in \Psi$. By Zorn's lemma, there is a maximal element P in the set Ψ.

Assume that $n \in M - P$, $a \in R - \Omega_n(P)$, $\Omega_n(P)Rb \subseteq \Omega_n(P)$ for $b \in R - \Omega_n(P)$. We prove that $aRbn$ is not in P. Since P is maximal element of Ψ, it follows that both $((P : M) + Ra) \cap K$ and $(P + Rbn) \cap Km$ are not empty. Let $l \in ((P : M) + Ra) \cap K$ and $km \in (P + Rbn) \cap Km$. Hence $l = q + da$, $km = p + tbn$ for some $q \in (P : M)$, t, $d \in R$, $k \in K$ and $p \in P$. Therefore, $lk \in K$ and so $lkm = (q + da)(p + tbn) = qp + qtbn + dap + datbn \in Km$. Since $qp + qtbn + dap \in P$ and $P \cap Km = \emptyset$, we get that $dactbn \notin P$ and so $aRbn$ is not in P. □

Proposition 3.4. *Let M be a finitely generated R-module, $K = \{a_i \in R : a_0 = a \text{ and } a_{i+1} \in a_i R a_i,\ i \in \mathbb{N}\}$ be a set of R and N a submodule of M such that $N \cap Km = \emptyset$ for $m \in M$. Then there is a one-sided prime submodule P of R-module M containing N but not am. (i.e. $P \cap Km = \emptyset$)*

Proof. Consider the set

$$\Psi = \{L : L \cap Km = \emptyset,\ N \subseteq L \leq M\}.$$

Let Λ be a chain of Ψ. Then $A = \cup_{A_i \in \Lambda} A_i$ and so $A \in \Psi$. Then by Zorn's Lemma, there is a maximal element P of Ψ. Take r_1, $r_2 \in R - (P : M)$ and $n \in M - P$ such that $\Omega_n(P)Rr_2 \subseteq \Omega_n(P)$. By the maximality, choose elements $k_a m \in (P + Rr_2 n) \cap Km \neq \emptyset$ and $k_b \in [(P : M) + Rr_1] \cap K \neq \emptyset$. Hence $k_b = q + dr_1$, $k_a m = p + tr_2 n$ for some $q \in (P : M)$, t, $d \in R$, $k \in K$ and $p \in P$. Assume that $a \geq b$. Therefore, there exists the elements x, y such that $k_{a+1} = xk_b y k_a \in K$ and so

$$\begin{aligned} k_{a+1}m &= xk_b y k_a m = x(q + dr_1)y(p + tr_2 n) \\ &= xqyp + xqytr_2 n + xdr_1 yp + xdr_1 ytr_2 n \in Km \end{aligned}$$

Since $xqyp + xqytr_2 n + xdr_1 yp \in P$ and $P \cap Km = \emptyset$, we get that $xdr_1 ytr_2 n \notin P$. This means that P is a one-sided prime submodule of M. □

Proposition 3.5. *Let N be a submodule of an R-module M. If either $O\text{-}rad_M(N)$ or M is cyclic, then $O\text{-}rad_M(N) \subseteq W_M(N)$.*

Proof. Let $O\text{-}rad_M(N) = Rm$ for $m \in M$. Let am be in $O\text{-}rad_M(N)$ with $a \in R$ but not a strongly nilpotent element on N. There is a sequence $K = \{a_i \in R : a_0 = a \text{ and } a_{i+1} \in a_iRa_i,\ i \in \mathbb{N}\}$ such that $N \cap Km = \emptyset$. Then there is a one-sided prime submodule P of M containing N but not am. This is a contradiction with $am \in O\text{-}rad_M(N)$.

Let $M = Rm$ where $m \in M$. Let am be in $O\text{-}rad_M(N)$ with $a \in R$ but not a strongly nilpotent element on N. There is a sequence $K = \{a_i \in R - \{0_R\} : a_0 = a \text{ and } a_{i+1} \in a_iRa_i,\ i \in \mathbb{N}\}$ such that $N \cap Km = \emptyset$. Then there is a one-sided prime submodule P of M containing N but not am. This is a contradiction with $am \in O\text{-}rad_M(N)$. □

Theorem 3.6. *Let N be a submodule of a module M. Then $W_M(N) = O\text{-}rad_M(N)$ if either $O\text{-}rad_M(N)$ or M is cyclic and one of the following conditions holds;*

1) *$axam \notin P$ whenever $xam \notin P$ where P is a one-sided prime submodule.*

2) *Every one-sided prime submodule P is a maximal submodule.*

Proof. It is enough to show that $W_M(N) \subseteq O\text{-}rad_M(N)$.

Let $am \in W_M(N)$ but not in $O\text{-}rad_M(N)$. Then there is a one-sided prime submodule P of M containing N such that am is not in P. For the one-sided prime submodule P, we have two cases;

a) Let $\Omega_m(P)Ra \subseteq \Omega_m(P)$. Since $aRam$ is not in P, there is a non zero element $a_1 = at_0a \in aRa$ such that $a_1m \notin P$. Then $\Omega_m(P)Ra_1 \subseteq \Omega_m(P)Ra \subseteq \Omega_m(P)$ and so we get that a_1Ra_1m is not in P, there is a non zero element $a_2 = a_1t_1a_1 \in a_1Ra_1$ such that $a_2m \notin P$. By using this method, we get the sequence $\eta(a)$ of a is the set $\eta(a) = \{a_i : a_{i+1} \in a_iRa_i \text{ and } a_0 = a,\ i \in \mathbb{N}\}$ but $\eta(a)m$ does not contain any element of P since for all $i \in \mathbb{N}$, $a_im \notin P$. Therefore am is not a strongly nilpotent element of M on N, a contradiction.

b) Let $\Omega_m(P)Ra \nsubseteq \Omega_m(P)$.

i) Let the condition in (1) hold. There are elements $p_0 \in \Omega_m(P)$ and $x \in R$ such that $(p_0x)\,am \notin P$ and so choose $a_1m = a\,(p_0x)\,am \notin P$ by the condition (1).

ii) Let the condition in (2) hold. Then P is a maximal submodule of M and $P + Ram = M$. Hence $aM = aP + aRam$ for $a \in R$ and so $am - ap = alam \notin P$. Now choose $a_1m = atam$.

Since $\Omega_m(P)PRa_1 \nsubseteq \Omega_m(P)$, with the method in (b), we can take $a_2m = a_1ta_1m \notin P$, where $t \in R$.

Therefore, we have the sequence $\eta(a)$ of a the set $\eta(a) = \{a_0, a_1, a_2, ... : a_{i+1} \in a_iRa_i \text{ and } a_0 = a,\ i \in \mathbb{N}\}$ but $\eta(a)m$ does not contain any element of P since for all $i \in \mathbb{N}$, $a_im \notin P$. Therefore am is not a strongly nilpotent element of M on N. □

Theorem 3.7. *Let R be a left Noetherian ring and let P be a submodule of an R-module M. Suppose that P is maximal among all submodules in M that are not finitely generated. Then P is a one-sided prime submodule of M.*

Proof. Suppose that $\Omega_m(P) \neq R$, $a \in R - \Omega_m(P)$ and $b \in R - \Omega_m(P)$ such that $aRbm \subseteq P$ and $\Omega_m(P)Rb \subseteq \Omega_m(P)$ for $m \in M$. Then $P + Rbm$ is different from P and $P + Rbm$ is finitely generated. Let $\{p_1 + r_1bm, ..., p_t + r_tbm\}$ be a generating set for $P + Rbm$ where $p_i \in P$ and $r_i \in R$.

Define the set $K = \{y \in R : ybm \in P\}$. Then clearly, K is a finitely generated left ideal because R is a left Noetherian ring.

Take an element x in $P \subsetneq P + Rbm$.

$x = u_1(p_1 + r_1bm) + ... + u_t(p_t + r_tbm)$ for some $u_i \in R$ and so

$$x - (u_1p_1 + ... + u_tp_t) = (u_1r_1 + ... + u_tr_t)bm$$

Hence $(u_1r_1 + ... + u_tr_t) \in K$. This means that $x \in Rp_1 + ... + Rp_t + Kbm$ and from otherside, we have $Rp_1 + ... + Rp_t + Kbm \subseteq P$. Then $P = Rp_1 + ... + Rp_t + K$ which implies that P is finitely generated, a contradiction. □

It is well known that if every prime submodule in a module M is finitely generated, then M satisfies ascending chain condition on submodules. Since the class of one-sided prime submodule is different from the class of prime submodule in a module M, we get the result as follows:

Corollary 3.8. *Let R be a left Noetherian ring and let M be an R-module. If every one-sided prime submodule in a module M is finitely generated, then M satisfies ascending chain condition on submodules.*

Proof. Let every one-sided prime submodule in a module M be finitely generated. Define the set

$$\Omega = \{N_i : N_i \text{ is a submodule of } M \text{ but not finitely generated}\}.$$

$\Omega \neq \oslash$, $J = \cup N_i$ is not finitely generated submodule in M and J is upper bound in the set Ω. By Zorn's Lemma, there is a maximal element P in the set Ω. By Theorem 3.7, P is a one-sided prime submodule of M and then M satisfies ascending chain condition on submodules. □

The following theorem may be regarded as a generalization of Kaplansky's Theorem for one-sided prime submodules.

Theorem 3.9. *Let M be a cyclic R-module satisfying the ascending chain condition on O-radical submodules. Then any O-radical submodule in M is the intersection of a finite number of one-sided prime submodules. In particular any submodule in M is the intersection of a finite number of one-sided prime submodules.*

Proof. Let $M = Rm$ and take an O-radical submodule P such that $P = \Omega_m(P)m$. If not, let a submodule P be maximal among these for which the assertion fails. Then it is clear that P is not a one-sided prime submodule. Take $a \in R - \Omega_m(P)$ and $b \in R - \Omega_m(P)$ such that $aRbm \subseteq P$ and $\Omega_m(P)Rb \subseteq \Omega_m(P)$. Let J be a left O-radical ideal of $\Omega_m(P) + Ra$ and K an O-radical submodule of $P + Rbm$. Since P is maximal, Jm and K are each expressible as a finite intersection of one-sided prime submodules. We reach a contradiction proving that $P = Jm \cap K$.

Let $x \in Jm \cap K$. By Proposition 3.5, x is in $W_M(Jm) \cap W_M(K)$ and so x is a strongly nilpotent on $Jm = (\Omega_m(P) + Ra)m$ and $K = P + Rbm$. If $T = \{a_i : a_{i+1} \in a_iRa_i$ and $a_0 = a$, $i \in \mathbb{N}\}$, then there exits $a_nm \in ((\Omega_m(P) + Ra)m) \cap Tm$ and so $a_tm \in ((\Omega_m(P) + Ra)m) \cap Tm$ for all $t \geq n$. Similarly, there exists $a_mm \in (P+Rbm)\cap Tm$ and so $a_vm \in (P+Rbm)\cap Tm$ for all $v \geq m$ for some $n, m \in \mathbb{N}$. Now assume that $n \leq m$. Then we observe that $a_{m+1} = la_nka_m \in T$ for some $l, k \in R$ and so $a_{m+1}m$ in $Tm\cap((\Omega_m(P)+Ra)(P+Rbm))) \subseteq Tm\cap P$. Since P is O-radical submodule, $a_{m+1}m \in W_M(P) = P = rad_M(P)$. The $W_M(P) = W_R(P : m)m = rad_R(P : m)m$ with [15] and so x is in P, which means that $P = Jm \cap K$. This is a contradiction with our assumption. Therefore, any O-radical submodule in M is the intersection of a finite number of one-sided prime submodules. □

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