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Application of Topological Degree Method in Quantitative Behavior of Fractional Differential Equations

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Abstract. In the present manuscript we incorporate fractional order Caputo derivative to study a class of non-integer order differential equation. For existence and uniqueness of solution some results from fixed point theory is on our disposal. The method used for exploring these existence results is topological degree method and some auxiliary conditions are developed for stability analysis. For further elaboration an illustrative example is provided in the last part of the research article.

1. Introduction

Researchers in every branch of mathematics are struggling to expound new results, refine the existing and formulate new ideas. More explicitly the field of differential equations is getting more comprehension. As we face more complicated problems, researchers formulate various kinds of differential equations to model these phenomena. In classical integer order differential equations it was observed that the experimental out come did not match with analytical results. Therefore, mathematicians have to think over the theory presented by L'Hostipal and that was discussed with great Euler to find out the applications of the newly formulated theory of fractional order operators. Initially, fractional calculus was purely theoretical mathematical notion. Later on, after long efforts and discussions, it was concluded that fractional calculus has tremendous applications in several branches of science; like in MRI screening, phenomena of signal processing, earthquakes' nonlinear oscillation. Moreover, this theory has also attracted the interests of people who are working in econometrics, geophysics, engineering disciplines and other related fields; for comprehensive study on applications of this new emerging field readers may study the articles [1, 2, 10, 11, 13, 14, 16, 19] and references therein. From theoretical point of view a differential equation can be studied either via it's qualitative behavior or it can be studied through it's quantitative nature. In the first case one needs not to deal with it's well posdness, but in the phase plane possible solutions set's general aspects are studied. In the later case the differential equations' analytical solutions are explored. This analytical solution can be exact or it can be numerical solution. The degree of difficulty level of a differential equation is related to its study. If a differential equation's exact solution is not possible, then we need to use numerical

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methods to exhibit its approximate solution(s). In some cases if a differential equation contains more complicated nonlinear terms then we need to discuss the qualitative behavior of the differential equation. The existence of solutions, it's uniqueness, stability analysis or other aspects of a differential equation has been studied via the tools of classical fixed point theory, measure of non-compactness and related results (see[3, 4, 6–9, 12, 17, 18, 20–23]).

The preceding study of fractional differential equations created a motivation to study the present problem. This manuscript is devoted to study the existence of solution, it's uniqueness for a Cauchy problem of fractional order. Since there are several ways for solving such kinds of differential equation, but we will use most applicable method, known as topological degree method. In order to make the present study a comprehensive note we provide some valuable work carried out in the past. In this regard J. Wang and his coauthors in 2012 [24], studied the quantitative nature of the following model. The model under their consideration is of fractional order along with non-local subsidiary boundary conditions,

$$\begin{cases} {}^{c}\dfrac{d\rho(s)}{ds^{\varsigma}} = I(s,\rho(s)); 0 \leq s \leq T \\ \rho(0) + g(\rho) = \rho_0, \end{cases}$$

here the symbol ${}^{c}\frac{d}{dt^{\varsigma}}$ represents Caputo fractional derivative of order $\rho \in (0,1)$. Along with the preceding model the authors also carried out some existence as well as uniqueness for the succeeding fractional order boundary value problem,

$$\begin{cases} {}^{c}\dfrac{d\rho(s)}{ds^{\varsigma}} = I(s,\rho(s)); 0 \leq s \leq T \\ c_1\rho(s)(0) + c_2 g(T) = c_3, \end{cases}$$

here c_1, c_2, c_3 are any arbitrary constants.

Afterward another interesting problem was solved by El-Shahed and his coauthors in [5]. The research conducted in this paper is mainly devoted to solubility of the following non-integer order differential equation along with multiple points boundary conditions,

$$\begin{cases} {}^{c}\dfrac{d\rho(s)}{ds^{\varsigma}} + I(s,\rho(s)) = 0, \quad here\ \ t \in [1\ \ l],\ while\ \varsigma \in [n-1\ \ n),\ n \in (2,\ \infty) \\ \rho(a) = \rho\rho_1(\lambda_1) + \ldots + \rho\rho_{m-2}(\lambda_{m-2}),\ \rho^{'}(a) = \rho^{''}(a) = \rho^{'''} = \cdots = \rho^{n-1}(a) = 0,\ \rho(l) = \displaystyle\sum_{k=1}^{m-2} \gamma_k \rho(\lambda) \end{cases}$$

where $\lambda_i \in (0,\ l)$ for $i = 1, \ldots, m-1$ and the mathematical terms $\sum_{k=1}^{m-2} \rho v_k$ and $sum_{k=1}^{m-2} \gamma_k$ lies below unity.

The fixed point theorem of Schauder's is used by the authors in [3], to explore solution of non-integer order differential equations of the following shape

$$\begin{cases} {}^{c}\dfrac{d\rho(s)}{ds^{\varsigma}} + I\left(s, \rho(s), {}^{c}\dfrac{d\delta(s)}{ds^{\varsigma}}\right) = 0;\ t \in (0,\ \ 1),\ \varsigma \in [3,\ \ 4], \\ \rho(0) = \rho^{'}(0) = \rho^{''}(0) = 0, \rho(1) = \rho(\xi),\ \xi \in (0,\ \ 1), \end{cases}$$

${}^{c}\frac{d\delta(s)}{ds^{\varsigma}}$ is reserved for the differential operator of Caputo version of the solution, ρ.
In [9], the authors used another tool known as applied generalization method to study the solution's existence theory and numerical solution to the following fractional order boundary value problem,

$$\begin{cases} {}^{c}\dfrac{d\rho(s)}{ds^{\varsigma}} + I(s,\rho(s)) = 0; t \in (0,1), \rho \in (n-1, n], n \geq 2, \\ \rho^{'}(0) = \rho^{''}(0) = \rho^{'''} = \cdots = \rho^{n-1}(0) = 0, \rho(1) = \eta\rho(\xi),\ \xi, \eta \in (0,1). \end{cases}$$

Some results from fixed point theory are used as main tools to explore existence results and multiplicity of the possible positive solutions. Moreover, the authors illustrated their obtained results by providing some examples.

Besides the existence results for a variety of non-integer order differential equations some people devoted themselves to study the qualitative aspects as well as functional types of stability analysis of some models. In this regard the authors of [25–27] studied the local stability analysis as well as M. Leffler type of stability for a class of non-integer order differential, integro-differential equations.

Furthermore, some researcher paid ample attentions to the study of functional type stability. This new theory is known as Hyers-Ullam's type of stability analysis and some remarkable results have been carried out for Integral order differential equations with both kinds of initial and boundary value problems see[28–31]. Extending these kinds of problems for fractional order few research articles exist in the literature, we refer the reader to study [32–35] as well as work cited in. This new area of research is also closed to the existence theory of integral and differential equations of arbitrary order. The fact behind this assertion is that qualitative nature of these kinds of differential equations depends on the classical stability and existence results. Due to the huge applications, Hyers-Ullam's type of stability and its further extended form have capture many fields of science. These include problems in control theory, physics, biology, chemistry, dynamics and economics.

Purpose of describing the aforesaid literature review and introduction is two fold; first to provide glimpses to readers of recently done work and secondly these articles provide motivation to study a class of Cauchy problem of non-local nature with subsidiary conditions on the independent variable. The model to be studied in the present article is given as,

$$\begin{cases} {}^{c}\mathfrak{D}^{\omega}\rho(t) = I(t,\rho(t)),\ 0 \le s \le 1,\ \omega \in (1\ 2], \\ \rho(0) = \rho_0,\ \rho(1) = g(\rho), \end{cases} \tag{1}$$

where the function g is a non-local operator from space of continuous function defined on interval [0, 1] into R, and I is continuous function with domain $[0,1] \times R$ and range R. The mathematical object, ${}^{c}\mathfrak{D}^{\omega}$ is differential operator defined by Caputo, of order ω where $\omega \in (0, 1)$.

Organization of the paper is given as; section 2 is devoted to describe some preliminaries definitions and results to be used in proving the main results of the paper. Section 3 concerns with derivation of Green function, existence as well as uniqueness of the solution via famous Schauder fixed point theorem. In order to show the authenticity of our obtained results we provide an example.

2. Basic Results & Fundamental Definitions

This section of the manuscript is reserved for providing some fundamental definitions, useful theorems and other related materiel from the theory of fractional calculus and mathematical analysis. Beside this we also provide some results from fixed point theory to transfer our model into fixed point problem. For beginner of the field we provide some useful books, articles therefore the interested readers are referred to see [8, 9, 12, 15, 24].

Definition 2.1. *Given a function g which is Lebesgue integral on the interval [0, T], then Riemann Liuville type of integral of the function g of non-integer order, $0 < \rho < 1$ is given by;*

$$\mathfrak{I}_a^{\rho} g(\tau) = \frac{1}{\Gamma(\rho)} \int_0^{\tau} (\tau - s)^{\rho-1} g(t)\, dt.$$

Definition 2.2. *Given a function g which is defined on the interval [0, T], then Caputo type of differential operator of the function, g of non-integer order, $\rho \in \mathbb{R}$ is defined as;*

$${}^{c}\mathfrak{D}_a^{\alpha} g(\tau) = \frac{1}{\Gamma(n-\rho)} \int_0^{\tau} (\tau - s)^{n-\rho-1} g^{(n)}(t)\, dt,$$

where $n = [\alpha] + 1$ and $[\rho]$ shows the flour function at the value ρ.

Theorem 2.3. *If ψ is a function, then the relation between fractional integral and differential operators is given by;*

$$\Im^{\rho}({}^{c}\mathfrak{D}^{\rho})\psi(t) = \psi(t) + \sum_{l=0}^{n-1} a_l s^l,$$

for any $a_l \in \mathbb{R}$, $l = 0, 1, 2, \ldots, n-1$, where $n = [\rho] + 1$ and $[\rho]$ denotes the floor function at ρ.

Definition 2.4. *Assume that Ω is a subset of X and G is a continuous function defined on Ω into X which is also bounded. Then the function G is said to be $\rho-$Lipschitz if we can find a constant $c \geq 0$ such that*

$$\rho(G(A)) \leq c\rho(A), \ \ A \subset \Omega.$$

If $c < 1$, then the function G is a strict $\rho-$contraction while G is $\rho-$condensing if

$$\rho(G(A)) < \rho(A), \ \ A \subset \Omega.$$

In practice it is known that $G : \Omega \to X$ is Lipschitz if we can find $c > 0$ such that

$$\|G(t) - G(s)\| \leq c\|t - s\|, \ \ t, s \in \Omega.$$

Definition 2.5. *For a continuous function with domain [0 1] and counter domain R the model* (4) *is stable in the sense of Hyers-Ulam definition if for every positive real number, ϵ the following inequality hold*

$$|{}^{c}\mathfrak{D}^{\rho}v(s) - I(s, v(s))| \leq \epsilon, \ \ s \in [0, 1] \tag{2}$$

and we can find a solution v^* belongs to the space of continuous functions defined on [0 1] so that

$$|v(s) - v^*(s)| \leq k_I\epsilon, \ \ 0 \leq s \leq 1. \tag{3}$$

Definition 2.6. *For a continuous function with domain [0 1] and counter domain R the model* (4) *is stable in the sense of generalized Hyers-Ulam definition if for every positive real number, ϵ and for every continuous function defined on positive real line into itself, the following inequality hold*

$$|{}^{c}\mathfrak{D}^{\rho}v(s) - I(s, v(s))| \leq \varpi(s)\epsilon, \ \ s \in [0, 1] \tag{4}$$

and we can find a solution v^* belongs to the space of continuous functions defined on [0 1] so that

$$|v(s) - v^*(s)| \leq k_I\varpi(s)\epsilon, \ \ 0 \leq s \leq 1. \tag{5}$$

Remark 2.7. *A continuous function λ with domain [0 1] and range R is solution to the inequality* (4) *if and on if we can find another continuous function say ψ that depends on the solution λ so that*

$(i)|\psi(s)| \leq \epsilon, s \in [0, 1],$
$(ii){}^{c}\mathfrak{D}^{\rho}\lambda(s) = I(s, \lambda(s)) + \psi(s), \ \ s \in [0, 1].$

Proposition 2.8. *[24] Assume two ρ-Lipschitz functions say B_1 and B_2 with domain $\wedge$ and range set Y with Lipschitzen C_1 and C_2 respectively, then the sum of B_1 and B_2 is also ρ-Lipschitz functions with constants $C_1 + C_2$.*

Proposition 2.9. *If a function, B_1 with domain $\wedge$ and range set Y is compact, then B_1 is ρ-Lipschitz function with Lipschitzen zero.*

Proposition 2.10. *If a function B_1 with domain $\wedge$ and range set Y is ρ-Lipschitz function with constant C, then B_1 is ρ-Lipschitz function with Lipschitzen C.*

The succeeding theorem shows the existence as well as basic properties of the topological degree for ρ-condensing perturbations of the identity. Assume the space

$$\Delta = \{J - B_1 : \Lambda \subset Y \text{ is bounded and open set, } B_1 \text{ is } \rho - \text{Lipschitz function defined on closure of } \Lambda\}$$

A function $Deg : \Lambda_1 \to \Lambda_2$ exists, that fulfills the following properties;

Theorem 2.11. *[36] We recall some basic properties of proposed degree theory. Let for the family of admissible triplets given by*

$$F = \Big\{(I - F, \Omega, z) : \Omega \subset \ Z \ \text{be an open and bounded set}, F \in C_\beta(\Omega), \ z \in Z - (I - F(BND(\Omega)))\Big\}$$

There exists one degree function $D : \Theta \to Z$, which satisfies the properties:

(P_1) *$Deg(I, \Lambda_1, x) = 1$, at each $y \in \Omega$; This property is called Normalization,*

(P_2) *For any pair of non-overlapping open sets, $W_1, \ W_2 \subset \Lambda_1$ and for every $x \notin (J - D)(\Lambda - (W_1 \cup W_2))$, one obtain that $Deg(J - D, \ \Lambda, \ x) = Deg(J - F\Lambda, \ W_1, \ x) + Deg(J - F\Lambda, \ W_2, \ x)$; It is called additivity property on domain,*

(P_3) *$Deg(J - M(s, x), \Lambda, x)$ is independent of $s \in J$ for every bounded and continuous function $M : J \times \Lambda \to \Lambda$ that fulfill the inequality $\rho(M(J \times A) < \rho(A))$; $\vee \ A \subset \Lambda_1$, with property $\rho(A) > 0$ as well as for each continuous function $f : J \to \Lambda_2$ which satisfy the condition that $f \neq f - M(s, x)$: at each $s \in J$, it is called Homotopy invariance property,*

(P_4) *$Deg(J - D, \ \Lambda_1, x) \neq 0$ implies that $x \in (J - D)(\Lambda_1)$, i.e the function D satisfy existence property.*

(P_5) *$Deg(J - D, \Lambda, x) = Deg(J - D, \ W_1, x)$ for every open set $W_1 \subset \Lambda_1$ at each $x \notin (J - D)(\Lambda - W_1)$. It is called Excision property.*

If a mathematical structure is equipped with a degree function defined on I, by means of this degree theory we study the usability of the "a priori estimate method".

Theorem 2.12. *[36] Consider a function $H : W \to W$ which is ρ- contraction and*

$$\Xi = \{w \in W : \ \text{if there exists } \ \lambda \in [0, 1], \ \text{such that } w = \lambda H(w)\}.$$

If Ξ is contain in a ball inside the set W, then we can find a positive real number ρ such that $\Xi \subset B_\rho(0)$, thus

$$D\Big(I - \varpi H, B_\rho(0), (0)\Big) = 1, \ \ \text{for all} \ \ 0 \leq \varpi \leq 1.$$

Therefore, H has at least one fixed point while set of all fixed points for the function H is a proper subset of $B_r(0)$.

3. Different Aspects of the Model

The present part of the article focus on the existence theory of our proposed model. Moreover, under some axillary conditions about uniqueness of underlying fractional differential equation (1), will be studied. For establishing main results about existence theory and other aspects we suppose some hypotheses which will be used in the sequel.

(H_1) For any continuous function $\chi_1(\cdot)$ and $\chi_2(\cdot)$, with domain [0 1] and range set R, we can find some constant values $0 \leq C_{\Theta_1}, \ C_{\Theta_2} \leq \ 1$ such that

$$d(\Theta_1(\chi_1), \ \Theta_1(\chi_2)) \leq C_{\Theta_1} \|\chi_1 - \chi_2\|, \ d(\Theta_2(\chi_1), \ \Theta_2(\chi_2)) \leq C_{\Theta_2} \|\chi_1 - \chi_2\|,$$

where d is usual metric on R. Where by R we mean the set of real numbers.

(H_2) For any continuous function ρ with domain [0 1] and range set R we can find some constant values $C_{\theta_3},\ C_{\theta_4}, Z_1, Z_2 > 0, \alpha \in [0,1)$ so that

$$|\Theta_1(\rho)| \leq C_{\theta_3}\|\rho\|_c^{\alpha} + Z_1,\ |\Theta_2(\rho)| \leq C_{\theta_4}\|\rho\|^{\alpha} + Z_2.$$

(H_3) For any arbitrary order pair (p,q) from $J \times \mathbb{R}$, we can find constants $C_{\theta_5},\ C_{\theta_6} > 0$ *and* $\alpha_2 \in [0,1)$ such that

$$|f(t,u)| \leq C_{\theta_5}\|u\|^{\alpha_2} + C_{\theta_6}.$$

Initially, we prove existence result for our proposed BVP with arbitrary input function. Also we provide result for relationship between Lipschitz continuous and $\rho-$Lipschitz continuous functions. Later on solution of our actual model will be studied. At the last Hyers-Ulam stability analysis will be elaborated while providing some theorem. To illustrate our carried out existence results an example is given to show the authenticity of the obtained results

Theorem 3.1. *The non-local Cauchy problem of fractional order,*

$$\begin{cases} {}^{cp}\mathcal{D}^{\xi}\psi(t) = \epsilon(t),\ t \in I = [0,1], \\ \psi(0) = \psi_0,\ \psi(1) = \tau(\psi), \end{cases}$$

possesses a unique solution ψ, that gets the shape $\psi(t) = \int_0^1 \mathfrak{G}(t,s)\epsilon(s)ds$, here the function, $\mathfrak{G}(t,s)$ is the non-singular function, which is known as, Green function and is provided by

$$\mathfrak{G}(t,s) = \frac{1}{\Gamma(\xi)} \begin{cases} -t(1-s)^{\xi-1} + (t-s)^{\xi-1},\ s \leq t \in (0,\ 1), \\ -t(1-s)^{\xi-1},\ t \leq s \in (0\ 1). \end{cases} \tag{6}$$

Proof. Using Theorem 2.3 and applying integral operator I^{ψ} of fractional order to the fractional differential equation $D^{\alpha}\psi(t) = h(t)$, one can get

$$\psi(t) = \theta_0 + \theta_1 t + I^{\xi}\epsilon(t). \tag{7}$$

Use of the auxiliary boundary conditions $\psi(0) = \psi_0, \psi(1) = \tau(\psi)$ in (7) while estimating $\theta_0 = \psi_0$ and $\theta_1 = \tau(\psi) - \psi_0 + I^{\xi}\epsilon(1)$, we get

$$\begin{aligned} \psi(t) &= \psi_0 + t\left(\tau(\psi) - \psi_0 - I^{\xi}\epsilon(1)\right) + I^{\xi}\epsilon(t) \\ &= (1-t)\psi_0 + t\tau(\psi) - tI^{\xi}\epsilon(1) + I^{\xi}\epsilon(t) \end{aligned}$$

Implies

$$\begin{aligned} \psi(t) &= (1-t)\psi_0 + t\tau(\psi) - \frac{t}{\Gamma(\xi)}\int_0^1 (1-s)^{\xi-1}\epsilon(s)ds - \frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\epsilon(s)ds = \\ &= (1-t)\psi_0 + t\tau(\psi) + \int_0^1 \mathfrak{G}(t,s)\epsilon(s)ds. \end{aligned} \tag{8}$$

Therefore, in light of relation (8) the proposed class of fractional order differential equation (1)exhibit a solution, which is given by;

$$\psi(t) = (1-t)\psi_0 + t\tau(\psi) + \int_0^1 \mathfrak{G}(t,s)f(s,\psi(s))ds. \tag{9}$$

□

We now transform the fractional order integral equation (9) to another operator equation form. For this purpose, we define some operators;

$$\Psi, \text{ with domain and range } C(J,R), \text{ defined by } \Psi\psi(\theta) = (1-\theta)\psi_0 + \theta\tau(\psi) \tag{10}$$

while

$$\Xi, \text{ with domain and range } C(J,R) \text{ defined by } \Xi\psi(t) = \int_0^1 \mathfrak{G}(t,s)\epsilon(s)ds. \tag{11}$$

$$Q, \text{ with domain and range } C(J,R), \text{ defined as } Q(\psi) = \Psi(\psi) + \Xi(\psi). \tag{12}$$

It is obvious that the operator Q is well defined in terms of set theory. Thus, the relevant integral equation (9) can be easily transform to the following form,

$$\psi = Q(\psi) = \Psi(\psi) + \Xi(\psi). \tag{13}$$

hence equation (1) solution's set is equivalent to the existence of fixed point of equation (13).

Theorem 3.2. *The mapping Ξ is ρ-Lipschitz continuous with constant Lipschitzen π whenever Ξ, with domain and range $C(J,R)$ is Lipschitz continuous having the constant $0 \le \pi_\tau < 1$. Furthermore, Ξ satisfy the inequality*

$$\|\Xi(\psi)\| \le |\psi_0| + C_\tau\|\psi\|^{q_1} + Z_1.$$

Proof. *We presume $\psi, v \in C([0,1],R)$ while using the hypothesis (H_1), then we assume*

$$|\Xi\psi - \Xi v| = |t(\tau(\psi) - \tau(v))|$$

for $t \le 1$ we have,

$$\|\Xi\psi - \Xi v\| \le \pi_\tau\|\psi - v\|.$$

Thus, Ξ satisfy the condition of $\rho-$Lipschitz continuous function with constant π_τ. In addition to the above, we establish growth condition via (H_2), for this we assume

$$\begin{aligned}|\Xi(\psi)| &= |(1-t)\psi_0 + t\tau(\psi)| \\ &\le |(1-t)\psi_0| + |t\tau(\psi)| \\ &\le |\psi_0| + |\tau(\psi)|,\end{aligned}$$

therefore, one can obtain

$$\|\Xi\psi\| \le |\psi_0| + C_\tau\|\psi\|^{q_1} + Z_1. \tag{14}$$

□

Theorem 3.3. *Assume an operator Ψ with domain and range $C(J,R)$, then the operator Ψ obeys the following estimation*

$$\|\Psi\psi\| \le \frac{1}{\Gamma(\xi+1)}[C_f\|\psi\|^{q_2} + Z_2].$$

Proof. Assume a sequence of bounded sets $\{\psi_n\}_{n\in\mathbb{N}}$ as well as a collection of continuous functions, $B_k = \{\psi : \psi \in C(J, R), \|\psi\| \le k\} \subset C(J, R)$ such that $\psi_n \to \psi, n \to \infty$ in B_k. So it need to provide the proof of,

$\|\Psi(\psi_n) - \Psi(\psi)\| \to 0$ as *n approaches towards* ∞. Therefore, we suppose

$$\begin{aligned}
\left|\Psi(\psi_n) - \Psi(\psi)\right| &= \left|\int_0^1 \mathfrak{G}(t,s)[I(s,\psi_n(s)) - I(s,\psi(s))]ds\right| \\
&= \left|\frac{1}{\Gamma(p)}\int_0^1 (1-s)^{p-1}[I(s,\psi_n(s)) - I(s,\psi(s))]ds \right.\\
&\left. - \frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}[I(s,\psi_n(s)) - I(s,\psi(s))]ds\right| \\
&\le \frac{1}{\Gamma(\xi)}\int_0^1 (1-s)^{\xi-1}|I(s,\psi_n(s)) - I(s,\psi(s))|ds \\
&+ \frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}|I(s,\psi_n(s)) - I(s,\psi(s))|ds.
\end{aligned} \tag{15}$$

Since I is a continuous function, which results that $I(s,\psi_n(s)) \to I(s,\psi(s))$, $n \to \infty$. Using Lebesgue Dominated convergence theorem we obtain, $\frac{1}{\Gamma(\xi)}\int_0^1 (1-s)^{\xi-1}[I(s,\psi_n(s)) - I(s,\psi(s))]ds \to 0$, as $n \to \infty$. In the same lines, one can treat the second term in (15). Hence, the right hand side of relation (15) approaches towards the value, 0 as n tends to infinity. Therefore, we obtain $\|\Psi(\psi_n) - \Psi(\psi)\| \to 0$, as $n \to \infty$. In conclusion Ψ is a continuous map. To show the condition, known as growth condition, we suppose $B_k \subset C(J, R)$ is contained in a ball of radius ρ, where $0 < \rho < \infty$. If $\Psi : B_k \to B_k$, we have to show that $\Psi(B_k)$ is contained in a ball of radius, ρ $0 < \rho < \infty$. Therefore, if $\psi \in B_k = \{\|\psi\| \le k : \psi \in C(J, R)\}$, then one obtain;

$$\begin{aligned}
|\Psi(\psi(t))| &= \left|\int_0^1 \Psi(t,s)I(s,\psi(s))ds\right| \\
&\le \left|-\frac{t}{\Gamma(\xi)}\int_0^1 (1-s)^{\xi-1}I(s,\psi(s))ds + \frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}I(s,\psi(s))ds\right| \\
&\le \frac{1}{\Gamma(\xi)}\int_0^1 (1-s)^{\xi-1}|I(s,\psi(s))|ds
\end{aligned} \tag{16}$$

which implies that, $\|\Psi(\psi)\| \le \frac{1}{\Gamma(\xi+1)}[C_I\|\psi\|^{q_2} + Z_2]$.

Hence, we obtained the required results. □

Theorem 3.4. *The operator* Ψ *from* $C(J, R)$ *into itself is compact, consequently* Ψ *is* ρ*-Lipschitiz with Lipschitizen 0.*

Proof. In order to prove that Ψ is a compact set, we assume the inclusion $D \subset B_k \subset C(J, R)$ and suppose that it is contained in a ball of radius $r < \infty$. Our goal is to prove that $\Psi(D)$ is relatively compact in the space of all continuous functions with domain [0 1] and rang set, R. Use of relation (16) and a sequence, $\{\psi_n\}$ is taken in the set D, then we obtain the succeeding estimation,

$$\|\Psi(\psi_n)\| \le \frac{1}{\Gamma(\xi+1)}[C_f\|\psi\|^{q_2} + Z_2], \text{ for each } \psi_n \in D.$$

Thus, $\Psi(D)$ is bounded in $C(J, R)$. To show that $\Psi(D)$ is equicontinuous, lets assume $0 \le t_1 \le t_2 \le 1$, then one obtain

$$\begin{aligned}
|\Psi(\psi_n(t_2)) - \Psi(\psi_n(t_1))| &\leq \frac{(t_1 - t_2)}{\Gamma(\xi)} \int_0^1 (1-s)^{\xi-1} |I(s, \psi_n(s))| ds \\
&+ \frac{1}{\Gamma(\xi)} \int_0^{t_1} \left((t_2 - s)^{\xi-1} - (t_1 - s)^{\xi-1}\right) |I(s, \psi_n(s))| ds \\
&+ \frac{1}{\Gamma(\xi)} \int_{t_1}^{t_2} (t_1 - s)^{\xi-1} |I(s, \psi_n(s))| ds \\
&\leq \frac{(t_1 - t_2)}{\Gamma(\xi+1)} [C_I \|\psi_n\|^{q_2} + M_2] + \frac{1}{\Gamma(\xi+1)} [C_I \|\psi\|^{q_2} + M_2] \left((t_2 - t_1)^{\xi} - t_1^{(}\xi) + t_2^{\xi}\right) \\
&= \frac{[C_I \|\psi\|^{q_2} + Z_2]}{\Gamma(\xi+1)} \left((t_1 - t_2) + (t_2 - t_1)^{\xi} + t_2^{\xi} - t_1^{\xi}\right).
\end{aligned}$$

If t_1 approaches to t_2, then $|\Psi(\psi_n(t)) - \Psi(\psi(t))| \to 0$, as $n \to \infty$, thus Ψ is an equicontinuous. Therefore, the conclusion obtained from discussion is that $\Psi(D) \subseteq C(J, R)$, which fulfills the premises of Arzela-Ascola theorem. Hence, $\Psi(D)$ is relatively compact in the space of all continuous functions defined on [0 1] into set of real numbers. So the mapping Ψ is ρ-Lipschitz with Lipschitzen zero. □

Theorem 3.5. *We assume the hypotheses* (H_1) *to* (H_3)*, then the solution's set of* (1) *possesses at least one element* $\psi \in C(J, R)$ *and it is contained in a ball inside the space* $C(J, R)$ *with respect to usual norm.*

Proof. Suppose the operators, $\Xi, \Psi, Q : C(J, R) \to C(J, R)$ which are continuous with domain and counter domain given in (10), (11), (12). Furthermore, for the constant k_τ, $0 \leq k_\tau < 1$, Ξ is ρ-Lipschitz. Thus, the operator Q is strict ρ-contraction with constant k_τ. In addition to the above, we assume $W_0 = \{\psi \in C(J, R) : \exists\, \lambda \in [0, 1],$ such that $\psi = \lambda Q\psi\}$. Now to show that the set, W_0 is contained in a ball, let λ, $0 \leq \lambda \leq 1$ so that $\psi = \lambda Q\psi$ from (16) and (5), one get

$$\begin{aligned}
\|\psi\| = \|\lambda Q\psi\| &= \lambda(\|\Xi\psi + \Psi(\psi)\|) \leq \lambda(\|\Xi\psi\| + \|\Psi(\psi)\|) \\
&\leq |\psi_0| + C_\tau \|\psi\|^{q_1} + M_1 + \frac{1}{\Gamma(p+1)} [C_I \|\psi\|^{q_2} + Z_2].
\end{aligned} \tag{17}$$

We need to prove that W_0 is bounded. We prove it via contradiction method and suppose that $\|\psi\| = \mathcal{R}$ such that $\mathcal{R} \to \infty$. Division of both sides of the relation (17) by $\|\psi\|$ give the following,

$$1 \leq \frac{1}{\|\psi\|} \left[|\psi_0| + C_g \|\psi\|^{q_1} + M_1 + \frac{1}{\Gamma(\xi+1)} [C_I \|\psi\|^{q_2} + Z_2] \right]$$

using $\mathcal{R} \to \infty$, then we have

$1 \leq 0.$

Thus, we reached to an ill-defined circumstances, which is due to our wrong supposition. Therefore, the set W_0 is bounded and the fixed point's set, W_0 of the operator contains at least one point. Moreover, this set is also bounded $C(J, R)$. □

To show that solution's set is singleton, we assume another hypothesis,

(H_4) A positive constant L_I exist such that
$|I(t, \psi) - I(t, v)| \leq L_I |\psi - v|$, for every $t \in J$ for each $\psi, v \in R$.

Theorem 3.6. *Suppose that* $(H_1) - (H_4)$ *are true, then fractional order differential equation* (1) *possesses a unique solution* $\psi \in C(J, R)$ *if and only if the quantity* $k_\tau + \frac{L_f}{\Gamma(\xi+1)}$ *lies below unity.*

Proof. Take continuous functions ψ, v and also suppose the relation

$$\begin{aligned}|Q\psi(t) - Qv(t)| &\leq |t(\tau(\psi) - \tau(v))| + \int_0^1 \mathfrak{G}(t,s)|[I(s,\psi(s)) - I(s,v(s))]|ds \\ &\leq |(\tau(\psi) - \tau(v))| + \int_0^1 \mathfrak{G}(t,s)[L_I|\psi - v|]ds, \quad t \leq 1.\end{aligned}$$

Furthermore, we take the maximum over $[0,1]$, then one can obtain

$$\begin{aligned}\|Q\psi - Q(v)\| &\leq k_\tau\|\psi - v\| + \frac{1}{\Gamma(\xi+1)}L_I\|\psi - v\| \\ &\leq \left(k_\tau + \frac{L_I}{\Gamma(\xi+1)}\right)\|\psi - v\|,\end{aligned}$$

thus fixed point's set of the proposed system is singleton. This shows the fact that the obtained result is the required unique solution of (1). □

Theorem 3.7. *Suppose the hypotheses from* (H_1) *to* (H_4)*, then for* $\left(k_\tau + \frac{L_I}{\Gamma(\alpha+1)}\right) \neq 1$ *the boundary value problem* (1) *is Hyers-Ulam stable and generalized Hyers-Ulam stable.*

Proof. Assume a continuous function ψ in the space $C([0,1],R)$ which satisfy the boundary value problem (1) and also consider a continuous function v that is the required unique solution of boundary value problem provided as

$$\begin{cases} {}^{cp}D^{\rho}\psi(t) = I(t,\psi(t)), \quad t \in [0,1] \\ \psi(0) = v(0), \quad \psi(1) = \tau(\psi) = \tau(v). \end{cases}$$

Then, the form obtained by the solution is as,

$$v(t) = (1-t)\psi_0 + t\tau(\psi) + \int_0^1 \mathfrak{G}(t,s)I(s,\psi(s))ds, \ t \in [0,1].$$

Applying the proceeding relation one can obtain

$$|v(t) - (1-t)\psi_0 + t\tau(v) + \int_0^1 \mathfrak{G}(t,s)I(s,v(s))ds| \leq \epsilon, \ t \in [0,1]. \tag{18}$$

In addition, we also use (18)

$$\begin{aligned}|v(t) - \psi(t)| &= \left|v(t) - \left((1-t)\psi_0 + t\tau(\psi) + \int_0^1 \mathfrak{G}(t,s)I(s,\psi(s))ds\right)\right| \\ &\leq \left|v(t) - \left(t\tau(v) + (1-t)\psi_0 + \int_0^1 \Psi(t,s)I(s,v(s))ds\right)\right| \\ &+ \left|\left(t\tau(v) + (1-t)\psi_0 + \int_0^1 \Psi(t,s)I(s,v(s))ds\right) - \left(t\tau(\psi) + (1-t)\psi_0[R_{mn}]\int_0^1 \Psi(t,s)I(s,\psi(s))ds\right)\right| \\ &\leq \epsilon + k_\tau|v(t) - \psi(t)| + \int_0^1 |\mathfrak{G}(t,s)||I(s,v(s)) - I(s,\psi(s))|ds \\ &\leq \epsilon + \left(k_\tau + \frac{L_I}{\Gamma(\alpha+1)}\right)|\psi(t) - v(t)| \\ &\leq \frac{\epsilon}{1-\Upsilon}.\end{aligned}$$

Since, $\Upsilon \neq 1$ and we assume that $1 - \left(k_\tau + \frac{L_I}{\Gamma(\alpha+1)}\right) = c_I$, then one can get

$$\|\psi(t) - v(t)| \leq c_I \epsilon, \quad c_I \neq 0.$$

Thus, the fractional order boundary value problem (1) Hyer-Ulam stable. □

Remark 3.8. *Following in the same lines as in the preceding theorem it is not difficult to provide the proof of the fact that the system* (1) *is generalized Hyer-Ulam stable.*

Example 3.9.

$$\begin{cases} {}^c\mathfrak{D}^{\frac{3}{2}}\psi(s) = \dfrac{e^{-5t}}{40}\cos|\psi(s)|, \ 0 \leq s \leq 1, \\ \psi(0) = 1, \ \psi(1) = \displaystyle\sum_{i=1}^{10} \frac{1}{3}\psi(s_i). \end{cases} \tag{19}$$

From (19), *we see that* $k_\tau = C_\tau = \frac{1}{3}, C_I = \frac{1}{40} = L_I, Z_1 = Z_2 = 0, \xi = \frac{3}{2}$. *Then in light of Theorem 3.6 one get* $k_\tau + \frac{L_I}{\Gamma(\xi+1)} = \frac{5\sqrt{\pi}+1}{15\sqrt{\pi}} < 1$. *Hence the solution of* (19) *is unique. It is not difficult to show that solution's set is contained in a ball which shows the validity of Theorem 3.5. Thus, under the conditions of Theorem 3.7, boundary value problem* (3.9) *is stable in the sense of Hyers- Ulam definition as well as stable in the sense of Generalized Hyers-Ulam definition.*

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