Filomat 34:2 (2020), 391–398
https://doi.org/10.2298/FIL2002391I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Lorentz-Marcinkiewicz Property of Direct Sum of Operators

Pembe Ipek Al[a]

[a] *Department of Mathematics, Karadeniz Technical University, 61080, Trabzon, Turkey*

Abstract. In this paper, the relations between Lorentz-Marcinkiewicz property of the direct sum of operators in the direct sum of Hilbert spaces and its coordinate operators are studied. Then, the obtained results are supported by applications.

1. Introduction

The general theory of singular numbers and operator ideals was given by Pietsch [14], [15] and the case of linear compact operators was investigated by Gohberg and Krein [6]. However, the first result in this area can be found in the works of Schmidt [16] and von Neumann, Schatten [17] who used these concepts in the theory of non-selfadjoint integral equations.

Later on, the main aim of mini-workshop held in Oberwolfach (Germany) was to present and discuss some modern applications of the functional-analytic concepts of s−numbers and operator ideals in areas like numerical analysis, theory of function spaces, signal processing, approximation theory, probability of Banach spaces and statistical learning theory (see [4]).

Let $\mathcal{H}$ be a Hilbert space, $S_\infty(\mathcal{H})$ be the class of linear compact operators in $\mathcal{H}$ and $s_n(T)$ be the $n-th$ singular numbers of the operator $T \in S_\infty(\mathcal{H})$. In [2] and [3], the Lorentz-Marcinkiewicz operator ideals are defined as

$$S_{\varphi,q}(\mathcal{H}) = \left\{ T \in S_\infty(\mathcal{H}) : \sum_{n=1}^{\infty} \varphi^q(n) n^{-1} s_n^q(T) < \infty \right\},\ 0 < q < \infty$$

and

$$S_{\varphi,\infty}(\mathcal{H}) = \left\{ T \in S_\infty(\mathcal{H}) : \sup_{n \geq 1} \varphi(n) s_n(T) < \infty \right\}$$

where, $\varphi : (0,\infty) \to (0,\infty),\ \varphi \in C(0,\infty),\ \varphi(1) = 1$ and $\widetilde{\varphi}(t) = \sup\limits_{s>0} \dfrac{\varphi(ts)}{\varphi(s)} < \infty,\ \widetilde{\varphi} : (0,\infty) \to \mathbb{R}$.

The infinite direct sum of Hilbert spaces and the infinite direct sum of operators have been studied in [5]. Namely, the infinite direct sum of Hilbert spaces $H_n,\ n \geq 1$ and the infinite direct sum of operators A_n in $H_n,\ n \geq 1$ are defined as

$$H = \bigoplus_{n=1}^{\infty} H_n = \left\{ u = (u_n) : u_n \in H_n,\ n \geq 1,\ \sum_{n=1}^{\infty} \|u_n\|_{H_n}^2 < +\infty \right\},$$

2010 *Mathematics Subject Classification*. Primary 47A05; Secondary 47A10
Keywords. direct sum of Hilbert spaces and operators, compact operators, Lorentz-Marcinkiewicz operator classes
Received: 25 October 2019; Accepted: 04 June 2020
Communicated by Miodrag M. Spalević
Email address: ipekpembe@gmail.com (Pembe Ipek Al)

$$A = \bigoplus_{n=1}^{\infty} A_n,$$
$$D(A) = \{u = (u_n) \in H : u_n \in D(A_n),\ n \geq 1,\ Au = (A_n u_n) \in H\}.$$

Recall that H is a Hilbert space with the norm induced by the inner product

$$(u, v)_H = \sum_{n=1}^{\infty} (u_n, v_n)_{H_n},\ u, v \in H.$$

Our aim in this paper is to study the relations between Lorentz-Marcinkiewicz property of the direct sum of operators in the direct sum of Hilbert spaces and Lorentz-Marcinkiewicz property of its coordinate operators.

It should be noted that the analogous problems in special cases have been investigated in [9].

The problem of belonging to Schatten-von Neuman classes of the resolvent operators of the normal extensions of the minimal operator generated by the direct sum of differential-operator expression for first order with suitable operator coefficients in the direct sum of Hilbert spaces of Hilbert-valued vector functions in finite interval has been studied in [8].

In [7] and [10], the same problem for normal and hyponormal extensions of the minimal operators generated by corresponding differential-operator expressions under some conditions to operator coefficients in a finite interval has been investigated.

Later on, some more general Schatten-von Neumann classes of the compact operators in Hilbert spaces have been defined and characterized in [11] in terms of Berezin symbols. In [1], the question raised by Nordgren and Rosenthal about the Schatten-von Neumann class membership of operators in standard reproducing kernel Hilbert spaces in terms of their Berezin symbols has been answered.

2. Lorentz-Marcinkiewicz property of direct sum of operators

Let H_n be a Hilbert space, $A_n \in L(H_n)$ for $n \geq 1$ and

$$H = \bigoplus_{n=1}^{\infty} H_n,\ A = \bigoplus_{n=1}^{\infty} A_n.$$

Recall that, in order to $A \in L(H)$ the necessary and sufficient condition is $\sup\limits_{n\geq 1} \|A_n\| < \infty$. Moreover, $\|A\| = \sup\limits_{n\geq 1} \|A_n\|$ (see [12]).

It is known that if $A_n \in S_\infty(H_n)$ for $n \geq 1$, then the necessary and sufficient condition for $A \in S_\infty(H)$ is $\lim\limits_{n\to\infty} \|A_n\| = 0$ (see [13]).

The following result on singular numbers of the operator $A \in S_\infty(H)$

$$\{s_m(A) : m \geq 1\} = \bigcup_{n=1}^{\infty} \{s_m(A_n) : m \geq 1\}$$

can be found in [9].

Throughout this paper for the simplicity we assume that:

(1) for any $n,\ k \geq 1$ with $n \neq k$, $\{s_m(A_n) : m \geq 1\} \cap \{s_m(A_k) : m \geq 1\} = \emptyset$ or $\{0\}$;

(2) for any $n \geq 1$ in the sequence $(s_m(A_n))$, if for some $k > 1,\ s_k(A_n) > 0$, then $s_k(A_n) < s_{k-1}(A_n)$.

Note that the following proposition is true.

Proposition 2.1. *For $n \geq 1$ there is a strongly increasing sequence $k_m^{(n)} : \mathbb{N} \to \mathbb{N}$ such that $s_{k_m^{(n)}}(A) = s_m(A_n)$ holds for $m \geq 1$ and $\bigcup\limits_{n=1}^{\infty} \bigcup\limits_{m=1}^{\infty} \left\{k_m^{(n)}\right\} = \mathbb{N}$. Moreover, it is clear that $k_m^{(n)} \geq m$ for any $n, m \geq 1$.*

Indeed, in the Hilbert space $H = \bigoplus_{n=1}^{\infty} H_n = l_2(\mathbb{R}),\ H_n = (\mathbb{R}, |\cdot|)$ consider the following infinite matrices with reel entries in forms

$$A = \begin{pmatrix} a_1 & & & & & \\ & a_2 & & & & \\ & & a_3 & & 0 & \\ & & & \ddots & & \\ & 0 & & & a_n & \\ & & & & & \ddots \end{pmatrix} : H \to H$$

and

$$B = \begin{pmatrix} b_1 & & & & & \\ & b_2 & & & & \\ & & b_3 & & 0 & \\ & & & \ddots & & \\ & 0 & & & b_n & \\ & & & & & \ddots \end{pmatrix} : H \to H,$$

where for any $n,\ m \geq 1,\ n \neq m,\ a_n \neq a_m,\ a_n > 0$ and $b_n = \dfrac{a_n + a_{n+1}}{2}$ with property $\lim_{n\to\infty} a_n = 0$.

In this case, $A,\ B \in S_\infty(H)$ and the singular numbers of the operators $A,\ B$ are given in the following forms

$$\{s_m(A_n) : m \geq 1\} = \{a_n : n \geq 1\},$$
$$\{s_m(B_n) : m \geq 1\} = \{b_n : n \geq 1\},$$

respectively. Then, by [13] it implies that $T = A \oplus B \in S_\infty(H \oplus H)$ and $\{s_m(T) : m \geq 1\} = \{a_n, b_n : n \geq 1\}$. In this case, we have

$$k_m^{(1)} = 2m - 1, m \geq 1,$$
$$k_m^{(2)} = 2m, m \geq 1.$$

Firstly, let prove the following theorem.

Theorem 2.2. *Let $0 < q < \infty$. $A \in S_{\varphi,q}(H)$ if and only if the series*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varphi^q \left(k_m^{(n)}\right) \left(k_m^{(n)}\right)^{-1} s_m^q(A_n)$$

is convergent.

Proof. Let $A \in S_{\varphi,q}(H)$. This means that the series

$$\sum_{m=1}^{\infty} \varphi^q(m) m^{-1} s_m^q(A)$$

is convergent. From the structure of the set of the singular numbers of the operator A and the important theorem on the convergent of the rearrangement series it is obtained that the series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varphi^q \left(k_m^{(n)}\right) \left(k_m^{(n)}\right)^{-1} s_m^q(A_n)$$

is convergent.

Conversely, if the series in the theorem is convergent, then the series

$\sum\limits_{m=1}^{\infty} \varphi^q(m)m^{-1}s_m^q(A)$, which is the rearrangement of the above series, is convergent. So, $A \in S_{\varphi,q}(H)$.

Now, in Theorem 2.3-2.5, we will investigate the problem of belonging to Lorentz-Marcinkiewicz operator classes of its coordinate operators, if the direct sum of operators belongs to Lorentz-Marcinkiewicz operator classes.

Theorem 2.3. *Let $A \in S_\infty(H)$, $\psi(t) = t^{-1/q}\varphi(t) : (0,\infty) \to (0,\infty)$ be a monotone increasing function and $0 < q < \infty$. If $A \in S_{\varphi,q}(H)$, then $A_n \in S_{\varphi,q}(H_n)$ for $n \geq 1$.*

Proof. Under the assumptions in the theorem, we get

$$\begin{aligned}
\sum_{m=1}^{\infty} \varphi^q(m)m^{-1}s_m^q(A) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \varphi^q\Big(k_m^{(n)}\Big)\Big(k_m^{(n)}\Big)^{-1} s^q_{\left(k_m^{(n)}\right)}(A) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \varphi^q\Big(k_m^{(n)}\Big)\Big(k_m^{(n)}\Big)^{-1} s_m^q(A_n) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\varphi^q\Big(k_m^{(n)}\Big)}{\Big(k_m^{(n)}\Big)} \frac{m}{\varphi^q(m)} \varphi^q(m)m^{-1}s_m^q(A_n) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\psi^q\left(k_m^{(n)}\right)}{\psi^q(m)} \varphi^q(m)m^{-1}s_m^q(A_n) \\
&\geq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \varphi^q(m)m^{-1}s_m^q(A_n) \\
&\geq \sum_{m=1}^{\infty} \varphi^q(m)m^{-1}s_m^q(A_n)
\end{aligned}$$

Therefore, $A_n \in S_{\varphi,q}(H_n)$ for $n \geq 1$.

Theorem 2.4. *Let $A \in S_\infty(H)$, $\varphi(t) : (0,\infty) \to (0,\infty)$ be a monotone increasing function, $\sup\limits_{m\geq 1}\left(\frac{k_m^{(n)}}{m}\right) \leq \gamma < \infty$ for $n \geq 1$ and $0 < q < \infty$. If $A \in S_{\varphi,q}(H)$, then $A_n \in S_{\varphi,q}(H_n)$ for $n \geq 1$.*

Proof. Under the assumptions in the theorem, we have

$$\begin{aligned}
\sum_{m=1}^{\infty} \varphi^q(m)m^{-1}s_m^q(A_n) &= \sum_{m=1}^{\infty} \frac{\varphi^q(m)}{\varphi^q\left(k_m^{(n)}\right)} \frac{\left(k_m^{(n)}\right)}{m} \varphi^q\left(k_m^{(n)}\right)\left(k_m^{(n)}\right)^{-1} s_m^q(A_n) \\
&\leq \sup_{m\geq 1}\left(\frac{k_m^{(n)}}{m}\right) \sum_{m=1}^{\infty} \varphi^q\left(k_m^{(n)}\right)\left(k_m^{(n)}\right)^{-1} s_m^q(A_n) \\
&\leq \gamma \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \varphi^q\left(k_m^{(n)}\right)\left(k_m^{(n)}\right)^{-1} s^q_{k_m^{(n)}}(A) \\
&= \gamma \sum_{j=1}^{\infty} \varphi^q(j)j^{-1}s_j^q(A) < \infty.
\end{aligned}$$

Therefore, $A_n \in S_{\varphi,q}(H_n)$ for $n \geq 1$.

Now, we will investigate in the case of $q = \infty$.

Theorem 2.5. *Let $\varphi : (0,\infty) \to (0,\infty)$ be a monotone increasing function. If $A \in S_{\varphi,\infty}(H)$, then $A_n \in S_{\varphi,\infty}(H_n)$ for $n \geq 1$.*

Proof. Since $A \in S_{\varphi,\infty}(H)$, we have $\sup\limits_{m\geq 1} \varphi(m)s_m(A) < \infty$. Hence,

$$\sup_{m\geq 1} \varphi\left(k_m^{(n)}\right) s_m(A_n) < \infty.$$

On the other hand, one can easily check that

$$\begin{aligned}
\sup_{m\geq 1} \varphi(m)s_m(A_n) &= \sup_{m\geq 1} \varphi(m)\frac{\varphi\left(k_m^{(n)}\right)}{\varphi\left(k_m^{(n)}\right)} s_m(A_n) \\
&\leq \sup_{m\geq 1} \frac{\varphi(m)}{\varphi\left(k_m^{(n)}\right)} \sup_{m\geq 1} \varphi\left(k_m^{(n)}\right) s_m(A_n) \\
&\leq \sup_{m\geq 1} \varphi\left(k_m^{(n)}\right) s_m(A_n) \\
&= \sup_{m\geq 1} \varphi\left(k_m^{(n)}\right) s_{k_m^{(n)}}(A) < \infty.
\end{aligned}$$

Then, $A_n \in S_{\varphi,\infty}(H_n)$ for $n \geq 1$.

Now, in Theorem 2.6-2.8, we will investigate the problem of belonging to Lorentz-Marcinkiewicz operator classes of the direct sum of operators, if its coordinate operators belong to Lorentz-Marcinkiewicz operator classes.

Theorem 2.6. *Let $\varphi(t) : (0, \infty) \to (0, \infty)$ be a monotone decreasing function, the series $\sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty} \varphi^q(m)m^{-1}s_m^q(A_n)$ be convergent and $0 < q < \infty$. If $A_n \in S_{\varphi,q}(H_n)$ for $n \geq 1$, then $A \in S_{\varphi,q}(H)$.*

Proof. The validity of this claim is clear from the following inequality

$$\begin{aligned}
\sum_{m=1}^{\infty} \varphi^q(m)m^{-1}s_m^q(A) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \varphi^q\left(k_m^{(n)}\right)\left(k_m^{(n)}\right)^{-1} s^q_{k_m^{(n)}}(A) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\varphi^q\left(k_m^{(n)}\right)}{\varphi^q(m)} \frac{m}{\left(k_m^{(n)}\right)} \varphi^q(m)m^{-1}s_m^q(A_n) \\
&\leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \varphi^q(m)m^{-1}s_m^q(A_n) < \infty.
\end{aligned}$$

Theorem 2.7. *Let $\sup\limits_{m\geq 1} \dfrac{\varphi^q\left(k_m^{(n)}\right)}{\varphi^q(m)} \leq \gamma_n < \infty$, $\sum\limits_{m=1}^{\infty} \varphi^q(m)m^{-1}s_m^q(A_n) \leq \beta_n$ for $n \geq 1$, $\sum\limits_{n=1}^{\infty} \gamma_n\beta_n$ be convergent and $0 < q < \infty$. If $A_n \in S_{\varphi,q}(H_n)$ for $n \geq 1$, then $A \in S_{\varphi,q}(H)$.*

Proof. Under the assumptions in the theorem, we have

$$\begin{aligned}
\sum_{m=1}^{\infty} \varphi^q(m) m^{-1} s_m^q(A) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \varphi^q\left(k_m^{(n)}\right)\left(k_m^{(n)}\right)^{-1} s^q_{k_m^{(n)}}(A) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\varphi^q\left(k_m^{(n)}\right)}{\varphi^q(m)} \frac{m}{\left(k_m^{(n)}\right)} \varphi^q(m) m^{-1} s^q_{k_m^{(n)}}(A) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\varphi^q\left(k_m^{(n)}\right)}{\varphi^q(m)} \frac{m}{\left(k_m^{(n)}\right)} \varphi^q(m) m^{-1} s^q_{m}(A_n) \\
&\leq \sum_{n=1}^{\infty} \gamma_n \sum_{m=1}^{\infty} \varphi^q(m) m^{-1} s_m^q(A_n) \\
&\leq \sum_{n=1}^{\infty} \gamma_n \beta_n < \infty.
\end{aligned}$$

Therefore, $A \in S_{\varphi,q}(H)$.

Now, we will investigate in the case of $q = \infty$.

Theorem 2.8. *Let* $\alpha_n = \sup_{m\geq 1} \varphi(m) s_m(A_n) < \infty$, $\gamma_n = \sup_{m\geq 1} \dfrac{\varphi\left(k_m^{(n)}\right)}{\varphi(m)}$ *for* $n \geq 1$ *and* $\sup_{n\geq 1} \gamma_n \alpha_n < \infty$. *If* $A_n \in S_{\varphi,\infty}(H_n)$ *for* $n \geq 1$, *then* $A \in S_{\varphi,\infty}(H)$.

Proof. From $\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\left\{k_m^{(n)}\right\} = \mathbb{N}$, we get

$$\begin{aligned}
\sup_{m\geq 1} \varphi(m) s_m(A) &= \sup_{n,m\geq 1} \varphi\left(k_m^{(n)}\right) s_{k_m^{(n)}}(A) \\
&= \sup_{n,m\geq 1} \varphi\left(k_m^{(n)}\right) s_m(A_n) \\
&\leq \sup_{n\geq 1}\left(\sup_{m\geq 1}\left(\frac{\varphi\left(k_m^{(n)}\right)}{\varphi(m)}\right)\sup_{m\geq 1}\left(\varphi(m) s_m(A_n)\right)\right) \\
&= \sup_{n\geq 1} \alpha_n \gamma_n < \infty.
\end{aligned}$$

Then, $A \in S_{p,\infty}(H)$.

Remark 2.9. *Using this method, the analogous researches for the following operators*

$$B = \begin{pmatrix} 0 & B_1 & & & & & \\ & 0 & B_2 & & & & \\ & & 0 & B_3 & & 0 & \\ & & & \ddots & \ddots & & \\ & 0 & & & 0 & B_n & \\ & & & & & \ddots & \ddots \end{pmatrix} : H = \bigoplus_{n=1}^{\infty} H_n \to H$$

and

$$C = \begin{pmatrix} 0 & & & & & \\ C_1 & 0 & & & & \\ & C_2 & 0 & & & 0 \\ & & \ddots & \ddots & & \\ & 0 & & C_n & 0 & \\ & & & & \ddots & \ddots \end{pmatrix} : H = \bigoplus_{n=1}^{\infty} H_n \to H$$

can be studied.

3. Examples

In this section, we provide some examples as applications of our theorems.

Example 3.1 In the Hilbert space $H = \bigoplus_{n=1}^{\infty} H_n = l_2(\mathbb{C}),\ H_n := (\mathbb{C}, |\cdot|),\ n \geq 1$, consider the following diagonal infinite matrix with complex entries

$$A = \begin{pmatrix} a_1 & & & & & \\ & a_2 & & & & \\ & & a_3 & & 0 & \\ & & & \ddots & & \\ & 0 & & & a_n & \\ & & & & & \ddots \end{pmatrix} : H \to H$$

under the condition $|a_n| < r < 1,\ n \geq 1$. Then, $\lim_{n\to\infty} a_n = 0$. Thus, $A \in S_\infty(H)$. If we define

$$A_n := a_n I \text{ for } n \geq 1,$$

then

$$s_m(A_n) = |\lambda(A_n)| = \{|a_n|,\ 0\},\ m \geq 1,$$

where I is the identity operator in the corresponding Hilbert space. Hence, the singular numbers of the operator A are given as

$$\{s_m(A) : m \geq 1\} = \{|a_n| : n \geq 1\}.$$

Moreover, for $n \geq 1$, $0 < q < \infty$ and for any function φ in the definition Lorentz-Marcinkiewicz we get

$$\sum_{m=1}^{\infty} \varphi^q(m) m^{-1} s_m^q(A_n) = \sum_{m=1}^{\infty} \varphi^q(m) m^{-1} |a_n|^q = |a_n|^q.$$

Then $A_n \in S_{\varphi,q}(H_n),\ n \geq 1,\ 0 < q < \infty$. Therefore, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \varphi^q(m) m^{-1} s_m^q(A_n) = \sum_{n=1}^{\infty} |a_n|^q < \infty.$$

Hence, by Theorem 2.6, $A \in S_{\varphi,q}(H)$.

Example 3.2 Let $H_n := (\mathbb{C}^2, |\cdot|_2),\ H := \bigoplus_{n=1}^{\infty} H_n = l_2(\mathbb{C}^2),\ A_n = \begin{pmatrix} 0 & \alpha^{2n-1} \\ \alpha^{2n} & 0 \end{pmatrix}$ for $n \geq 1$, $0 < |\alpha| < 1$ and $A = \bigoplus_{n=1}^{\infty} A_n : H \to H$. Then $A \in S_\infty(H)$ (see [13]).

For $n \geq 1$ we get

$$\|A_n\| = |\alpha|^{2n-1},$$

$$\{s_m(A_n) : m \geq 1\} = \{|\alpha|^{2n-1},\ |\alpha|^{2n}\}$$

and

$$\{s_m(A) : m \geq 1\} = \{|\alpha|^n :\ n \geq 1\}.$$

Moreover, for $n \geq 1$ and $0 < q < \infty$ we obtain

$$\sum_{m=1}^{\infty} \varphi^q(m) m^{-1} s_m^q(A_n) = |\alpha|^{(2n-1)q} + \varphi^q(2) 2^{-1} |\alpha|^{2nq} < \infty.$$

Thus, $A_n \in S_{\varphi,q}(H_n),\ n \geq 1,\ 0 < q < \infty$. Therefore, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varphi^q(m) m^{-1} s_m^q(A_n) &= \sum_{n=1}^{\infty} \left(|\alpha|^{(2n-1)q} + \varphi^q(2) 2^{-1} |\alpha|^{2nq} \right) \\ &= \frac{|\alpha|^q}{1 - |\alpha|^{2q}} \left(1 + \varphi^q(2) 2^{-1} |\alpha|^q \right) < \infty. \end{aligned}$$

Hence, by Theorem 2.6, $A \in S_{\varphi,q}(H)$.

Acknowledgment. This article is dedicated to Professor Gradimir V. Milovanovic on the occasion of his 70th anniversary.

References

[1] I. Chalendar, E. Fricain, M. Gürdal, M. T. Karaev, Compactness and Berezin symbols, Acta Universitatis Szegediensis 78(1) (2012) 315–329.
[2] F. Cobos, On the Lorentz-Marcinkiewicz, Mathematische Nachrichten 126 (1986) 281–300.
[3] F. Cobos, Duality and Lorentz-Marcinkiewicz operator spaces, Mathematica Scandinavica 63 (1988) 261–267.
[4] F. Cobos, D. D. Haroske, T. Kühn, T. Ullrich, Mini-workshop: modern applications of s-numbers and operator ideals, Mathematisches Forschungs Institute Oberwolfach, Oberwolfach, Germany, February 8–February 14, 2015, 369–397.
[5] N. Dunford, J. T. Schwartz, Linear Operators I, Interscience Publishers, New York, 1958.
[6] I. C. Gohberg, M. G. Krein, Introduction to the Theory of Linear Non-Selfadjoint Operators in Hilbert Space, American Mathematical Society, Rhode Island, 1969.
[7] Z. I. Ismailov, Compact inverses of first-order normal differential operators, Journal of Mathematical Analysis and Applications 320 (2006) 266–278.
[8] Z. I. Ismailov, Multipoint normal differential operators for first order, Opuscula Mathematica 29 (2009) 399–414.
[9] Z. I. Ismailov, E. Otkun Çevik, E. Unluyol, Compact inverses of multipoint normal differential operators for first order. Electronic Journal of Differential Equations 89 (2011) 1–11.
[10] Z. I. Ismailov, E. Unluyol, Hyponormal differential operators with discrete spectrum, Opuscula Mathematica 30 (2010) 79–94.
[11] M. T. Karaev, M. Gürdal, U. Yamancı, Special operator classes and their properties, Banach Journal of Mathematical Analysis 7(2) (2013) 74–88.
[12] M. A. Naimark, S. V. Fomin, Continuous direct sums of Hilbert spaces and some of their applications, Uspekhi Matematicheskikh Nauk 10 (1955) 111–142 (article in Russian).
[13] E. Otkun Çevik, Z. I. Ismailov, Spectrum of the direct sum of operators, Electronic Journal of Differential Equations 210 (2012) 1–8.
[14] A. Pietsch, Operators Ideals, North-Holland Publishing Company, Amsterdam, 1980.
[15] A. Pietsch, Eigenvalues and s−Numbers, Cambridge University Press, Londan, 1987.
[16] E. Schmidt, Zur theorie der linearen und nichtlinearen integralgleichungen, Mathematische Annalen 64(2) (1907) 433–476.
[17] J. von Neumann, R. Schatten, The cross-space of linear transformations, Mathematische Annalen 47 (1946) 608–630.