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Steffensen Type Inequalities for Anti-Symmetrized Monotone Functions

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Abstract. Several Steffensen type inequalities are derived for anti-symmetrized monotone functions, i. e. functions f which anti-symmetrical transform $\tilde{f}(x) = \frac{1}{2}\left[f(x) - f(a+b-x)\right]$, $x \in [a,b]$, is monotone.

1. Introduction

In 1918 Steffensen [6] proved the following result:

Theorem 1.1. *Let f and g be integrable functions defined on the interval $[a,b]$ such that f is nonincreasing and $0 \leq g \leq 1$ and denote $\lambda = \int_a^b g(t)\,dt$. Then we have*

$$\int_{b-\lambda}^{b} f(t)\,dt \leq \int_a^b f(t)g(t)\,dt \leq \int_a^{a+\lambda} f(t)\,dt, \tag{1}$$

with the inequalities reversed if f is nondecreasing.

Steffensen published his result in an actuarial journal and the inequalities went mostly unnoticed for the first few decades. However, since then the inequality has been extensively studied, extended, refined and applied in many areas. For a recent, comprehensive presentation of the research related to the Steffensen inequality one can see the monograph [5].

Milovanović and Pečarić [2] weakened the assumptions on the function g and showed that these assumptions are sufficient and necessary, i. e. the weakest assumption under which inequality (1) holds.

Theorem 1.2. *Let f and g be integrable functions defined on the interval $[a,b]$ and let $\lambda = \int_a^b g(t)\,dt$.*

a) The second inequality in (1) holds for every nonincreasing function f if and only if

$$\int_a^x g(t)\,dt \leq x - a \text{ and } \int_x^b g(t)\,dt \geq 0 \text{ for every } x \in [a,b]. \tag{2}$$

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b) The first inequality in (1) holds for every nonincreasing function f if and only if

$$\int_x^b g(t)\,dt \leq b - x \text{ and } \int_a^x g(t)\,dt \geq 0 \text{ for every } x \in [a,b]. \tag{3}$$

The following result, due to Pečarić [3], extends the Steffensen inequality by introducing one more function.

Theorem 1.3. *Let f, g and h be integrable functions defined on the interval $[a,b]$ such that h is positive, the function $x \mapsto \frac{f(x)}{h(x)}$ is nonincreasing and $0 \leq g \leq 1$. Then*

$$\int_a^b f(t)g(t)\,dt \leq \int_a^{a+\lambda} f(t)\,dt, \tag{4}$$

where λ is the solution of the equation

$$\int_a^{a+\lambda} h(t)\,dt = \int_a^b h(t)g(t)\,dt. \tag{5}$$

If the function $x \mapsto \frac{f(x)}{h(x)}$ is nondecreasing, then the inequality in (4) is reversed.

Applying Theorem 1.3 with $h = 1$ one recovers the second inequality in (1). Analogous extension of the first inequality can be derived from Theorem 1.3 with substitutions $g(x) \to 1 - g(x)$ and $\lambda \to a + b - \lambda$. For alternative proofs, interpretations and extensions of these results see also [1, 4].

In this paper we will obtain several Steffensen type inequalities for functions which anti-symmetrical transform $\tilde{f}(x) = \frac{1}{2}\left[f(x) - f(a+b-x)\right]$, $x \in [a,b]$, is monotone. The anti-symmetrical transform of a monotone function is again monotone, while the reverse implication does not need to hold.

2. Main results

The *symmetrical transform of a function f on the interval* $[a,b]$, denoted by $\check{f}_{[a,b]}$ or simply by $\check{f}$ if the interval $[a,b]$ is clear from the context, is defined by

$$\check{f}(x) = \frac{1}{2}\left[f(x) + f(a+b-x)\right], \quad \text{for } x \in [a,b].$$

Similarly, the *anti-symmetrical transform of f on the interval* $[a,b]$, denoted by $\tilde{f}_{[a,b]}$ or simply by $\tilde{f}$, is defined by

$$\tilde{f}(x) = \frac{1}{2}\left[f(x) - f(a+b-x)\right], \quad \text{for } x \in [a,b].$$

For every function one has $f = \check{f} + \tilde{f}$.

Definition 2.1. *We say that a function $f : [a,b] \to \mathbb{R}$ is anti-symmetrized nonincreasing (resp. nondecreasing) if its anti-symmetrical transform $\tilde{f}$ is nonincreasing (resp. nondecreasing).*

Obviously, if f is a monotone function, then its anti-symmetrical transform $\tilde{f}$, as a sum of two such functions, is monotone in the same direction. On the other hand, there are functions which are not monotone, yet their anti-symmetrical transform is. For example, let $f_0 : [a,b] \to \mathbb{R}$ be given by $f_0(x) = x^2$. Then

$$\tilde{f}_0(x) = \frac{1}{2}\left[x^2 - (a+b-x)^2\right] = (a+b)x - \frac{(a+b)^2}{2},$$

so if a and b are such that $a < 0 < b$ and $b - a > 0$, the function f_0 is not monotone, while $\tilde{f}_0$ is increasing.

We are going to prove several Steffensen type inequalities for anti-symmetrized monotone functions. The first result corresponds to the second inequality in (1) under the Milovanović-Pečarić's assumptions.

Theorem 2.2. *Let f and g be integrable functions defined on the interval $[a,b]$ such that f is anti-symmetrized nonincreasing and g satisfies (2) with $\lambda = \int_a^b g(t)\,dt$. Then*

$$2\int_a^b f(t)\tilde{g}(t)\,dt \le \int_a^{a+\lambda} f(t)\,dt - \int_{b-\lambda}^b f(t)\,dt, \tag{6}$$

with the inequality reversed if f is anti-symmetrized nondecreasing.

Proof. Applying the Steffensen inequality (1) for the nonincreasing function $\tilde{f}$ we get

$$\int_a^b \tilde{f}(t)g(t)\,dt \le \int_a^{a+\lambda} \tilde{f}(t)\,dt. \tag{7}$$

The left hand side of (7) is equal to

$$\begin{aligned}\int_a^b \tilde{f}(t)g(t)\,dt &= \frac{1}{2}\int_a^b f(t)g(t)\,dt - \frac{1}{2}\int_a^b f(a+b-t)g(t)\,dt \\ &= \frac{1}{2}\int_a^b f(t)g(t)\,dt - \frac{1}{2}\int_a^b f(y)g(a+b-y)\,dy = \int_a^b f(t)\tilde{g}(t)\,dt, \end{aligned}\tag{8}$$

while the right hand side is equal to

$$\int_a^{a+\lambda} \tilde{f}(t)\,dt = \frac{1}{2}\int_a^{a+\lambda} f(t)\,dt - \frac{1}{2}\int_a^{a+\lambda} f(a+b-t)\,dt = \frac{1}{2}\int_a^{a+\lambda} f(t)\,dt - \frac{1}{2}\int_{b-\lambda}^b f(y)\,dy. \tag{9}$$

Inserting these two identities in (7) we obtain the required inequality. □

Analogously we can obtain the corresponding result for the first inequality in (1).

Theorem 2.3. *Let f and g be integrable functions defined on the interval $[a,b]$ such that f is anti-symmetrized nonincreasing and g satisfies (3) with $\lambda = \int_a^b g(t)\,dt$. Then*

$$\int_{b-\lambda}^b f(t)\,dt - \int_a^{a+\lambda} f(t)\,dt \le 2\int_a^b f(t)\tilde{g}(t)\,dt, \tag{10}$$

with the inequality reversed if f is anti-symmetrized nondecreasing.

We can combine Theorems 2.2 and 2.3 in the following corollary.

Corollary 2.4. *Let f and g be integrable functions defined on the interval $[a,b]$ such that f is anti-symmetrized monotone and g satisfies (2) and (3) with $\lambda = \int_a^b g(t)\,dt$. Then*

$$2\left|\int_a^b f(t)\tilde{g}(t)\,dt\right| \le \left|\int_a^{a+\lambda} f(t)\,dt - \int_{b-\lambda}^b f(t)\,dt\right|.$$

We are going to prove two versions of Theorem 1.3 for anti-symmetrized monotone functions. The first one is given in the next theorem.

Theorem 2.5. *Let f, g and h be integrable functions defined on the interval $[a,b]$ such that h is positive, the function $x \mapsto \frac{\tilde{f}(x)}{h(x)}$ is nonincreasing and $0 \le g \le 1$. Then inequality (6) holds, where λ is the solution of the equation (5).*

If the function $x \mapsto \frac{\tilde{f}(x)}{h(x)}$ is nondecreasing, then the reversed inequality in (6) holds.

Proof. Due to (8), (9) and the assumptions of the theorem we have

$$\begin{aligned}
&\frac{1}{2}\int_a^{a+\lambda} f(t)\,dt - \frac{1}{2}\int_{b-\lambda}^{b} f(t)\,dt - \int_a^b f(t)\tilde{g}(t)\,dt \\
&\quad= \int_a^{a+\lambda} \tilde{f}(t)\,dt - \int_a^b \tilde{f}(t)g(t)\,dt \\
&\quad= \int_a^{a+\lambda} \tilde{f}(t)\left[1-g(t)\right]\,dt - \int_{a+\lambda}^{b} \tilde{f}(t)g(t)\,dt \\
&\quad\geq \frac{\tilde{f}(a+\lambda)}{h(a+\lambda)}\int_a^{a+\lambda} h(t)\left[1-g(t)\right]\,dt - \int_{a+\lambda}^{b} \tilde{f}(t)g(t)\,dt \\
&\quad= \frac{\tilde{f}(a+\lambda)}{h(a+\lambda)}\left[\int_a^b h(t)g(t)\,dt - \int_a^{a+\lambda} h(t)g(t)\,dt\right] - \int_{a+\lambda}^{b} \frac{\tilde{f}(t)}{h(t)}h(t)g(t)\,dt \\
&\quad= \int_{a+\lambda}^{b} h(t)g(t)\left[\frac{\tilde{f}(a+\lambda)}{h(a+\lambda)} - \frac{\tilde{f}(t)}{h(t)}\right]dt \geq 0.
\end{aligned}$$

□

The second extension of Theorem 1.3 to the anti-symmetrized monotone functions is given in the next theorem.

Theorem 2.6. *Let f, g and h be integrable functions defined on the interval $[a, b]$ such that h is positive, the function $z(x) = \frac{f(x)}{h(x)}$ is anti-symmetrized nonincreasing, $0 \leq g \leq 1$ and λ is the solution of the equation (5). Then*

$$2\int_a^b \tilde{z}(t)h(t)g(t)\,dt \leq \int_a^{a+\lambda} f(t)\,dt - \int_{b-\lambda}^{b} f(t)\frac{h(t)}{h(a+b-t)}\,dt. \tag{11}$$

If the function z is anti-symmetrized nondecreasing, then the reversed inequality in (11) holds.

Proof. We have

$$\begin{aligned}
\int_a^{a+\lambda} f(t)\,dt - \int_a^b z(t)h(t)g(t)\,dt &= \int_a^{a+\lambda} z(t)h(t)\,dt - \int_a^b z(t)h(t)g(t)\,dt \\
&= \int_a^{a+\lambda} z(t)h(t)\left[1-g(t)\right]\,dt - \int_{a+\lambda}^{b} z(t)h(t)g(t)\,dt
\end{aligned} \tag{12}$$

Since $\tilde{z}$ is nonincreasing we have

$$z(t) - z(a+b-t) = 2\tilde{z}(t) \geq 2\tilde{z}(a+\lambda) \tag{13}$$

for $t \le a+\lambda$, while the reverse inequality holds for $t \ge a+\lambda$, so

$$\begin{aligned}
\int_a^{a+\lambda} z(t)h(t)\left[1-g(t)\right] dt - \int_{a+\lambda}^b z(t)h(t)g(t)\,dt \ge 2\tilde{z}(a+\lambda)\int_a^{a+\lambda} h(t)\left[1-g(t)\right] dt \\
+ \int_a^{a+\lambda} z(a+b-t)h(t)\left[1-g(t)\right] dt - \int_{a+\lambda}^b z(t)h(t)g(t)\,dt \\
= 2\tilde{z}(a+\lambda)\left[\int_a^b h(t)g(t)\,dt - \int_a^{a+\lambda} h(t)g(t)\,dt\right] \\
- \int_{a+\lambda}^b \left[2\tilde{z}(t) + z(a+b-t)\right] h(t)g(t)\,dt + \int_a^{a+\lambda} z(a+b-t)h(t)\left[1-g(t)\right] dt \\
= 2\int_{a+\lambda}^b \left[\tilde{z}(a+\lambda) - \tilde{z}(t)\right] h(t)g(t)\,dt - \int_a^b z(a+b-t)h(t)g(t)\,dt \\
+ \int_a^{a+\lambda} z(a+b-t)h(t)\,dt \\
\ge - \int_a^b z(a+b-t)h(t)g(t)\,dt + \int_a^{a+\lambda} \frac{f(a+b-t)}{h(a+b-t)} h(t)\,dt
\end{aligned}$$

From (12) and the last inequality we conclude

$$\int_a^{a+\lambda} f(t)\,dt - \int_a^{a+\lambda} \frac{f(a+b-t)}{h(a+b-t)} h(t)\,dt \ge \int_a^b \left[z(t) - z(a+b-t)\right] h(t)g(t)\,dt,$$

i.e.

$$\int_a^{a+\lambda} f(t)\,dt - \int_{b-\lambda}^b f(t)\frac{h(a+b-t)}{h(t)}\,dt \ge 2\int_a^b \tilde{z}(t)h(t)g(t)\,dt,$$

□

When h is symmetric, i.e. $h(t) = h(a+b-t)$, then the assumptions on f and h in Theorems 2.5 and 2.6 are equivalent since $\tilde{z}(x) = \frac{\tilde{f}(x)}{h(x)}$. In that case $\int_a^b \tilde{z}(t)h(t)g(t)\,dt = \int_a^b \tilde{f}(t)g(t)\,dt = \int_a^b f(t)\tilde{g}(t)\,dt$, so inequalities (6) and (11) are the same.

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