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Inner Differentiability and Differential Forms on Tangentially Locally Linearly Independent Sets

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Abstract. The de Rham theorem gives a natural isomorphism between De Rham cohomology and singular cohomology on a paracompact differentiable manifold. We proved this theorem on a wider family of subsets of Euclidean space, on which we can define inner differentiability. Here we define this family of sets called tangentially locally linearly independent sets, propose inner differentiability on them, postulate usual properties of differentiable real functions and show that the integration over sets that are wider than manifolds is possible.

1. Introduction

The differentiable mappings are usually defined on open sets. On arbitrary set a function is differentiable, if there is a bigger open set that contains the set and the function is differentiable on it. However, this is only an agreement. In this paper we define inner differentiability on a wider family of subsets of Euclidean space called tangentially locally linearly independent - TLLI sets in order to give new highlight to the well known De Rham theorem, that gives a nice relationship between analysis and topology.

De Rham has shown in [6] that there exist isomorphism between de Rham cohomology and singular cohomology on a paracompact differentiable manifold. This is very important fact as singular cohomology, defined as in [7], is very topological theory and de Rham cohomology is much more analytical that is based on the existence of differential forms with prescribed properties, explained as in [4]. An important operation on differential forms, the exterior derivative, is used in the celebrated Stokes' theorem as formulated in its modern form in [2], that also shows the relationship between topology and analysis. In [8] we proved de Rham theorem on the tangentially locally linearly independent.

In this paper instead using usual definition of derivatives as limits for the differential forms we use algebraic approach to the derivative that is mentioned in [5] to define inner differentiability on TLLI sets. Therefore, in the second Section of the paper we consider the family of TLLI sets and some of their properties and in the third Section is defined the inner differentiability of real multivariate functions on these sets. This allows us to postulate in Section 4 the integration over class of sets called cuboidle sets that is wider class of manifolds by defining differential forms on TLLI sets. Section 5 concludes the paper.

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2. Tangentially locally linearly independent and full tangentially locally linearly independent sets

In this Section we state the definition of a wider family of subsets of Euclidean space than open sets called tangentially locally linearly independent - TLLI sets and their properties in order in the next section to define the inner differentiability of multivariate real functions.

Definition 2.1. *A set $M \subseteq \mathbb{R}^n$ is called tangentially locally linearly independent (TLLI), if for any arbitrary point $\underline{x}^0 = \left(x_1^0, ..., x_n^0\right) \in M$ is valid:*
if $D_1, ..., D_n$ are real functions on the set M and continuous at $\underline{x}^0$ such
$\sum_{i=1}^{n} \left(x_i - x_i^0\right) \cdot D_i\left(\underline{x}\right) = 0, \forall\, \underline{x} \in M$, *then* $D_i\left(\underline{x}^0\right) = 0, \forall\, i \in \{1, ..., n\}$.

Theorem 2.2. *If $M \subseteq \mathbb{R}^n$ is TLLI set, then all points from the set M are accumulation points of the set M, i.e. $M \subseteq M'$.*

Proof. Let $M \subseteq \mathbb{R}^n$ be TLLI and let assume the opposite statement of the theorem, i.e. $M \not\subset M'$. So, there is a point $\underline{y} \in M$, but $\underline{y} \notin M'$.
Let consider the functions $D_i : M \to \mathbb{R}$ defined by:

$$D_i\left(\underline{x}\right) = \begin{cases} 1, & \text{if} \quad \underline{x} = \underline{y} \\ 0, & \text{if} \quad \underline{x} \in M \backslash \left\{\underline{y}\right\}, \end{cases} \quad \forall\, i \in \{1, ..., n\} \quad .$$

Next we prove that these functions are continuous at the point $\underline{y} \in M$.

Let $\varepsilon > 0$ is an arbitrary real number. Since $\underline{y} \notin M'$ then there exists an open neighborhood $T_\delta\left(\underline{y}\right)$ at $\underline{y}$ such $T_\delta\left(\underline{y}\right) \cap M \subseteq \left\{\underline{y}\right\}$. Therefore, for any point $\underline{x} \in T_\delta\left(\underline{y}\right) \cap M$ is true that $\left\| D_i\left(\underline{x}\right) - D_i\left(\underline{y}\right) \right\| = 0 < \varepsilon, \quad \forall\, i \in \{1, ..., n\}$. So the functions D_i for all $i = 1, ..., n$ are continuous at the point $\underline{y} \in M$.

By the definition of the functions D_i, $i = \overline{1, n}$, $\sum_{i=1}^{n} (x_i - y_i) \cdot D_i\left(\underline{x}\right) = 0$ for all $\underline{x} \in M$ but $D_i\left(\underline{y}\right) = 1 \neq 0, \forall\, i \in \{1, ..., n\}$, which is in contradiction of the assumption that the set M is TLLI. Therefore statement of the theorem is valid. □

Example 2.3. *All lines in $\mathbb{R}^2$ are not TLLI sets.*

Proof. Let $\Pi = \left\{(x, y) \in \mathbb{R}^2 \,:\, ax + by = c\right\}$ be an arbitrary line in $\mathbb{R}^2$, where a, b, c are real numbers such that at least one of a and b is different than 0.
The functions $D_1 : \Pi \to \mathbb{R}$, $D_2 : \Pi \to \mathbb{R}$ defined by $D_1((x, y)) = a, D_2((x, y)) = b$ for all $(x, y) \in \Pi$ are continuous at a fixed point $\underline{x}^0 = \left(x^0, y^0\right) \in \Pi$, and

$$\begin{aligned} &\left(x - x^0\right) D_1\left((x, y)\right) + \left(y - y^0\right) D_2\left((x, y)\right) = \left(x - x^0\right) \cdot a + \left(y - y^0\right) \cdot b = \\ &= a \cdot x - a \cdot x^0 + b \cdot y - b \cdot y^0 = (a \cdot x + b \cdot y) - \left(a \cdot x^0 + b \cdot y^0\right) = \\ &= c - c = 0, \ \forall\ (x, y) \in \Pi . \end{aligned}$$

But $D_1\left(\underline{x}^0\right) = a$, $D_2\left(\underline{x}^0\right) = b$ and at least one of a and b is different than 0, so by definition the line is not TLLI set. □

Let $\underline{x}^0 \in \mathbb{R}^n$ be an arbitrary point. The line through the point $\underline{x}^0$ and parallel with the x_k- axis, $k \in \{1, ..., n\}$, is denoted by:

$$G_k\left(\underline{x}^0\right) = \left\{\left(x_1^0, ..., x_{k-1}^0, x_k, x_{k+1}^0, ..., x_n^0\right) \,:\, x_k \in \mathbb{R}\right\}, \ k \in \{1, ..., n\} .$$

Definition 2.4. *A set $M \subseteq \mathbb{R}^n$ is full TLLI if any point $\underline{x}^0 \in M$ is an accumulation point of all sets $M \cap G_k\left(\underline{x}^0\right)$, $k \in \{1, ..., n\}$.*

Theorem 2.5. *Any full TLLI set $M \subseteq \mathbb{R}^n$ is TLLI set.*

Proof. Let M be a full TLLI set and $\underline{x}^0 \in M$ is an arbitrary point. Let $D_1, ..., D_n$ be functions determined by the assumption of the Definition 2.1. Then for any $\underline{x} \in M \cap G_k\left(\underline{x}^0\right)$ where $k \in \{1, 2, ..., n\}$ is fixed arbitrarly chosen, the following statement is valid:

$$0 = \sum_{i=1}^{n} \left(x_i - x_i^0\right) \cdot D_i\left(\underline{x}\right) = \left(x_k - x_k^0\right) \cdot D_k\left(\underline{x}\right).$$

So, $D_k\left(\underline{x}\right) = 0$ for any $\underline{x} \in M \cap G_k\left(\underline{x}^0\right)$. Since M is ful TLLI the point $\underline{x}^0$ is an accumulattion point of the set $M \cap G_k\left(\underline{x}^0\right)$. Then there exists a sequence $\left(\underline{x}^m\right)_{m\in\mathbb{N}}$ in the set $M \cap G_k\left(\underline{x}^0\right)$ such that $\underline{x}^m \to \underline{x}^0$, $m \to \infty$. Since $D_k\left(\underline{x}\right) = 0$ for any $\underline{x} \in M \cap G_k\left(\underline{x}^0\right)$, then $D_k\left(\underline{x}^m\right) = 0$ for all $m \in \mathbb{N}$. The function D_k is continuous, so $0 = D_k\left(\underline{x}^m\right) \to D_k\left(\underline{x}^0\right)$, $m \to \infty$. Therefore, $D_k\left(\underline{x}^0\right) = 0$ for an arbitrary $k \in \{1, 2, ..., n\}$.
Finaly. since $k \in \{1, 2, ..., n\}$ an $\underline{x}^0$ are arbitrary, then the set M is TLLI. □

Notice that all open sets and all closed n-dimensional rectangular cuboids in the space $\mathbb{R}^n$ are full TLLI sets.

3. Derivatives of multivariate real functions without limits

The definition of derivative, avoiding limit of a quotient difference was one of the main discissions among mathematicians in eighties of the previous century, see [9] and [3]. In this Section in order to define inner differentialbility on TLLI sets we consider the algebraic approach to the derivatives given in [5].

Definition 3.1. *A multivariate real function $f : M \to \mathbb{R}$, defined on TLLI set $M \subseteq \mathbb{R}^n$ is differentiable at $\underline{x}^0 \in M$, if there exist n real-valued functions $D_1, ..., D_n$ on the set M and continuous at $\underline{x}^0 \in M$ such that:*

$$f\left(\underline{x}\right) = f\left(\underline{x}^0\right) + \sum_{i=1}^{n} \left(x_i - x_i^0\right) \cdot D_i\left(\underline{x}\right), \quad \forall\, \underline{x} \in M \tag{1}$$

Definition 3.2. *A multivariate real function $f : M \to \mathbb{R}$ is differentiable on the set $M \subseteq \mathbb{R}^n$, if it is differentiable at any point of the set M.*

Theorem 3.3. *Let $f : M \to \mathbb{R}$ be a real function on the TLLI set $M \subseteq \mathbb{R}^n$ and let f be differentiable at $\underline{x}^0 \in M$. Then the values $D_1\left(\underline{x}^0\right), ..., D_n\left(\underline{x}^0\right)$ are unique .*

It doesn't mean that the functions $D_1(\underline{x}), ..., D_n(\underline{x})$ are unique on the set M.

Proof. Let $D_1,...,D_n$ and $D_1',...,D_n'$ are functions for such the equation (1) is valid. Then,

$$\sum_{i=1}^{n} \left(x_i - x_i^0\right) \cdot \left(D_i\left(\underline{x}\right) - D_i'\left(\underline{x}\right)\right) = 0, \quad \forall\, \underline{x} \in M$$

Since the functions $D_1(\underline{x}) - D_1'(\underline{x}),...,D_n(\underline{x}) - D_n'(\underline{x})$ are continuous at the point $\underline{x}^0$ and the set M is TLLI then $D_i(\underline{x}^0) - D_i'(\underline{x}^0) = 0, \forall\, i \in \{1, ..., n\}$, i.e. $D_i(\underline{x}^0) = D_i'(\underline{x}^0), \forall\, i \in \{1, ..., n\}$. □

We say that theses unique values $D_1\left(\underline{x}^0\right), ..., D_n\left(\underline{x}^0\right)$ are partial derivatives of the function f at $\underline{x}^0$ and we employ the notation

$$D_i\left(\underline{x}^0\right) = \frac{\partial f}{\partial x_i}\left(\underline{x}^0\right) = f_{x_i}{}'\left(\underline{x}^0\right), \quad \forall\, i \in \{1, ..., n\}.$$

Theorem 3.4. *Let $f : M \to \mathbb{R}$ be a real function on the TLLI set $M \subseteq \mathbb{R}^n$ and let f be differentiable at $\underline{x}^0 \in M$, then f is continuous at $\underline{x}^0 \in M$.*

Proof. Because f is differentiable at $\underline{x}^0 \in M$, then there exist n real functions $D_1, ...D_n$ on the set M that are continuous at $\underline{x}^0 \in M$ such that:

$$f\left(\underline{x}\right) = f\left(\underline{x}^0\right) + \sum_{1=1}^{n} \left(x_i - x_i^0\right) \cdot D_i\left(\underline{x}\right), \quad \forall\, \underline{x} \in M.$$

Let $\left(\underline{x}^m\right)_{m\in\mathrm{N}}$ be a sequence in M such that $\underline{x}^m \to \underline{x}^0$, $m \to \infty$, i.e. $x_i^m \to x_i^0, m \to \infty$ for all $i \in \{1, ..., n\}$. Since the set M is TLLI,

$$\lim_{m\to\infty} f\left(\underline{x}^m\right) = \lim_{m\to\infty} \left(f\left(\underline{x}^0\right) + \sum_{i=1}^{n} \left(x_i^m - x_i^0\right) \cdot D_i\left(\underline{x}^m\right) \right) = f\left(\underline{x}^0\right).$$

Moreover, since the sequence is arbitrary then the function f is continuous at $\underline{x}^0 \in M$. □

Let $f : M \to \mathbb{R}$ be a real valued function on full TLLI set $M \subseteq \mathbb{R}^n$ and $\underline{x}^0 = \left(x_1^0, x_2^0, ..., x_n^0\right)$ be a fixed point of the set M.

We define n real univariate functions:
$g_k(x_k) = f\left(x_1^0, ..., x_{k-1}^0, x_k, x_{k+1}^0, ..., x_n^0\right)$ **for all** $k \in \{1, ..., n\}$.
The domain of these functions g_k for any $k \in \{1, ..., n\}$ is the set
$A_k = \left\{x_k \in \mathbb{R} : \left(x_1^0, ..., x_{k-1}^0, x_k, x_{k+1}^0, ..., x_n^0\right) \in M\right\} = M \cap G_k\left(\underline{x}^0\right)$.
Since A_k, $k = 1, .., n$ are TLLI sets in $\mathbb{R}$, then all points $x_k \in A_k$, $k = 1, ..., n$ are accumulation points of the sets A_k, $k = 1, .., n$, respectively.

In [5] are given the proofs of the last two theorems in this Section:

Theorem 3.5. *If the function $f : M \to \mathbb{R}$ is differentiable at $\underline{x}^0 \in M$, then all functions g_k, $k = 1, ..., n$ are differentiable at x_k^0, $k = 1, ..., n$, respectively, and $g_k'\left(x_k^0\right) = f_{x_k}'\left(\underline{x}^0\right)$.*

Definition 3.6. *Let $f : M \to \mathbb{R}$ be a real function on TLLI set $M \subseteq \mathbb{R}^n$. The function f is differentiable with respect to x_k at $\underline{x}^0 \in M$, if the function g_k is differentiable at x_k^0.*

Definition 3.7. *A function $f : M \to \mathbb{R}$ is continuously differentiable on full TLLI $M \subseteq \mathbb{R}^n$, if it is differentiable on M, and all its partial derivatives are continuous on M.*

If a real multivariate function f defined on TLLI set $M \subseteq \mathbb{R}^n$ is differentiable on M then a question about differentiability of its partial derivatives $f_{x_k}^{'}, k = 1, ..., n$ at a point $\underline{x}^0 \in M$ (with respect to all or some of the variables $x_k, k = 1, ..., n$) is raised.

Therefore, if the partial derivatives $f_{x_k}^{'}$ for some $k = 1, ..., n$ exist and they are differentiable at $\underline{x}^0 \in M$ with respect to some variables x_j, $j = 1, ..., n$ we say that there exist partial derivatives of second order of the function f at $\underline{x}^0 \in M$ with respect to some variables x_i and x_j they are denoted by $\left(f_{x_k}{'}\right)_{x_j}^{'}\left(\underline{x}^0\right) = f_{x_k x_j}^{'}\left(\underline{x}^0\right) = \frac{\partial^2 f}{\partial x_j \partial x_k}\left(\underline{x}^0\right) = f_{kj}^{'}\left(\underline{x}^0\right)$, where $k = 1, ..., n$ and $j = 1, ..., n$. If there exist partial derivatives of a second order of the function f on the whole set M then it is possible to discuss about their differentiability and partial derivatives of higher order.

Definition 3.8. *A real multivariate function is r- times differentiable at $\underline{x}^0 \in M$, where $r = 2, 3, ...,$ if there exist an open neighborhood U of that point such that the function f is $r - 1$- times differentiable on the set $U \cap M$ and all $r - 1$-partial derivatives of f are differentiable at $\underline{x}^0$.*
A function f is r- times differentiable on the set M if it is r- times differentiable at all points of the set M.
The partial derivtives from r-th order of the function f at $\underline{x}^0$ are denoted by $f_{x_{k_1} x_{k_2} ... x_{k_r}}^{'}\left(\underline{x}^0\right) = \frac{\partial^r f}{\partial x_{k_r} ... \partial x_{k_2} \partial x_{k_1}}\left(\underline{x}^0\right)$.

Theorem 3.9. *Let $f : M \to \mathbb{R}$ be a multivariate real function on a closed rectangular cuboid $M = \left\{ \underline{x} \in \mathbb{R}^n : \ a_k \leq x_k \leq b_k \ , \ a_k, b_k \in \mathbb{R} \ , \ k = 1, ..., n \right\}$, and let all partial derivatives of the function f be differentiable with respect to all variables at the point $\underline{x}^0 \in M$. Then, $f_{x_i x_j}\left(\underline{x}^0\right) = f_{x_j x_i}\left(\underline{x}^0\right)$, $i, j = 1, 2, ..., n$.*

4. Differential forms on TLLI sets

Definition 4.1. *Differential form of k order on the set M (or $k-$form in M) is a mapping ω , $\omega = \sum_{1 \leq i_1 < ... < i_k \leq n} a_{i_1 ... i_k}\left(\underline{x}\right) dx_{i_1} \wedge ... \wedge dx_{i_k}$, where $a_{i_1 ... i_k} : M \to \mathbb{R}$ are continuous real functions for any $k-$variation $\{i_1, i_2, ..., i_k\}$ of the set of n elements $\{1, 2, ..., n\}$, and we will denote by $\omega = \sum_{\underline{i}} a_{\underline{i}}\left(\underline{x}\right) dx_{\underline{i}}$, where $dx_{\underline{i}} = dx_{i_1} \wedge ... \wedge dx_{i_k}$ and $a_{\underline{i}} = a_{i_1 ... i_k}$ for any variation $\underline{i} = \{i_1, ..., i_k\}$, $1 \leq i_1 < ... < i_k \leq n$, such that it maps to any singular $k-$cube $\phi : I^k \to M$ (that is continuously differentiable function on cube, i.e. $\phi \in C^1$) a real number:*

$$\omega\left(\phi\right) = \int_{\phi} \omega = \sum_{\underline{i}} \int_{I^k} a_{\underline{i}}\left(\phi\left(\underline{t}\right)\right) \frac{\partial\left(\phi_{i_1}, ...\phi_{i_k}\right)}{\partial\left(t_1, ..., t_k\right)} dt_1 \wedge ... \wedge dt_k,$$

where $\frac{\partial\left(\phi_{i_1}, ...\phi_{i_k}\right)}{\partial(t_1, ..., t_k)} = \begin{vmatrix} \frac{\partial \phi_{i_1}}{\partial t_1} \cdots\cdots \frac{\partial \phi_{i_1}}{\partial t_k} \\ \vdots \qquad \vdots \\ \frac{\partial \phi_{i_k}}{\partial t_1} \cdots\cdots \frac{\partial \phi_{i_k}}{\partial t_k} \end{vmatrix}$ *is the Jacobian of $\phi = \left(\phi_1, \phi_2, ..., \phi_n\right)$.*

Definition 4.2. *We define the following statements,*

1. $\omega = 0$ *if and only if* $\omega\left(\phi\right) = 0$, *for any singular $k-$cube $\phi : I^k \to M, \ \phi \in C^1$,*
2. $\omega_1 = \omega_2$ *if and only if* $\omega_1\left(\phi\right) = \omega_2\left(\phi\right)$, *for any singular $k-$cube $\phi : I^k \to M, \ \phi \in C^1$,*
3. *If ω_1 and ω_2 are two $k-$forms on M, then the sum $\omega = \omega_1 + \omega_2$ is $k-$form on M such that $\omega\left(\phi\right) = \omega_1\left(\phi\right) + \omega_2\left(\phi\right)$, for any singular $k-$cube $\phi : I^k \to M, \ \phi \in C^1$,*
4. *For any number $c \in \mathbb{R}$, $c\omega$ is $k-$form on M such that $(c\omega)\left(\phi\right) = c \cdot \omega\left(\phi\right)$, for any singular $k-$cube $\phi : I^k \to M, \ \phi \in C^1$.*

Definition 4.3. *If $\Gamma = \sum_{\phi} n_{\phi}\phi$ is continuously differentiable $k-$chain on M, then the $k-$form on M ω maps a real number to the $k-$chain $\Gamma = \sum_{\phi} n_{\phi}\phi$*

$$\omega\left(\Gamma\right) = \int_{\Gamma} \omega = \sum_{\phi} \sum_{\underline{i}} n_{\phi} \int_{I^k} a_{\underline{i}}\left(\phi\left(\underline{t}\right)\right) \frac{\partial\left(\phi_{i_1}, ...\phi_{i_k}\right)}{\partial\left(t_1, ..., t_k\right)} dt_1 \wedge ... \wedge dt_k.$$

Notice, if $\phi : I^k \to M$ is degenerated singular $k-$cube, i.e. there exists singular $k-1-$cube $\phi^{'} : I^{k-1} \to M$ such that

$$\phi\left(t_1, ..., t_{i-1}, t_i, t_{i+1}, ...t_n\right) = \phi'\left(t_1, ..., t_{i-1}, t_{i+1}, ..., t_n\right)$$

for some integer i , $1 \leq i \leq n$, then for any $k-$ form ω on M is valid that $\omega\left(\phi\right) = 0$. So, we conclude that a $k-$form ω on M is a real function from the free abelian group of all nondegenerated continuously differentiable singular k- cubes, $C_k\left(M\right)$.
The set of all $k-$forms for any $k \leq n$ on M is denoted by $D^k\left(M\right)$, i.e.

$D^k\left(M\right) = \left\{\omega \ \middle| \ \omega : C_k\left(\mathrm{M}\right) \to \mathbb{R} \text{ is } k - \text{ form on } M \right\}$. If $k > n$ then $D^k\left(M\right) = 0$.

Definition 4.4. *Let ω be a $k-$form on the set M such that $\omega = a\left(\underline{x}\right)dx_{i_1} \wedge ... \wedge dx_{i_s} \wedge ... \wedge dx_{i_p} \wedge ... \wedge dx_{i_k}$, where $a : M \to \mathbb{R}$ is continuous real function on M. By $\overline{\omega}$ is denoted the $k-$form on M that is obtained by transposition of dx_{i_s} and dx_{i_p}, i.e. $\overline{\omega} = a\left(\underline{x}\right)dx_{i_1} \wedge ... \wedge dx_{i_p} \wedge ... \wedge dx_{i_s} \wedge ... \wedge dx_{i_k}$. Since $\int_\phi \overline{\omega} = -\int_\phi \omega$, i.e. $\overline{\omega}\left(\phi\right) = -\omega\left(\phi\right)$ for any singular $k-$cube $\phi : I^k \to M,\ \phi \in C^1$, the $k-$form $\overline{\omega}$ is called opposite $k-$form of ω.*

Notice, if the indices i_s and i_p are equal, then $\omega = \overline{\omega} = -\omega$, and so $\omega = 0$.

Therefore, if ω is a $k-$form $\omega = \sum_{\{i_1,...,i_k\}\in\{1,...,n\}} a_{i_1...i_k}\left(\underline{x}\right) dx_{i_1} \wedge ... \wedge dx_{i_k}$, then the $k-$variation with repetition $\{i_1, i_2, ..., i_k\}$ of n elements $\{1, 2, ..., n\}$ is enough to be just $k-$variation without repetition. Moreover, by transposition of the indices any $k-$form $\omega = \sum_{\{i_1,...,i_k\}\in\{1,...,n\}} a_{i_1...i_k}\left(\underline{x}\right) dx_{i_1} \wedge ... \wedge dx_{i_k}$ can be transformed into a form $\omega = \sum_{1\le i_1<...<i_k\le n} a_{i_1...i_k}\left(\underline{x}\right) dx_{i_1}\wedge...\wedge dx_{i_k}$, that we will call **standard differential** $k-$**form** and we will denote by $\omega = \sum_{\underline{i}} a_{\underline{i}}\left(\underline{x}\right) dx_{\underline{i}}$, where $dx_{\underline{i}} = dx_{i_1}\wedge...\wedge dx_{i_k}$ and $a_{\underline{i}} = a_{i_1...i_k}$ for any variation $\underline{i} = \{i_1, ..., i_k\}$, $1 \le i_1 < ... < i_k \le n$. By the definition, standard differential $k-$form of any $k-$form is unique.

Definition 4.5. *Let $dx_{\underline{i}}$ be a $p-$form and $dx_{\underline{j}}$ be a $q-$ form on M. A product of the differential forms $dx_{\underline{i}}$ and $dx_{\underline{j}}$ is $p + q-$form on M such that $dx_{\underline{i}} \wedge dx_{\underline{j}} = dx_{i_1} \wedge ... \wedge dx_{i_p} \wedge dx_{j_1} \wedge ... \wedge dx_{j_q}$ (not necessary being standard differential form).*

Notice, if $\underline{i} \cap \underline{j} \neq \emptyset$, then $dx_{\underline{i}} \wedge dx_{\underline{j}} = 0$. Let $\underline{i} \cap \underline{j} = \emptyset$, then the product of $dx_{\underline{i}}$ and $dx_{\underline{j}}$ is $p + q-$form on M in standard form $dx_{\underline{i}} \wedge dx_{\underline{j}} = (-1)^\alpha dx_{[\underline{i},\underline{j}]}$, where $\left[\underline{i}, \underline{j}\right]$ is notation of the indices $i_1, ..., i_p, j_1, ..., j_q$ increasingly ordered, and α is the number of negative differences between the indices $j_r - i_t$, $t \in \{1, ..., p\}$ and $r \in \{1, ..., q\}$. The proofs of the two next theorems are obtained in [1].

Theorem 4.6. *Let $dx_{\underline{i}}$ be $p-$form, $dx_{\underline{j}}$ is $q-$ form and $dx_{\underline{k}}$ is $r-$ form on M. Then*

$$dx_{\underline{i}} \wedge \left(dx_{\underline{j}} \wedge dx_{\underline{k}}\right) = \left(dx_{\underline{i}} \wedge dx_{\underline{j}}\right) \wedge dx_{\underline{k}}.$$

Definition 4.7. *Let $\omega = \sum_{\underline{i}} a_{\underline{i}}\left(\underline{x}\right) dx_{\underline{i}}$ be a $p-$form on M and $\lambda = \sum_{\underline{j}} b_{\underline{j}}\left(\underline{x}\right) dx_{\underline{j}}$ is $q-$form on M. A product of them is a $p + q-$ form $\omega \wedge \lambda = \sum_{\underline{i},\underline{j}} a_{\underline{i}} \cdot b_{\underline{j}}\left(\underline{x}\right) dx_{\underline{i}} \wedge dx_{\underline{j}}$.*

Proposition 4.8. *The following statements are valid:*

1. *Let ω, λ and σ be differential forms on M. Then $\omega \wedge (\lambda \wedge \sigma) = (\omega \wedge \lambda) \wedge \sigma$.*
2. *Let ω_1, ω_2 be any $k-$forms and λ be an arbitrary $p-$form on M. Then $(\omega_1 + \omega_2) \wedge \lambda = \omega_1 \wedge \lambda + \omega_2 \wedge \lambda$.*
3. *Let ω_1, ω_2 be any $k-$form and λ be an arbitrary $p-$form on M. Then $\lambda \wedge (\omega_1 + \omega_2) = \lambda \wedge \omega_1 + \lambda \wedge \omega_2$.*

From the last proposition we conclude that the set of all $k-$forms on M, $D^k(M)$ with respect to sum and product is a vector space.

Next we define an operator $d : D^k(M) \to D^{k+1}(M)$ and state some theorems about its properties that can be easily proved.

Definition 4.9. *Let $f : M \to \mathbb{R}$ be a $0-$form on M, where f is continuously differentiable function. Its differential is a $1-$form on M, $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$. Let $\omega = \sum_{\underline{i}} a_{\underline{i}}\left(\underline{x}\right) dx_{\underline{i}}$ is an arbitrary $k-$form on M, such that $a_{\underline{i}}$ continuously differentiable real function. Its differential is a $k + 1-$form on M, $d\omega = \sum_{\underline{i}} da_{\underline{i}} \wedge dx_{\underline{i}} = \sum_{\underline{i}} \sum_{j=1}^n \frac{\partial a_{\underline{i}}}{\partial x_j} dx_j \wedge dx_{\underline{i}}$.*

The proof of the next theorem is obtained in [1].

Theorem 4.10. *The mapping $d : D^k(M) \to D^{k+1}(M)$, $k \in \mathbb{Z}$ is linear, i.e.*

1. $d(\omega + \lambda) = d\omega + d\lambda$
2. $d(c\omega) = c \cdot d\omega.$

The statement of the following theorem is obtained from Calculus.

Theorem 4.11. *Let $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ are $0-$forms on M, where f and g are continuously differentiable functions, then $d(fg) = df \cdot g + f \cdot dg$.*

Theorem 4.12. *Let ω and λ are arbitrary k and $m-$ forms on M, respectively. Then,*

$$(\omega \wedge \lambda) = d\omega \wedge \lambda + (-1)^k \omega \wedge d\lambda. \tag{2}$$

Proof. The proof of this theorem is based on simple calculations considering two situations, first, assuming that $\omega = a_{\underline{i}} dx_{\underline{i}}$ and $\lambda = b_{\underline{j}} dx_{\underline{j}}$ and plugging theorems 4.10 and 4.11 and second, assuming in general that $\omega = \sum_{\underline{i}} a_{\underline{i}} dx_{\underline{i}}$ and $\lambda = \sum_{\underline{j}} b_{\underline{j}} dx_{\underline{j}}$ by plugging the result from the first assumption. □

Definition 4.13. *We say that ω is an exact differential k- form on M, then there exists $k-1$- form $\lambda \in D^{k-1}(M)$ such that $\omega = d\lambda$. We say that ω is a* ***closed*** *differential k- form on M if $d\omega = 0$.*

Definition 4.14. *We say a set $M \subseteq \mathbb{R}^n$ is cuboidle, if for any point $\underline{x} \in M$ there exists rectangular cuboid*

$$K = \left\{ \underline{y} \in \mathbb{R}^n \,\middle|\, a_i \le y_i \le b_i \,,\, a_i, b_i \in \mathbb{R} \,,\, i = \overline{1, n} \right\}$$

such that $\underline{x} \in K \subseteq M$.

A cuboidle set is TLLI set.

Theorem 4.15. *Let $\omega = \sum_{\underline{i}} a_{\underline{i}} d_{x_{\underline{i}}}$ be two times differentiable k- form on cuboidle set $M \subseteq \mathbb{R}^n$, i.e. for all indices $\underline{i}$ the functions $a_{\underline{i}} : M \to \mathbb{R}$ are two times differentiable on the set M. Then $dd\omega = 0$ on the set M.*

Proof. 1 case: Let ω be an arbitrary $0-$form $f : M \to \mathbb{R}$ on M, then

$$d^2\omega = d(d\omega) = d\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i=1}^{n} d\left(\frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}\right) dx_j \wedge dx_i$$

In the sum above we consider two terms $\frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i$ and $\frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j$. Because f is two times differentiable function on cuboidle set M, then for any point $\underline{x} \in M$ there exists rectangular cuboid

$$K = \left\{ \underline{y} \in \mathbb{R}^n \,\middle|\, a_i \le y_i \le b_i \,,\, a_i, b_i \in \mathbb{R} \,,\, i = \overline{1, n} \,, \right\}$$

such that $\underline{x} \in K \subseteq M$ and considering Theorem 3.9 the equation $\frac{\partial^2 f}{\partial x_j \partial x_i}(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ is true for any point $\underline{x} \in M$.

Therefore, $\frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i = (-1) \frac{\partial^2 f}{\partial x_j \partial x_i} dx_i \wedge dx_j = (-1) \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j$, and so all terms are cancelled between them, i.e. $d^2 f = 0$.

2 case: $d\left(dx_{\underline{i}}\right) = d\left(1 \cdot dx_{\underline{i}}\right) = d1 \wedge dx_{\underline{i}} = 0$.

3 case: $d\left(adx_{\underline{i}}\right) = da \wedge dx_{\underline{i}}$

$$d^2\left(adx_{\underline{i}}\right) = d\left(d\left(adx_{\underline{i}}\right)\right) = d\left(da \wedge dx_{\underline{i}}\right) \overset{(2)}{=} d^2 a \wedge dx_{\underline{i}} + (-1)^k da \wedge d^2 x_{\underline{i}} = 0.$$

Let $\omega = \sum_{\underline{i}} a_{\underline{i}} d_{x_{\underline{i}}}$ be two times differentiable k- form on a cuboidle set $M \subseteq \mathbb{R}^n$, then $d\omega = \sum_{\underline{i}} da_{\underline{i}} \wedge d_{x_{\underline{i}}}$, $d^2\omega = \sum_{\underline{i}} d^2\left(a_{\underline{i}} d_{x_{\underline{i}}}\right) = 0$.

Finally we conclude that $dd\omega = 0$ for any two times differentiable k- form ω on a cuboidle set $M \subseteq \mathbb{R}^n$. □

Theorem 4.16. *Let $\omega = \sum_{\underline{i}} a_{\underline{i}} d_{x_{\underline{i}}}$ be a differentiable k- form on a cuboidle set $M \subseteq \mathbb{R}^n$. If $\omega = \sum_{\underline{i}} a_{\underline{i}} d_{x_{\underline{i}}}$ is an exact k− form on the set M, then it is closed.*

Proof. Since ω is an exact $k-$ form on the set M, then there exists $k-1$ - form $\lambda \in D^{k-1}(M)$ such that $\omega = d\lambda$. Because ω is a differentiable k- form on a cuboidle set then λ is two times differentiable $k-1$- form on cuboidle set $M \subseteq \mathbb{R}^n$ and by Theorem 4.11 $dd\lambda = 0$ on the set M, Therefore, $d\omega = dd\lambda = 0$ on the set M, so ω is closed k- form on the set M. □

The converse statement of Theorem 4.16 is not always true, but if we assume additionally that the cuboidle set $M \subseteq \mathbb{R}^n$ is also convex set then any continuously differentiable closed k- form on M is exact as shown in [5].

5. Conclusion

In our paper we consider a family of sets in n dimensional real space so called TLLI sets that is wider than the family of open sets. Moreover, we define differentiability and differential forms on this family of sets. So we show that it is possible to integrate over singular cube not only in a manifold as we know by now but in a cuboidle set defined by the TLLI sets. At last we prove and state some theorems which are necessary for the definition of de Rham cohomology in order to complete the proof of the De Rham Theorem on a wider family than manifolds that we have shown in [6] .

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