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Reciprocal Power GCDQ Matrices and Power LCMQ Matrices Defined on Factor Closed Sets over Euclidean Domains

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Abstract. In this paper, we use a generalized form for the Jordan totient function in order to extend the Reciprocal power GCDQ matrices and power LCMQ matrices from the standard domain of natural integers to Euclidean domains. Structural theorems and determinantal arguments defined on both arbitrary and factor-closed q-ordered sets are presented over such domains. We illustrate our work in the case of Gaussian integers.

1. Introduction

Let $T = \{t_1, t_2, ..., t_m\}$ be a well ordered set of m distinct positive integers with $t_1 < t_2 < ... < t_m$. The *GCD* matrix defined on T is $(T)_{m\times m} = (t_i, t_j)$, where (t_i, t_j) is the greatest common divisor of t_i and t_j. The *LCM* matrix defined on T is $[T]_{m\times m} = \left[t_i, t_j\right]$, where $\left[t_i, t_j\right]$ is the least common multiple of t_i and t_j. If r is any real number, then the r^{th} power *GCD* matrix defined on T is $(T^r)_{m\times m} = (t_i, t_j)^r$, and the r^{th} power *LCM* matrix defined on T is $[T^r]_{m\times m} = \left[t_i, t_j\right]^r$. Moreover, set $T = \{t_1, t_2, ..., t_m\}$ is said to be factor-closed if $t \in T$ for any divisor t of $t_i \in T$, and it is gcd-closed if $(t_i, t_j) \in T$ for all t_i and t_j in T. In 1876, Smith [18] showed that if $T = \{1, 2, ..., m\}$, then $\det(T) = \prod_{i=1}^{m} \phi(i)$ and $\det[T] = \prod_{i=1}^{m} \phi(i)\pi(i)$, where ϕ is Euler's totient function and π is a multiplicative function such that $\pi(p^k) = -p$. Moreover, Smith showed that the results hold true for any factor-closed set. In 1989, Beslin and El-Kassar [3] extended the concept of *GCD* matrices and Smith's determinant to unique factorization domains. In 1989/92, Beslin and Ligh [2, 4, 5] factorized the *GCD* matrices defined on gcd-closed sets, and they computed their determinants. In addition, they obtained the structural theorems for the *LCM* matrices and showed that they are non-singular. Later, in 1992, Borque and Ligh [6] conjectured that the *LCM* matrices defined on gcd-closed sets are invertible. In 1996, Chun [7] introduced the concept of power *GCD* and *LCM* matrices, and presented their structures, determinants and inverses over the domain of natural integers. In 1996, Haukkanen and Sillanpaa [13] studied the *GCD* and *LCM* matrices for lcm-closed and gcd-closed sets. In 1997, Haukkanen [12], in his famous paper "On Smith's Determinant", gave a counter example for Bourque and Ligh's conjecture. In 1998, Hong [14, 15] showed inductively that $\det(T)$ divides $\det[T]$ if T is gcd-closed and $m \leq 3$, and he gave

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a counter example in case $m \geq 4$. In 2009, Hong et al. [16] generalized the power *GCD* matrices defined on factor-closed sets from the standard settings $\mathbb{Z}$ to unique factorization domains. El-Kassar et al. [8–11] extended many results concerning the *GCD* and *LCM* matrices defined on factor-closed sets to principal ideal domains. Recently, Awad et al. [1] gave a generalization for the power *GCD* and *LCM* *P*-matrices defined on gcd-closed sets over unique factorization domains, where the results found in literature are considered as special cases if the domain of natural integers is taken in particular. In this paper, we give a generalization for the Reciprocal power GCDQ and power LCMQ matrices defined on both arbitrary and factor-closed q-ordered sets of non-zero non-associate elements in any Euclidean domain S. In addition, some examples in $\mathbb{Z}[i]$ are given in order to describe what have been done.

2. Preliminaries

Definition 2.1. *The nonempty set $P = \{p_1, p_2, p_3, ...\}$ is said to be a prime residue system of an Euclidean domain S if P is a complete well-ordered set of non-zero non-associate prime elements in S.*

Definition 2.2. *If $T = \{t_1, t_2, ..., t_m\}$ is a set of non-zero non-associate elements in an Euclidean domain S with measure q and prime residue system $P = \{p_1, p_2, p_3, ...\}$, then the list of well-ordered primes $\{p_1, p_2, ..., p_i\}$ in P whose elements divide all the elements of T and its ordering is inherited from the well-ordering of P is said to be a complete prime residue system of T in S.*

Definition 2.3. *If $T = \{t_1, t_2, ..., t_m\}$ is a set of non-zero non-associate elements in an Euclidean domain S with measure q, then the q-ordering $<_q$ in S is a linear ordering defined via the following scheme: $t_i <_q t_j$ if $q(t_i) < q(t_j)$ and $t_i \approx t_j$ if $q(t_i) = q(t_j)$.*

Hence, if $T = \{t_1, t_2, ..., t_m\}$ such that $t_1 <_q t_2 <_q ... <_q t_m$, then T is q-ordered.

Definition 2.4. *Let S be an Euclidean domain with measure q and prime residue system P, and let x be a non-zero element in S with the unique prime factorization $x \approx up_1^{\alpha_1} p_2^{\alpha_2} ... p_m^{\alpha_m}$, where $p_i \in P$, $\alpha_i \in \mathbb{N}$, and u is a unit in S. Define the totally multiplicative function $\phi_s(x)$ as*

$$\phi_s(x) = \prod_{i=1}^{m} q\left(p_i^{\alpha_i - 1}\right)((q(p_i) - 1),$$

and $\phi_s(u) = 1$.

Theorem 2.5. *If $x \in S$ and $E(x)$ is a complete set of distinct non-associate divisors d of x in S, then $q(x) = \sum_{d \in E(x)} \phi_s(d)$.*

Proof. See [8] □

Definition 2.6. *Let $x = up_1^{\alpha_1} p_2^{\alpha_2} ... p_m^{\alpha_m}$ be a non-zero element in S. Define the Jordan totient function $J_{k,s}$ on $S - \{0\}$ to be the multiplicative function $J_{k,s}(x) = \prod_{i=1}^{m} q\,(p_i)^{k(\alpha_i - 1)} \left(q(p_i)^k - 1\right)$ with $J_{k,s}(x) = 1$ if x is unit.*

Theorem 2.7. *If $x \in S$ and $E(x)$ is a complete set of distinct non-associate divisors d of x in S, then $q\,(x)^k = \sum_{d \in E(x)} J_{k,s}(d)$.*

Proof. Since $J_{k,s}\,(x)$ is multiplicative, then $q(x) = \sum_{d \in E(x)} J_{k,s}(d)$ is also multiplicative. Hence,

$$\begin{aligned}
\sum_{d \in E(p_i^{\alpha_i})} J_{k,s}(d) &= 1 + q\,(p_i)^{k(1-1)}\,[q(p_i)^k - 1] + q\,(p_i)^{k(2-1)}\,[q(p_i)^k - 1] + ... + q\,(p_i)^{k(\alpha_i - 1)}\,[q(p_i)^k - 1] \\
&= 1 + q\,(p_i)^k - 1 + q\,(p_i)^{2k} - q\,(p_i)^k + ... + q\,(p_i)^{k\alpha_i} - q\,(p_i)^{k(\alpha_i - 1)} \\
&= q\left(p_i^{\alpha_i}\right)^k
\end{aligned}$$

□

Definition 2.8. *Let $P = \{p_1, p_2, ..., p_m\}$ be a complete prime residue system of an Euclidean domain S. An element $d \in P$ is said to be a P-divisor of $x \in S$ if d divides x. Moreover, if x and y are both non-zero elements in S, then the greatest common P-divisor of x and y in S is denoted by $(x, y)_p$ which is unique up to order and up to associates.*

Definition 2.9. *Let $T = \{t_1, t_2, ..., t_m\}$ be a q-ordered set of non-zero non-associate elements in an Euclidean domain S with measure q and prime residue system P, and let r be any real number. The r^{th} power GCD q-matrix defined on T is the $m \times m$ matrix$(T^r)_q$ whose ij^{th} entry is $(t_{ij})_r = q\left((t_i, t_j)_p\right)^r$, where $(t_i, t_j)_p$ is the greatest common P-divisor of t_i and t_j in S.*

3. Reciprocal Power GCDQ Matrices over Euclidean Domains

In this section, we study the factorizations and determinants of the r^{th} power Reciprocal GCDQ matrices defined on q-ordered sets of non-zero non-associate elements in an Euclidean domain S. Moreover, we present some examples in $\mathbb{Z}[i]$.

Definition 3.1. *If $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered set of non-zero non-associate elements in S, then the r^{th} power Reciprocal GCDQ matrix defined on T over S is the matrix $\left(1/(T^r)_q\right)$ whose ij^{th} entry is $(t_{ij})^{-r} = \dfrac{1}{(t_{ij})_r}$.*

3.1. Factorizations of Reciprocal Power GCDQ Matrices over Euclidean Domains

Theorem 3.2. *If $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered set of non-zero non-associate elements in S, then the Reciprocal power GCDQ matrix $\left(1/(T^r)_q\right) = E\,(1/G_r)\,E^T$, where $(1/G_r)$ is a diagonal matrix and E is a lower triangular incidence matrix.*

Proof. Let $D = \{y_1, y_2, ..., y_n\}$ be the minimal factor-closed set containing $T = \{t_1, t_2, ..., t_m\}$ in S, and let $E(t)$ be a complete set of non-associate divisors d for every t_i in T. Consider the $n \times n$ diagonal matrix $(1/G_r) = diag(J_{-r,s}(y_1), J_{-r,s}(y_2), \dots, J_{-r,s}(y_n))$, and the $m \times n$ incidence matrix $E = (e_{ij})$ such that $e_{ij} = 1$ if $y_j \in E(t_i)$ and 0 otherwise. Then,

$$(E\,(1/G_r)\,E^T)_{ij} = \sum_{k=1}^{n}(e_{ik}J_{-r,s}(y_k)e_{jk}) = \sum_{\substack{y_k \in E(t_i) \\ y_k \in E(t_j)}} J_{-r,s}(y_k) = \sum_{y_k \in E((t_i,t_j))} J_{-r,s}(y_k) = [q(t_i, t_j)]^{-r}.$$

□

Example 3.3. *If $T = \{1, 1+i, 2+i\}$ is a q-ordered factor-closed set in the Euclidean domain $\mathbb{Z}[i]$ with measure $q\,(a+bi) = a^2 + b^2$, then the 2^{nd} power Reciprocal GCDQ matrix has the following factorization:*

$$E\,(1/G_r)\,E^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{4} & 0 \\ 0 & 0 & -\frac{24}{25} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{4} & 1 \\ 1 & 1 & \frac{1}{25} \end{bmatrix} = \left(1/(T^2)_q\right)$$

Theorem 3.4. *If $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered set of non-zero non-associate elements in S, then the Reciprocal power GCDQ matrix $\left(1/(T^r)_q\right)$ can be decomposed into a product of an $m \times n$ matrix $(1/G_r)$ and its $n \times m$ incidence matrix $(1/B_r)$ for some positive integer $n \geq m$, where the non-zero entries of $(1/G_r)$ are $J_{-r,s}(d)$ for some divisor d in the minimal factor-closed set D containing T in S.*

Proof. Let $D = \{y_1, y_2, ..., y_n\}$ be the minimal factor-closed set containing $T = \{t_1, t_2, ..., t_m\}$ in S, and let $E(t)$ be a complete set of non-associate divisors d for every t_i in T. Consider the $m \times n$ matrix $(1/G_r) = (g_{ij})$ such that $g_{ij} = J_{-r,s}(y_j)$ if $y_j \in E(t_i)$ and 0 otherwise, and the $n \times m$ matrix $(1/B_r) = (b_{ij})$ such that $b_{ij} = 1$ if $g_{ji} \neq 0$ and 0 otherwise, which is an incidence matrix relative to $(1/G_r)$. Then,

$$((1/G_r)\,(1/B_r))_{ij} = \sum_{k=1}^{n}(g_{ik}b_{kj}) = \sum_{y_k \in E(t_i),\ y_k \in E(t_j)} J_{-r,s}(y_k) = \sum_{y_k \in E((t_i,t_j)_p)} J_{-r,s}(y_k) = [q(t_i, t_j)]^{-r}.$$

□

Example 3.5. *If T is defined as above, then*

$$(1/G_2)\,(1/B_2) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -\frac{3}{4} & 0 \\ 1 & 0 & -\frac{24}{25} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{4} & 1 \\ 1 & 1 & \frac{1}{25} \end{bmatrix} = \left(1/(T^2)_q\right)$$

Theorem 3.6. *If $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered set of non-zero non-associate elements in S, and if $D = \{y_1, y_2, ..., y_n\}$ is the minimal factor-closed set containing T in S, then the Reciprocal power GCDQ matrix can be decomposed into the product $\left(1/(T^r)_q\right) = (1/G_r)\,(1/G_r)^T$, where $(1/G_r)$ is an $m \times n$ matrix.*

Proof. Let $\overline{F}$ be an extension of the field of fractions F over S in which $J_{-r,s}(t)$ has a square root for every $t_i \in T$. Let $D = \{y_1, y_2, ..., y_n\}$ be the minimal factor-closed set containing T in S, and let $E(x)$ be a complete set of distinct non-associate divisors d of x in S. Define the $m \times n$ matrix $(1/G_r)$ whose ij^{th} entry is $g_{ij} = \sqrt{J_{-r,s}(y_j)}$ if $y_j \in E(t_i)$, and 0 otherwise. Then,

$$(G_{-r}G_{-r}^T)_{ij} = \sum_{k=1}^{n}(g_{ik}g_{jk}) = \sum_{\substack{y_k \in E(t_i) \\ y_k \in E(t_j)}} \sqrt{J_{-r,s}(y_k)}\,\sqrt{J_{-r,s}(y_k)} = \sum_{y_k \in E((t_i,t_j)_p)} J_{-r,s}(y_k) = [q(t_i, t_j)_p]^{-r} = (t_{ij})^{-r}.$$

□

Example 3.7. *If $T = \{1, 1+i, 2+i\}$, then*

$$(1/G_2)\,(1/G_2)^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \sqrt{-\frac{3}{4}} & 0 \\ 1 & 0 & \sqrt{-\frac{24}{25}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & \sqrt{-\frac{3}{4}} & 0 \\ 0 & 0 & \sqrt{-\frac{24}{25}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{4} & 1 \\ 1 & 1 & \frac{1}{25} \end{bmatrix} = \left(1/(T^r)_q\right)$$

3.2. Determinants of Reciprocal Power GCDQ Matrices over Euclidean Domains

Theorem 3.8. *If $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered factor-closed set of non-zero non-associate elements in S, then*

$$\det\left(1/(T^r)_q\right) = \prod_{i=1}^{m} J_{-r,s}(t_i).$$

Proof. Since $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered factor-closed set of non-zero non-associate elements in S, then $T \approx D = \{y_1, y_2, ..., y_m\}$ and $\left(1/(T^r)_q\right) = E\,(1/G_r)\,E^T$, where E is a lower triangular matrix with diagonal entries $e_{ij} = 1$. Thus,

$$\det\left(1/(T^r)_q\right) = \det\left(E\,(1/G_r)\,E^T\right) = \det(E)\det(1/G_r)\det(E^T) = \det(1/G_r) = \prod_{i=1}^{m} J_{-r,s}(t_i).$$

□

Note that the proof of the above theorem could be obtained by using the other factorizations.

Example 3.9. *If $T = \{1, 1+i, 2+i\}$, then* $\det\left(1/(T^2)_q\right) = J_{-2,s}(1) \times J_{-2,s}(1+i) \times J_{-2,s}(2+i) = 1 \times \frac{-3}{4} \times \frac{-24}{25} = \frac{18}{25}$

Theorem 3.10. *Let $D = \{y_1, y_2, ..., y_n\}$ be the minimal q-ordered factor-closed set of non-zero non-associate elements in S containing $T = \{t_1, t_2, ..., t_m\}$ with $m < n$, $y_1 <_q y_2 <_q ... <_q y_n$, and $t_1 <_q t_2 <_q ... <_q t_m$. For some indices k_i such that $1 < k_1 < k_2 < ... < k_m < n$, let $E_{r_{(k_1,k_2,...,k_m)}}$ be the submatrix consisting of the $k_1^{th}, k_2^{th}, ..., k_m^{th}$ columns of E. Then,*

$$\det\left(1/(T^r)_q\right) = \sum_{1 \le k_1 < k_2 < ... < k_m \le n} \left(\left(\det E_{r_{(k_1,k_2,...,k_m)}}\right)^2 \prod_{i=1}^{m} J_{-r,s}(y_{k_i})\right).$$

Proof. Let $E = (1/B_r)^T$ so that $\left(1/(T^r)_q\right) = (1/G_r)(1/B_r) = (1/G_r)E^T$. Hence, the ij^{th} entry of $(1/G_r)$ may be written as $g_{ij} = e_{ij}J_{-r,s}(y_j)$, where $e_{ij} = 1$ if $y_j \in E(t_i)$ and 0 otherwise. Define, for some indices k_i such that $1 < k_1 < k_2 < ... < k_m < n$, the submatrices $G_{r_{(k_1,k_2,...,k_m)}}$ and $E_{r_{(k_1,k_2,...,k_m)}}$ which consist of the $k_1^{th}, k_2^{th}, ..., k_m^{th}$ columns of matrix $(1/G_r)$, and matrix E, respectively. Then, $G_{r_{(k_1,k_2,...,k_m)}} = E_{r_{(k_1,k_2,...,k_m)}}D$, where D is a $m \times m$ diagonal submatrix of $(1/G_r)$ whose diagonal elements are $d_{ii} = J_{-r,s}(y_{k_i})$. Therefore,

$$\det(G_{r_{(k_1,k_2,...,k_m)}}) = \det(E_{r_{(k_1,k_2,...,k_m)}})\left(\prod_{i=1}^{m} d_{ii}\right).$$

Applying Cauchy-Binet formula we obtain that

$$\begin{aligned}
\det\left(1/(T^r)_q\right) = \det\left((1/G_r)E^T\right) &= \sum_{1\le k_1<k_2...<k_m\le n}\left(\left(\det G_{r_{(k_1,k_2,...,k_m)}}\right)\left(\det E_{r_{(k_1,k_2,...,k_m)}}\right)^T\right)\\
&= \sum_{1\le k_1<...<k_m\le n}\det(E_{r_{(k_1,k_2,...,k_m)}})\left(\prod_{i=1}^{m}J_{-r,s}(y_{k_i})\right)\left(\det E_{r_{(k_1,k_2,...,k_m)}}\right)^T\\
&= \sum_{1\le k_1<k_2...<k_m\le n}\left(\left(\prod_{i=1}^{m}J_{-r,s}(y_{k_i})\right)\left(\det E_{r_{(k_1,k_2,...,k_m)}}\right)^2\right)
\end{aligned}$$

□

Example 3.11. *Let $T = \{1, 2, 1 + 3i, 5\}$ be an arbitrary set in $\mathbb{Z}[i]$. Since T is not factor-closed in $\mathbb{Z}[i]$, let $D = \{1, 1 + i, 2, 2 + i, 1 + 2i, 1 + 3i, 5\}$ be the minimal factor-closed set containing T in $\mathbb{Z}[i]$. It $r = 2$, then*

$$\left(1/(T^2)_q\right) = \begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & \frac{1}{16} & \frac{1}{4} & 1\\ 1 & \frac{1}{4} & \frac{1}{100} & \frac{1}{25}\\ 1 & 1 & \frac{1}{25} & \frac{1}{625}\end{bmatrix},\ (1/G_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & \frac{-3}{4} & \frac{-3}{16} & 0 & 0 & 0 & 0\\ 1 & \frac{-3}{4} & 0 & \frac{-24}{25} & 0 & \frac{18}{25} & 0\\ 1 & 0 & 0 & \frac{-24}{25} & \frac{-24}{25} & 0 & \frac{576}{625}\end{bmatrix},\ E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1 & 1 & 0 & 1\end{bmatrix}$$

Hence,

$$\begin{aligned}
\det\left(1/(T^2)_q\right) &= \sum_{1\le k_1<k_2.....<k_4\le 7}\left(\left(\prod_{i=1}^{4}J_{-2,s}(y_{k_i})\right)\left(\det E_{(k_1,k_2,k_3,k_4)_2}\right)^2\right)\\
&= J_{-2,s}(1)J_{-2,s}(1+i)J_{-2,s}(2)J_{-2,s}(2+i)[\det E_{(1,2,3,4)_2}]^2\\
&+ J_{-2,s}(1)J_{-2,s}(1+i)J_{-2,s}(2)J_{-2,s}(1+2i)[\det E_{(1,2,3,5)_2}]^2\\
&+ J_{-2,s}(1)J_{-2,s}(1+i)J_{-2,s}(2)J_{-2,s}(5)[\det E_{(1,2,3,7)_2}]^2\\
&+ J-_{2,s}(1)J_{-2,s}(1+i)J_{-2,s}(2+i)J_{-2,s}(1+2i)[\det E_{(1,2,4,5)_2}]^2\\
&+ J_{-2,s}(1)J_{-2,s}(1+i)J_{-2,s}(2+i)J_{-2,s}(1+3i)[\det E_{(1,2,4,6)_2}]^2\\
&+ J_{-2,s}(1)J_{-2,s}(1+i)J_{-2,s}(2+i)J_{-2,s}(5)[\det E_{(1,2,4,7)_2}]^2\\
&+ J_{-2,s}(1)J_{-2,s}(1+i)J_{-2,s}(1+2i)J_{-2,s}(1+3i)[\det E_{(1,2,5,6)_2}]^2\\
&+ J_{-2,s}(1)J_{-2,s}(1+i)J_{-2,s}(1+3i)J_{-2,s}(5)[\det E_{(1,2,6,7)_2}]^2\\
&+ J_{-2,s}(1)J_{-2,s}(2)J_{-2,s}(2+i)J_{-2,s}(1+2i)[\det E_{(1,3,4,5)_2}]^2\\
&+ J_{-2,s}(1)J_{-2,s}(2)J_{-2,s}(2+i)J_{-2,s}(1+3i)[\det E_{(1,3,4,6)_2}]^2\\
&+ J_{-2,s}(1)J_{-2,s}(2)J_{-2,s}(2+i)J_{-2,s}(5)[\det E_{(1,3,4,7)_2}]^2\\
&+ J_{-2,s}(1)J_{-2,s}(2)J_{-2,s}(1+2i)J_{-2,s}(1+3i)[\det E_{(1,3,5,6)_2}]^2\\
&+ J_{-2,s}(1)J_{-2,s}(2)J_{-2,s}(1+3i)J_{-2,s}(5)[\det E_{(1,3,6,7)_2}]^2\\
&= \frac{6237}{12500}.
\end{aligned}$$

4. Power LCMQ Matrices over Euclidean Domains

In this section, we study the structure theorems and determinants of the r^{th} power *LCMQ* matrices defined on q-ordered sets of non-zero non-associate elements in an Euclidean domain S. Moreover, we present some examples in $\mathbb{Z}[i]$. Let $P = \{p_1, p_2, ..., p_m\}$ be a complete prime residue system of S. A non-zero element $d \in S$ is said to be a P-multiple of another element x in S if x divides d and d is associate to a product of elements in P. If x and y are both non-zero elements in S, then the least common P-multiple of x and y in S is denoted by $[x, y]_p$.

Definition 4.1. *Let $T = \{t_1, t_2, ..., t_m\}$ be a q-ordered set of non-zero non-associate elements in an Euclidean domain S with measure q and a prime residue system P. Define the r^{th} power LCMQ matrix defined on T to be the $m \times m$ square matrix $[T^r]_q$ such that its ij^{th} entry is $(t_{ij})_r = \left(q[t_i, t_j]_p\right)^r$.*

4.1. Structures of the Power LCMQ Matrices Defined on Factor Closed Sets over Euclidean Domains

Theorem 4.2. *let $T = \{t_1, t_2, ..., t_m\}$ be a q-ordered set of non-zero non-associate elements in S, then the power LCMQ matrix $[T^r]_q$ can be written, up to associates, as $[T^r]_q = D_r E G_{-r} E^T D_r$.*

Proof. Let $D = \{y_1, y_2, ..., y_n\}$ be the minimal q-ordered factor-closed set of non-zero non-associate elements in S containing $T = \{t_1, t_2, ..., t_m\}$ with $m \leq n$. Consider the $n \times n$ diagonal matrix

$$(1/G_r) = diag(J_{-r,s}(y_1), J_{-r,s}(y_2), ..., J_{-r,s}(y_n)),$$

and the $m \times m$ diagonal matrix

$$D_r = diag(q[t_1]^r, q[t_2]^r, ..., q[t_m]^r).$$

If $E = (e_{ij})$ is an $m \times n$ incidence matrix defined as $e_{ij} = 1$ if $y_j \in E(t_i)$ and 0 otherwise, then

$$(D_r E A_{-r} E^T D_r)_{ij} \approx (D_r(T^{-r})D_r) = q[t_i^r](T^{-r})_{ij} q[t_j^r] = \frac{q[t_i^r]q[t_j^r]}{[q(t_i, t_j)]^r} = q\left[\frac{t_i^r t_j^r}{(t_i, t_j)^r}\right] = [q[t_i, t_j]_p]^r = (t_{ij})^r.$$

□

Example 4.3. *Let $T = \{1, 1+i, 2+i\}$ be a factor-closed set in $\mathbb{Z}[i]$, then the determinant of the 2^{nd} power LCMQ matrix is:*

$$[T^2]_q = \begin{bmatrix} 1 & 4 & 25 \\ 4 & 4 & 100 \\ 25 & 100 & 25 \end{bmatrix}$$

and has the following decomposition

$$D_2 T_{-2} D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{4} & 1 \\ 1 & 1 & \frac{1}{25} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 25 \\ 4 & 4 & 100 \\ 25 & 100 & 25 \end{bmatrix} = [T^2]_q$$

4.2. Determinants of the Power LCMQ Matrices on Factor Closed Sets Over Euclidean Domains

Theorem 4.4. *Let $T = \{t_1, t_2, ..., t_m\}$ be a q-ordered factor-closed set of non-zero non-associate elements in an Euclidean domain S with measure q, then*

$$\det\left([T^r]_q\right) = \prod_{i=1}^{m} J_{-r,s}(t_i) q(t_i)^{2r}.$$

Proof. Since $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered factor-closed set in S, then $T \approx D = \{y_1, y_2, ..., y_m\}$, and $[T^r]_q = D_r E G_{-r} E^T D_r$, where E is a lower triangular matrix with diagonal entries $e_{ii} = 1$ and $\det(E) = 1$. Therefore,

$$\begin{aligned}\det\left([T^r]_q\right) &= \det(D_r E G_{-r} E^T D_r) = \det(D_r)\det(E)\det((1/G_r))\det(E^T)\det(D_r)\\ &= \prod_{i=1} q(t_i)^r \times \det((1/G_r)) \times \prod_{i=1}^{m} q(t_i)^r = \prod_{i=1}^{m} J_{-r,s}(t_i) q(t_i)^{2r}.\end{aligned}$$

□

Example 4.5. *Let $T = \{1, 1+i, 2+i\}$ be a factor-closed set in $\mathbb{Z}[i]$, then the determinant of the 2^{nd} power LCMQ matrix*

$$[T^2]_q = \begin{bmatrix} 1 & 4 & 25 \\ 4 & 4 & 100 \\ 25 & 100 & 25 \end{bmatrix}$$

is

$$\begin{aligned}\det\left(\left[T^2\right]_q\right) &= J_{-2,s}(1)q(1)^{2\times2} J_{-2,s}(1+i)q(1+i)^{2\times2} J_{-2,s}(2+i)q(2+i)^{2\times2}\\ &= 1 \times 1^4 \times \left(-\frac{3}{4}\right) \times 2^4 \times \left(-\frac{24}{25}\right) \times 5^4 = 7200.\end{aligned}$$

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