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Some Existence Theorems for Semilinear Neumann Problems with Landesman-Lazer Condition Revisited

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Abstract. In this paper, existence theorems are established for Neumann problems for semilinear elliptic equations at resonance together with Landesman-Lazer condition revisited. Our existence results follow as an application of the Saddle point Theorem together with a standard eigenspace decomposition.

1. Introduction and main results

In the paper, we are concerned with the following Neumann boundary value problems

$$\begin{cases} -\Delta u = \mu_k u + g(u) - h(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Δ is the Laplacian operator, $\Omega \subset R^N (N \geq 1)$ is a bounded domain with smooth boundary and outer normal vector $n = n(x)$, $\frac{\partial u}{\partial n} = n(x) \cdot \nabla u$, the function $g : R \to R$ is a bounded continuous function with $G(u) = \int_0^u g(s)ds$ as its primitive, $h \in L^2(\Omega)$ and $\mu_k, k \geq 1$, is the k-th eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Let $m \geq 1$ be a multiplicity of μ_k. Then we set the eigenvalues of (2) be the increasing sequence:.

$$\mu_1 < \mu_2 \leq \cdots \leq \mu_{k-1} < \mu_k = \cdots = \mu_{k+m-1} < \mu_{k+m} \leq \mu_{k+m+1} \leq \cdots \to \infty.$$

Define the functional φ on $H^1(\Omega)$ by

$$\varphi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\mu_k \int_\Omega |u|^2 dx - \int_\Omega G(u)dx + \int_\Omega hudx, \; u \in H^1(\Omega),$$

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where the Sobolev space $H^1(\Omega)$ is the usual space of $L^2(\Omega)$ functions with weak derivative in $L^2(\Omega)$, endowed with the norm defined by

$$\|u\| = \left(\int_\Omega |u(x)|^2 dx + \int_\Omega |\nabla u(x)|^2 dx\right)^{\frac{1}{2}}$$

for all $u \in H^1(\Omega)$. It's well known that finding solutions of problem (1) is equivalent to finding critical points of φ in $H^1(\Omega)$.

There exists a lot of published literatures related to the solvability conditions for Neumann boundary value problems, see [2][17][21][22][31][32] and there references. For problem (1), the common solvability conditions were the periodicity condition, see [26], the monotonicity condition, see[23][24], the sign condition, see[14][15], the Landesman-Lazer type condition, see[16][18][29][30][33].

We focus on the so called Landesman-Lazer condition, introduced by Lazer and Leach [20] in 1969 in the case

$$g(t, x) = \lambda_N x + h(x) - e(t),$$

where $\lambda_N = (\frac{2\pi N}{T})^2$ and h is bounded. In the settings of [20], this condition ensures existence of one periodic solution for the following problem

$$\begin{cases} u'' + g(t, x) = 0, \\ u(0) = u(T), u'(0) = u'(T), \end{cases}$$

In the case when $g(t, x) = \lambda_N x + h(t, x)$, it can be written as follows:

$$\int_{\{v>0\}} \liminf_{x\to+\infty} h(t, x)v(t)dt + \int_{\{v<0\}} \limsup_{x\to-\infty} h(t, x)v(t)dt > 0,$$

for every v solving the homogeneous equation

$$x'' + \lambda_N x = 0.$$

Just as an intuitive idea, one can qualitatively think that a suitable shape for $h(t, x)$ to satisfy such a condition requires that h is positive for $x \to +\infty$ and negative for $x \to -\infty$.

This paper [20] opened the way towards what today is usually called the Landesman-Lazer condition, introduced one year later to a semilinear elliptic problem by Landesman and Lazer [19], and read as follows:

$(LL)_\pm$ for any nontrivial ϕ in the eigenspace associated with μ_k,

$$g(\mp\infty)\int_\Omega \phi^+ dx - g(\pm\infty)\int_\Omega \phi^- dx < \int_\Omega \bar{h}\phi dx < g(\pm\infty)\int_\Omega \phi^+ dx - g(\mp\infty)\int_\Omega \phi^- dx.$$

After the pioneering works [19][20], this type of conditions has inspired several authors in the attempt of finding the right abstract formulation and providing different generalizations. Contributions in this direction were given, among others, by [1][3][4][5][7][8][9][11][12][13] for a quite rich bibliography about the subject see [10]. In particular, in [28], Tang defined the function $F(t) = 2G(t)/t - g(t)$ and the constants $\underline{F}(+\infty) = \liminf_{t\to+\infty} F(t), \overline{F}(-\infty) = \limsup_{t\to-\infty} F(t)$ to prove that a resonance problem about the first eigenvalue of a linear operator

$$\begin{cases} u''(x) + m^2 u + g(x, u) = h(x), \quad x \in (0, \pi) \\ \quad u(0) = u(\pi) = 0, \end{cases} \tag{3}$$

is solvable under the Landesman-Lazer type condition:

$$\begin{aligned} &\int_0^\pi [\overline{F}(-\infty)(\sin x)^+ - \underline{F}(+\infty)(\sin x)^-]dx \\ &< \int_0^\pi h \sin x dx < \int_0^\pi [\underline{F}(+\infty)(\sin x)^+ - \overline{F}(-\infty)(\sin x)^-]dx. \end{aligned} \tag{4}$$

Later in 2001, Tomiczek [33] studied two-point boundary value problems (3) and introduced a rather general sufficient condition so called potential Landesman-Lazer type:

(p-LL)$_\pm$ for any nontrivial ϕ in the eigenspace associated with μ_k,

$$G^\mp \int_\Omega \phi^+ dx - G^\pm \int_\Omega \phi^- dx < \int_\Omega \bar{h}\phi dx < G^\pm \int_\Omega \phi^+ dx - G^\mp \int_\Omega \phi^- dx,$$

as a generalization to conditions (4) and (LL)$_\pm$, where $G^\pm = \lim_{s\to\pm\infty} \frac{G(s)}{s}$ and in [33], $\mu_k = m^2, \phi = \sin x$. In addition, in 2001, Tang [30] considered the Neumann boundary value problem (1) under the condition similar to (4) and obtained the following results:

Theorem A[30] Suppose that $g \in C(R, R)$ such that

$$0 \le \liminf_{|t|\to\infty} \frac{g(t)}{t} \le \limsup_{|t|\to\infty} \frac{g(t)}{t} < \mu_2.$$

Assume that $h \in L^q(\Omega)$ satisfying

$$\overline{F}(-\infty) < \frac{1}{|\Omega|} \int_\Omega h(x)dx < \underline{F}(+\infty), \tag{5}$$

where $q > \frac{2N}{N+2}$ if $N \ge 3$ ($q > 1$, if $N = 1, 2$),$|\Omega|$ is the volume of Ω,

$$\underline{F}(+\infty) = \liminf_{t\to+\infty} F(t), \ \ \overline{F}(-\infty) = \limsup_{t\to-\infty} F(t),$$

and

$$F(t) = 2G(t)/t - g(t), \text{ for } t \ne 0, \ F(0) = g(0).$$

Then the problem (1), where $k = 1$, has at least one solution in the Sobolev space $H^1(\Omega)$.

Theorem B [30] Suppose that $g \in C(R, R)$ such that

$$\lim_{|t|\to\infty} \frac{g(t)}{t} = 0.$$

Assume that $h \in L^q(\Omega)$ satisfying either

$$\left| \int_\Omega h\phi dx \right| < \frac{1}{2}(\underline{F}(-\infty) - \overline{F}(+\infty)), \tag{6}$$

or

$$\left| \int_\Omega h\phi dx \right| < \frac{1}{2}(\underline{F}(+\infty) - \overline{F}(-\infty)), \tag{7}$$

for any nontrivial ϕ in the eigenspace associated with μ_k, with $\|\phi\|_1 = 1$, where $q > \frac{2N}{N+2}$ if $N \ge 3$ ($q > 1$, if $N = 1, 2$), Then the problem (1), where $k > 1$, has at least one solution in the Sobolev space $H^1(\Omega)$.

The purpose of this paper is to introduce a rather generalization of (LL)$_\pm$ and (p-LL)$_\pm$ for the existence of a solution of problem (1). For readers' convenience, we first give the following statements.

The corresponding eigenfunctions, (ϕ_n), form an orthogonal basis for both $L^2(\Omega)$ and $H^1(\Omega)$. Assume that every ϕ_n with respect to the L^2 norm $\|\phi_n\|_2 = 1, n = 1, 2, \cdots$. We split the space $H^1(\Omega)$ into the following three subspaces spanned by the eigenfunctions of (2) as follows:

$$\hat{H} = span\{\phi_1, \cdots, \phi_{k-1}\},$$

$$\bar{H} = span\{\phi_k, \cdots, \phi_{k+m-1}\},$$

$$\tilde{H} = span\{\phi_{k+m}, \phi_{k+m+1}, \cdots\}.$$

Then

$$H^1(\Omega) = \hat{H} \oplus \bar{H} \oplus \tilde{H}$$

with dim$\hat{H} = k-1$, dim$\bar{H}$= m, dim $\tilde{H} = \infty$. Of course, if $k = 1$ then $m = 1$ (μ_1 is a simple eigenvalue) and $\hat{H} = \emptyset$. We also split an element $u \in H^1(\Omega)$ as $u = \hat{u} + \bar{u} + \tilde{u}$, and split a function $h \in L^2(\Omega)$ as $h = \bar{h} + h^{\perp}$, where $\hat{u} \in \hat{H}, \bar{u} \in \bar{H}, \tilde{u} \in \tilde{H}$ and

$$\int_{\Omega} h^{\perp} v dx = 0, \ \text{ for any } v \in \bar{H}.$$

The generalization of (LL)$_{\pm}$ and (p-LL)$_{\pm}$ for the existence of a solution of problem (1), reads as follows:

(GLL)$_{\pm}$ If $\{u_n\} \subset H^1(\Omega)$ is a sequence such that $\|u_n\|_2 \to \infty$ and there exists $\phi_0 \in \bar{H}$, $\frac{u_n}{\|u_n\|_2} \to \phi_0$ in $L^2(\Omega)$ as $n \to \infty$, then

$$\lim_{n\to\infty} \left(\int_{\Omega} G(u_n)dx - \int_{\Omega} \bar{h}u_n dx \right) = \pm\infty.$$

Suppose $\|u_n\|_2 \to \infty$ and $\frac{u_n}{\|u_n\|_2} \to \phi_0$ for some eigenfunction ϕ_0. Then an easy computation yields, by l'Hospital's rule,

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{\|u_n\|_2} \left(\int_{\Omega} G(u_n)dx - \int_{\Omega} \bar{h}u_n dx \right) \\ = & \lim_{n\to\infty} \int_{\Omega} \left(\frac{G(u_n)}{u_n} dx - \bar{h} \right) \frac{u_n}{\|u_n\|_2} dx \\ = & \int_{\Omega} (g(+\infty) + \bar{h})\phi_0^+ dx - \int_{\Omega} (g(-\infty) + \bar{h})\phi_0^- dx \end{aligned}$$

and directly

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{\|u_n\|_2} \left(\int_{\Omega} G(u_n)dx - \int_{\Omega} \bar{h}u_n dx \right) &= \lim_{n\to\infty} \int_{\Omega} \left(\frac{G(u_n)}{u_n} dx - \bar{h} \right) \frac{u_n}{\|u_n\|_2} dx \\ &= \int_{\Omega} (G^+ + \bar{h})\phi_0^+ dx - \int_{\Omega} (G^- + \bar{h})\phi_0^- dx, \end{aligned}$$

where $G^{\pm} = \lim_{s\to\pm\infty} \frac{G(s)}{s}$. Due to the last two expressions above, either (LL)$_{\pm}$ or (p-LL)$_{\pm}$ imply (GLL)$_{\pm}$. In addition, from [33], we know (p-LL)$_{+}$ is more general than the condition (4). That is, (GLL)$_{\pm}$ are more general than conditions (LL)$_{\pm}$, (p-LL)$_{\pm}$ and (4).

In this paper, we consider Neumann boundary value problems (1) under the Landesman-Lazer type condition (GLL)$_{\pm}$, and obtain the existence theorems by saddle point theorem together with a standard eigenspace decomposition. The main results in the paper are next summarized.

Theorem 1. Under the hypothesis (GLL)$_{-}$, the problem (1) has at least one solution in the Sobolev space $H^1(\Omega)$.

Theorem 2. Under the hypothesis (GLL)$_{+}$, the problem (1) has at least one solution in the Sobolev space $H^1(\Omega)$.

Remark 3. Compared with conditions (LL)$_{\pm}$, (p-LL)$_{\pm}$ and (5)-(7), the advantages of (GLL)$_{\pm}$ are illustrated by some examples.

(i) The verification of $(GLL)_\pm$ does not require the existence of limits $g(\pm\infty)$ at all. Set $g(s) = \arctan s + \pi\cos s$. An easy calculation yields that

$$\lim_{|s|\to\infty} G(s) = \lim_{|s|\to\infty}\left(s\arctan s - \frac{1}{2}\ln(1+s^2) + \pi\sin s\right) = \infty,$$

which means $(GLL)_+$ holds for $h \in L^2(\Omega)^\perp$, where

$$L^2(\Omega)^\perp = \left\{h \in L^2(\Omega) : \int_\Omega h\phi dx = 0 \text{ for all } \phi \in \bar{H}\right\} \subseteq L^2(\Omega).$$

However, the limits $g(\pm\infty)$ do not exist. That is, the condition $(LL)_\pm$ do not apply.

(ii) $(GLL)_\pm$ hold for $h \in L^2(\Omega)^\perp$. However, both $(LL)_\pm$ and $(p\text{-}LL)_\pm$ do not apply even if the limits $g(\pm\infty)$ exist. Set $g(s) = \frac{\mathrm{sgn} s}{(e+|s|)\ln(e+|s|)}$. Then we easily obtain

$$\lim_{|s|\to\infty} G(s) = \lim_{|s|\to\infty}\ln(\ln(e+|s|)) = +\infty,$$

which means $(GLL)_+$ holds for $h \in L^2(\Omega)^\perp$. However, we also get $g(\pm\infty) = 0$ and $G^\pm = 0$ which, respectively, imply the conditions $(LL)_\pm$ and $(p\text{-}LL)_\pm$ are empty.

(iii) $(GLL)_\pm$ hold for $h \in L^2(\Omega)^\perp$. However, all of the conditions (5)-(7), $(LL)_\pm$ and $(p\text{-}LL)_\pm$ do not apply. Set $g(s) = \frac{2s}{1+s^2} + 2\cos s$. Then we easily obtain

$$\lim_{|s|\to\infty} G(s) = \lim_{|s|\to\infty}\ln(1+s^2) + 2\sin s = +\infty,$$

$$\lim_{|s|\to\infty}\frac{G(s)}{s} = \lim_{|s|\to\infty}\frac{\ln(1+s^2)+2\sin s}{s} = 0,$$

and

$$F(s) = \frac{2G(s)}{s} - g(s) = \frac{\ln(1+s^2)+2\sin s}{s} - \frac{2s}{1+s^2} - 2\cos s.$$

Obviously it holds

$$\underline{F}(-\infty) = \underline{F}(+\infty) = -2,\ \overline{F}(+\infty) = \overline{F}(-\infty) = 2,$$

which implies that conditions (5), (6) and (7) are empty. That is, they do not apply. Moreover, $(GLL)_+$ holds for $h \in L^2(\Omega)^\perp$. However, the conditions $(LL)_\pm$ and $(p\text{-}LL)_\pm$ do not apply since the limits $g(\pm\infty)$ do not exist and the condition $(p\text{-}LL)_\pm$ is empty by $G^\pm = 0$.

The functions $g(s)$ and $h(x)$ satisfy our Theorems but not satisfying the corresponding results published in the literature so far, such as Theorems A and B.

2. Proof of Theorems

The methods to prove the theorems are variational basically based upon minmax methods together with a standard eigenspace decomposition. To make the statements precise, let us introduce some notation.

It is well known that, by Sobolev's inequality, there exists a constant $M > 0$ such that

$$\|u\|_{L^2(\Omega)} \le M\|u\|. \tag{8}$$

Since the function g is a bounded continuous, we can easy prove that φ is continuously differentiable in $H^1(\Omega)$, in a way similar to Theorem 1.4 in [25]. To prove Theorems 1 and 2, we recall an abstract critical point theorem, i.e., the Saddle point Theorem under the (PS) condition, the readers can refer to [27].

Lemma 1 Let H be a Banach space with a decomposition $H = H^- + H^+$, where H^- and H^+ are two subspaces of H with $\dim H^- < +\infty$. Assume that $\varphi : X \longrightarrow R$ is a C^1-function, satisfying (PS) condition and

(a) there exist constants $\rho > 0$ and α such that $\varphi|_{\partial B_\rho} \leq \alpha$,

(b) there exist a constant $\beta > \alpha$ such that $\varphi|_{H^+} \geq \beta$,
Then the functional φ possesses a critical point in H.

In addition, we need the following lemmas.

Lemma 2 There exist $C_1 > 0, C_2 > 0$ such that for any $u \in H$ we have

$$\int_\Omega |\nabla \hat{u}|^2 dx - \mu_k \int_\Omega |\hat{u}|^2 dx \leq -C_1 \|\hat{u}\|^2, \tag{9}$$

$$\int_\Omega |\nabla \tilde{u}|^2 dx - \mu_k \int_\Omega |\tilde{u}|^2 dx \geq C_2 \|\tilde{u}\|^2. \tag{10}$$

Proof The inequalities (9) and (10) follow from the variational characterization of μ_k.

Lemma 3 There exist $C_3 > 0, C_4 > 0, C_5 > 0$ such that for any $u \in H$ we have

$$\left| \int_\Omega g(u)\hat{u} dx - \int_\Omega h\hat{u} dx \right| \leq C_3 \|\hat{u}\|, \tag{11}$$

$$\left| \int_\Omega g(u)\tilde{u} dx - \int_\Omega h\tilde{u} dx \right| \leq C_4 \|\tilde{u}\|, \tag{12}$$

$$\left| \int_\Omega G(u) dx - \int_\Omega hu dx \right| \leq C_5 \|u\|_2. \tag{13}$$

Proof The inequalities (11),(12) and (13) follow from the Hölder inequality, the boundedness of g and the fact $h \in L^2(\Omega)$.

Lemma 4 Under the assumption $(GLL)_\pm$, the functional φ satisfies (PS) condition. That is, $\{u_n\}$ possesses a convergent subsequence if $\{u_n\}$ is a sequence of H such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$.

Proof Step 1. We claim that $\{u_n\}$ is bounded in $L^2(\Omega)$. We argue by contradiction. So, suppose that $\|u_n\|_2 \to \infty$ as $n \to \infty$. Put $v_n = \frac{u_n}{\|u_n\|_2}$. Then $\|v_n\|_2 = 1$. So, by boundedness of $\{\varphi(u_n)\}$ and $\|u_n\|_2 \to \infty$, it holds

$$\begin{aligned} \frac{\varphi(u_n)}{\|u_n\|_2^2} &= \frac{1}{2} \int_\Omega |\nabla v_n|^2 dx - \frac{1}{2} \mu_k \int_\Omega |v_n|^2 dx - \int_\Omega \frac{G(u_n)}{\|u_n\|_2^2} dx + \frac{1}{\|u_n\|_2^2} \int_\Omega hu_n dx \\ &= \frac{1}{2} \|v_n\|^2 - \frac{\mu_k + 1}{2} - \int_\Omega \frac{G(u_n)}{\|u_n\|_2^2} dx + \frac{1}{\|u_n\|_2^2} \int_\Omega hu_n dx \to 0. \end{aligned} \tag{14}$$

Due to (13), we easily obtain

$$\int_\Omega \frac{G(u_n)}{\|u_n\|_2^2} dx + \frac{1}{\|u_n\|_2^2} \int_\Omega hu_n dx \to 0.$$

It follows from (14) that

$$\|v_n\|^2 \to \mu_k + 1,$$

which means $\{v_n\}$ is bounded in H. Passing to a subsequence, if necessary, we may assume that there exists $v \in H$ such that

$$v_n \rightharpoonup v \text{ in } H \text{ and } v_n \to v \text{ in } L^2(\Omega).$$

For arbitrary $w \in H$, then we obtain

$$\int_\Omega \nabla v_n \nabla w dx \to \int_\Omega \nabla v \nabla w dx \text{ by } v_n \rightharpoonup v \text{ in } H,$$

$$\int_\Omega v_n w dx \to \int_\Omega v w dx \;\text{ by } v_n \to v \;\text{ in } L^2(\Omega),$$

$$\frac{1}{\|u_n\|_2}\int_\Omega g(u_n) w dx \to 0 \text{ and } \frac{1}{\|u_n\|_2}\int_\Omega h w dx \to 0,$$

by the boundedness of $g, h \in L^2(\Omega)$ and the hypothesis $\|u_n\|_2 \to \infty$. Moreover, by $\varphi'(u_n) \to 0$ and $\|u_n\|_2 \to \infty$, one has

$$\begin{aligned} 0 \leftarrow \frac{(\varphi'(u_n), w)}{\|u_n\|_2} &= \int_\Omega \nabla v_n \nabla w dx - \mu_k \int_\Omega v_n w dx \\ &\quad - \frac{1}{\|u_n\|_2}\int_\Omega g(u_n) w dx + \frac{1}{\|u_n\|_2}\int_\Omega h w dx. \end{aligned} \tag{15}$$

Thus by (15), for arbitrary $w \in H$, we have

$$\int_\Omega \nabla v \nabla w dx - \mu_k \int_\Omega v w dx = 0,$$

which means $v = \phi_0 \in \bar{H}$ is an eigenfunction corresponding to μ_k. Obviously,

$$v_n = \frac{u_n}{\|u_n\|_2} \to \phi_0 \text{ in } L^2(\Omega).$$

An easy computation yields, by (9) and (11),

$$\begin{aligned} (\varphi'(u_n), \hat{u}_n) &= \int_\Omega |\nabla \hat{u}_n|^2 dx - \mu_k \int_\Omega |\hat{u}_n|^2 dx - \int_\Omega g(u_n)\hat{u}_n dx + \int_\Omega h \hat{u}_n dx \\ &\leq -C_1\|\hat{u}_n\|^2 + C_3\|\hat{u}_n\|. \end{aligned} \tag{16}$$

Due to (16) and $\varphi'(u_n) \to 0$, we obtain $\|\hat{u}_n\|$ is bounded.

Similarly, it holds

$$\begin{aligned} (\varphi'(u_n), \tilde{u}_n) &= \int_\Omega |\nabla \tilde{u}_n|^2 dx - \mu_k \int_\Omega |\tilde{u}_n|^2 dx - \int_\Omega g(u_n)\tilde{u}_n dx + \int_\Omega h \tilde{u}_n dx \\ &\geq C_2\|\tilde{u}_n\|^2 - C_4\|\tilde{u}_n\|, \end{aligned}$$

which implies $\|\tilde{u}_n\|$ is bounded by $\varphi'(u_n) \to 0$.

Now we rewrite $\varphi(u_n)$ as follows:

$$\begin{aligned} \varphi(u_n) &= \underbrace{\frac{1}{2}\int_\Omega |\nabla \hat{u}_n|^2 dx - \frac{1}{2}\mu_k \int_\Omega |\hat{u}_n|^2 dx}_{A} + \underbrace{\frac{1}{2}\int_\Omega |\nabla \tilde{u}_n|^2 dx - \frac{1}{2}\mu_k \int_\Omega |\tilde{u}_n|^2 dx}_{B} \\ &\quad \underbrace{- \int_\Omega G(u_n) dx + \int_\Omega \bar{h} u_n dx}_{C} + \underbrace{\int_\Omega h^\perp \hat{u}_n dx + \int_\Omega h^\perp \tilde{u}_n dx}_{D}. \end{aligned} \tag{17}$$

Since $\|\hat{u}_n\|$ and $\|\tilde{u}_n\|$ are bounded, A, B and D are bounded. Moreover, since $\|u_n\|_2 \to \infty$, $\frac{u_n}{\|u_n\|_2} \to \phi_0$, and $(\mathrm{GLL})_\pm$ holds, we have

$$- \int_\Omega G(u_n) dx + \int_\Omega \bar{h} u_n dx \to -\infty \;\text{ and }\; +\infty,$$

by $(GLL)_+$ and $(GLL)_-$ respectively. That is, $C \to \pm\infty$. Thus, by (17) it holds

$$\varphi(u_n) \to \pm\infty.$$

Obviously it contradicts the assumption of the boundedness of $\varphi(u_n)$. So $\{u_n\}$ is bounded in $L^2(\Omega)$.

Step 2. We claim that $\{u_n\}$ is bounded in H. In fact, we again use the following equation:

$$\begin{aligned}\varphi(u_n) &= \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \frac{1}{2}\mu_k \int_\Omega |u_n|^2 dx - \int_\Omega G(u_n)dx + \int_\Omega h u_n dx \\ &= \frac{1}{2}\|u_n\|^2 - \frac{\mu_k + 1}{2}\int_\Omega |u_n|^2 dx - \int_\Omega G(u_n)dx + \int_\Omega h u_n dx. \qquad (18)\end{aligned}$$

Since $\{u_n\}$ is bounded in $L^2(\Omega)$, $\int_\Omega |u_n|^2 dx$, $\int_\Omega G(u_n)dx$ and $\int_\Omega h u_n dx$ are bounded. Moreover $\varphi(u_n)$ is bounded, thus by (18), we have $\|u_n\|$ must be also bounded.

Step 3. We claim $\{u_n\}$ has a strongly convergent subsequence in H. In fact, since $\|u_n\|$ is bounded in H, $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$ for the convenience, such that

$$u_n \rightharpoonup u \text{ in } H \text{ and } u_n \to u \text{ in } L^2(\Omega).$$

Then one has

$$-\mu_k \int_\Omega u_n(u_n - u)dx - \int_\Omega g(u_n)(u_n - u)dx + \int_\Omega h(u_n - u)dx \to 0.$$

Moreover, it holds

$$\begin{aligned}0 \leftarrow (\varphi'(u_n), u_n - u) &= \int_\Omega \nabla u_n \nabla(u_n - u)dx - \mu_k \int_\Omega u_n(u_n - u)dx \\ &\quad - \int_\Omega g(u_n)(u_n - u)dx + \int_\Omega h(u_n - u)dx.\end{aligned}$$

So we deduce that

$$\int_\Omega \nabla u_n \nabla(u_n - u)dx \to 0.$$

That is,

$$\int_\Omega |\nabla u_n|^2 dx - \int_\Omega \nabla u_n \nabla u dx \to 0.$$

Due to the weak convergence $u_n \rightharpoonup u$ in H, it holds

$$\int_\Omega \nabla u_n \nabla u dx - \int_\Omega \nabla u \nabla u dx \to 0.$$

Thus we get

$$\int_\Omega |\nabla u_n|^2 dx - \int_\Omega |\nabla u|^2 dx \to 0,$$

which , together with $u_n \to u$ in $L^2(\Omega)$, implies

$$\int_\Omega |\nabla u_n|^2 dx + \int_\Omega |u_n|^2 dx = \|u_n\|^2 \to \|u\|^2 = \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 dx.$$

The uniform convexity of H then implies that $u_n \to u$ in H. Hence the functional φ satisfies (PS) condition.

Proof of Theorem 1. Under the assumption $(GLL)_-$, we set $H = H^1(\Omega) = H^- \oplus H^+$, where $H^- = \hat{H}$ is a finite dimension subspace and $H^+ = \bar{H} + \tilde{H}$.

On the one hand, we claim that there is a constant β such that

$$\inf_{u\in H^+}\varphi(u)\geq\beta.$$

If not, there exists a sequence $\{u_n\}\subset H^+$ such that

$$\lim_{n\to\infty}\varphi(u_n)=-\infty. \tag{19}$$

Then $\|u_n\|_2\to\infty$, and for $v_n=\frac{u_n}{\|u_n\|_2}\in H^+$, by (19) we obtain

$$\begin{aligned}0\geq\frac{\varphi(u_n)}{\|u_n\|_2^2} &= \frac{1}{2}\int_\Omega|\nabla v_n|^2dx-\frac{1}{2}\mu_k\int_\Omega|v_n|^2dx-\int_\Omega\frac{G(u_n)}{\|u_n\|_2^2}dx+\int_\Omega h\frac{u_n}{\|u_n\|_2^2}dx\\ &= \frac{1}{2}\|v_n\|^2-\frac{\mu_k+1}{2}-\int_\Omega\frac{G(u_n)}{\|u_n\|_2^2}dx+\int_\Omega h\frac{u_n}{\|u_n\|_2^2}dx.\end{aligned} \tag{20}$$

However, by (13), we know

$$-\int_\Omega\frac{G(u_n)}{\|u_n\|_2^2}dx+\int_\Omega h\frac{u_n}{\|u_n\|_2^2}dx\to 0. \tag{21}$$

So by (20) and (21), we get

$$\|v_n\|^2\to\mu_k+1,$$

which implies $\|v_n\|$ is bounded. Passing to a subsequence, if necessary, we may assume that there is $v\in H^+$ such that

$$v_n\rightharpoonup v \text{ in } H \text{ and } v_n\to v \text{ in } L^2(\Omega).$$

Due to the weak lower semicontinuity of the norm in H, we know

$$\liminf_{n\to\infty}\int_\Omega|\nabla v_n|^2dx\geq\int_\Omega|\nabla v|^2dx. \tag{22}$$

Thus by (20), (21) and (22), we have

$$\int_\Omega|\nabla v|^2dx-\mu_k\int_\Omega|v|^2dx\leq 0,$$

which, together with (10), implies that $v=\phi_0\in\bar{H}$ is an eigenfunction associated with μ_k. Clearly,

$$v_n=\frac{u_n}{\|u_n\|_2}\to\phi_0 \text{ in } L^2(\Omega).$$

For all $u_n=\tilde{u}_n+\bar{u}_n\in H^+$, one has

$$\begin{aligned}\varphi(u_n) &= \frac{1}{2}\int_\Omega|\nabla\tilde{u}_n|^2dx-\frac{1}{2}\mu_k\int_\Omega|\tilde{u}_n|^2dx-\int_\Omega G(u_n)dx+\int_\Omega\bar{h}u_ndx+\int_\Omega h^\perp\tilde{u}_ndx\\ &\geq C_2\|\tilde{u}\|^2-\|h^\perp\|_2\|\tilde{u}\|_2-\int_\Omega G(u_n)dx+\int_\Omega\bar{h}u_ndx,\end{aligned} \tag{23}$$

which, together with $(GLL)_-$, yields

$$\varphi(u_n)\to+\infty \text{ as } n\to+\infty.$$

Obviously it contradicts with (9). That is, the conclusion is verified.

On the other hand, for $\hat{u}\in H^-$, we have

$$\begin{aligned}\varphi(\hat{u}) &= \frac{1}{2}\int_\Omega|\nabla\hat{u}|^2dx-\frac{1}{2}\mu_k\int_\Omega|\hat{u}|^2dx-\int_\Omega G(\hat{u})dx+\int_\Omega h\hat{u}dx\\ &\leq -C_1\|\hat{u}\|^2-\int_\Omega G(\hat{u})dx+\int_\Omega h\hat{u}dx,\end{aligned}$$

which implies

$$\varphi(\hat{u}) \to -\infty \text{ as } \|\hat{u}\| \to +\infty.$$

Hence there exist constants α and $R > 0$ such that

$$\sup_{u \in \partial D} \varphi(u) < \alpha < \beta,$$

where $D = \{u \in H^- | \ \|u\| \leq R\}$.

Therefore, the hypotheses (a) and (b) in Lemma 1 are satisfied. Recall that the functional φ satisfies (PS) condition in Lemma 4, the proof of Theorem 1 is finished via Lemma 1.

Proof of Theorem 2. Under the assumption $(\mathrm{GLL})_+$, we put $H = H^1(\Omega) = H^- \oplus H^+$, where

$$H^- = \hat{H} \oplus \bar{H} \text{ and } H^+ = \tilde{H}.$$

On the one hand, we claim that

$$\lim_{\|u\| \to \infty} \varphi(u) = -\infty, \ \ u \in H^-.$$

If not, there exist a sequence $\{u_n\}$ in H^- and a constant C_6 such that $\|u_n\| \to \infty$ and

$$\varphi(u_n) \geq C_6. \tag{24}$$

Since H^- is a finite dimension space, the two norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent on H^-. In fact, for all $u \in H^-$, one has

$$\int_\Omega |\nabla u|^2 dx - \mu_k \int_\Omega |u|^2 dx \leq 0.$$

Thus it holds

$$\|u\|^2 = \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 dx \leq (1 + \mu_k) \int_\Omega |u|^2 dx = (1 + \mu_k)\|u\|_2^2. \tag{25}$$

Obviously, by the definition of the two norms, one has

$$\|u\|_2^2 = \int_\Omega |u|^2 dx \leq \|u\|^2. \tag{26}$$

Due to (25) and (26), the two norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent on H^-. Then it holds

$$\|u_n\|_2 \to \infty.$$

Put $v_n = \frac{u_n}{\|u_n\|_2} \in H^-$. Since H^- is a finite dimension space, there exists $v \in H^-$ satisfying

$$v_n \to v \text{ both in } H \text{ and } L^2(\Omega). \tag{27}$$

Moreover, by (13), we know

$$-\int_\Omega \frac{G(u_n)}{\|u_n\|_2^2} dx + \int_\Omega \frac{h u_n}{\|u_n\|_2^2} dx \to 0. \tag{28}$$

Then via (27) and (28) we obtain

$$\begin{aligned} 0 &\leq \liminf_{n \to \infty} \frac{\varphi(u_n)}{\|u_n\|_2^2} \\ &= \liminf_{n \to \infty} \left[\frac{1}{2} \int_\Omega |\nabla v_n|^2 dx - \frac{1}{2}\mu_k \int_\Omega |v_n|^2 dx - \int_\Omega \frac{G(u_n)}{\|u_n\|_2^2} dx + \int_\Omega \frac{h u_n}{\|u_n\|_2^2} dx \right] \\ &= \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \frac{1}{2}\mu_k \int_\Omega |v|^2 dx. \end{aligned}$$

However, we all know

$$\frac{1}{2}\int_{\Omega}|\nabla v|^2dx-\frac{1}{2}\mu_k\int_{\Omega}|v|^2dx\leq 0.$$

Thus it holds

$$\frac{1}{2}\int_{\Omega}|\nabla v|^2dx=\frac{1}{2}\mu_k\int_{\Omega}|v|^2dx,$$

which implies that $v=\phi_0\in\bar{H}$ is an eigenfunction associated with μ_k. Clearly,

$$v_n=\frac{u_n}{\|u_n\|_2}\to\phi_0 \text{ in } L^2(\Omega).$$

For all $u_n=\hat{u}_n+\bar{u}_n\in H^-$, one has

$$\begin{aligned}\varphi(u_n) &= \frac{1}{2}\int_{\Omega}|\nabla\hat{u}_n|^2dx-\frac{1}{2}\mu_k\int_{\Omega}|\hat{u}_n|^2dx-\int_{\Omega}G(u_n)dx+\int_{\Omega}\bar{h}u_ndx+\int_{\Omega}h^{\perp}\hat{u}_ndx\\ &\leq -C_1\|\hat{u}_n\|^2+\|h^{\perp}\|_2\|\hat{u}_n\|_2-\int_{\Omega}G(u_n)dx+\int_{\Omega}\bar{h}u_ndx,\end{aligned}$$

which, together with $(\text{GLL})_+$, implies

$$\varphi(u_n)\to-\infty \text{ as } n\to+\infty.$$

This contradicts (24). The conclusion is verified.

On the other hand, by (8), (10) and (13), for $u\in H^+$, we have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_{\Omega}|\nabla u|^2dx-\frac{1}{2}\mu_k\int_{\Omega}|u|^2dx-\int_{\Omega}G(u)dx+\int_{\Omega}hudx\\ &\geq C_3\|u\|^2-C_5\|u\|_2\geq C_3\|u\|^2-C_7\|u\|,\end{aligned}$$

which implies

$$\varphi(u)\to+\infty \text{ as } \|u\|\to+\infty.$$

Hence there exist constants α and $R>0$ such that

$$\sup_{u\in\partial D}\varphi(u)<\alpha<\beta,$$

where $D=\{u\in H^-|\ \|u\|\leq R\}$.

Therefore, the hypotheses (a) and (b) in Lemma 1 are satisfied. Recall that the functional φ satisfies (PS) condition in Lemma 4, Theorem 2 is proved via Lemma 1.

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