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Idealization of j-Approximation Spaces

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Abstract. The current work concentrates on generating different topologies by using the concept of the ideal. These topologies are used to make more thorough studies on generalized rough set theory. The rough set theory was first proposed by Pawlak in 1982. Its core concept is upper and lower approximations. The principal goal of the rough set theory is reducing the vagueness of a concept to uncertainty areas at their borders by increasing the lower approximation and decreasing the upper approximation. For the mentioned goal, different methods based on ideals are proposed to achieve this aim. These methods are more accurate than the previous methods. Hence it is very interesting in rough set context for removing the vagueness (uncertainty).

1. Introduction

The observation that one cannot distinguish objects on the basis of given information about them is the starting point of the rough set theory. In other words, imperfect information causes indiscernibility of objects. The indiscernibility relation induces an approximation space made of equivalence classes of indiscernible objects. The originally rough set was described by a pair of approximation operator, called a lower and an upper approximation in term of these equivalence classes. An equivalence relation is sometimes difficult to be obtained in real-world problems due to the vagueness incompleteness of human knowledge. From this point of view, many proposals have been introduced for generalizing and interpreting the rough sets for more details see, [8, 9, 12, 18, 24, 25]. Lin [17] and Yao [30] studied the rough sets using neighbourhood systems for the interpretation of granules. Abd El-Monsef et al. [2] introduced mixed neighbourhood systems to approximate the rough sets. In 2014, Abd El-Monsef et al. [1] applied the concept of "j-neighborhood space" (in briefly, j-NS) to generalize the classical rough set theory by using different general topologies induced from binary relations. An interesting and natural research topic in the rough set theory is to study the rough set theory via topology [3, 14–16, 20–23, 28, 31]. Ideal is a fundamental concept in the topological spaces and plays an important role in the study of topological problems. Kuratowski [13] and Vaidyanathaswamy [27] were the first who studied the notion of the ideal topological spaces. Few researchers [6, 10, 26] interesting in applying the concept of the ideals in the rough set theory.

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One of the primary motivations of this paper is there exists a close relationship between topologies and rough sets. So, the main contribution of the present work is to generate different topologies by using the notion of ideals. It is showed that these topologies are finner than the previous one [1]. Additionally, these topologies are used to define and generalize the main concepts of rough set. This paper explores rough set theory from the point of view of topology. It generalize the notions of rough sets based on the topological space which generated by ideals. The current approximations are defined by using a closure operator and a interior operator of the topologies induced by ideals. The topologies induced by ideals are stronger than the topologies generated by j-NS which were used in the previous methods to define the approximations. The main aim of rough set is to reduce the boundary region by increasing the lower approximation and decreasing the upper approximation. This aim is achieved as it appears through the following sections.

This paper is organized as follows. After the introduction. Section 2 presents the main concepts of Abd El-Monsef et al.'s approach [1] and the necessary definitions required in the sequel to the present work. The main purpose of Section 3 is to generate different topologies by using ideal $\mathcal{I}$. The relationships among these topologies are presented. The current topologies are compared to the previous one in [1] and shown to be more general. If $\mathcal{I} = \{\phi\}$, then the current definitions are coincided with Abd El-Monsef et al.'s [1] definitions. So, Abd El-Monsef et al.'s [1] definitions are special case of the current definitions. In Section 4, a new approximations namely $\mathcal{I}$-j-approximations are constructed by using the generated topologies which are introduced in the previous section. Moreover, the basic properties of this new type of approximations are presented. These approximations are extended the notation of j-approximations [1]. Theorem 4.1 and Corollary 4.1 are introduced the comparisons between the current approximations and the previous one [1]. Theorem 4.1 shows that the present method reduced the boundary region by increasing the $\mathcal{I}$-j-lower approximations and decreasing the $\mathcal{I}$-j-upper approximations with the comparison of previous method 2.5 [1]. Moreover, Corollary 4.1 shows that the current accuracy is greater than the previous one. At the end of this section, the relationships among the $\mathcal{I}$-j-lower, $\mathcal{I}$-j-upper approximations, $\mathcal{I}$-j-boundary regions and $\mathcal{I}$-j-accuracy are summarized in Table 2. Finally, new method is suggested for new approximations based on ideal in Section 5. The method is depended on using the properties of ideal in the definition of j-approximation spaces instead of using the usual properties of interior and closure. This method is satisfied all properties of the previous method [1]. Comparisons between this type of approximations, the approximations in the previous section and Abd El-Monsef et al.'s approximations 2.5 [1] are studied. The conclusion of this work is discussed in Section 6.

2. Preliminaries

The aim of this section is to present the basic concepts and properties of ideals, j-neighborhood space and j-approximations.

Definition 2.1. *[7] A non-empty collection $\mathcal{I}$ of subsets of a set X is called an ideal on X, if it satisfies the following conditions*

1. $A \in \mathcal{I}$ *and* $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
2. $A \in \mathcal{I}$ *and* $B \subseteq A \Rightarrow B \in \mathcal{I}$.

i.e., $\mathcal{I}$ is closed under finite unions and subsets.

Definition 2.2. *[1] Let R be an arbitrary binary relation on a non-empty finite set U. The j-neighborhood of* $x \in U(N_j(x)), j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$ *is defined as:*

1. *r-neighborhood:* $N_r(x) = \{y \in U : xRy\}$.

2. *l-neighborhood:* $N_l(x) = \{y \in U : yRx\}$.
3. $< r >$*-neighborhood:* $N_{<r>}(x) = \bigcap_{x \in N_r(y)} N_r(y)$.
4. $< l >$*-neighborhood:* $N_{<l>}(x) = \bigcap_{x \in N_l(y)} N_l(y)$
5. *i-neighborhood:* $N_i(x) = N_r(x) \cap N_l(x)$.
6. *u-neighborhood:* $N_u(x) = N_r(x) \cup N_l(x)$.
7. $< i >$*-neighborhood:* $N_{<i>}(x) = N_{<r>}(x) \cap N_{<i>}(x)$.
8. $< u >$*-neighborhood:* $N_{<u>}(x) = N_{<r>}(x) \cup N_{<i>}(x)$.

Remark 2.1. *It should be noted that the concept of j-neighborhood of* $x \in U(N_j(x))$, $j \in \{r, l, < r >, < l >, i, u, < i >\}$, *in [1] is the same as the notion of*

1. *the after set and fore sets in [5] if* $j = r, l$ *respectively.*
2. *the intersection of after set and fore sets and their union in [29] if* $j = i, u$ *respectively.*
3. *the minimal right set and the minimal left set in [4] if* $j =< r >, < l >$ *respectively.*
4. *the intersection of minimal right set and minimal left set in [11] if* $j =< i >$.

Definition 2.3. *[1] Let R be an arbitrary binary relation on a non-empty finite set U and* $\xi_j : U \to P(U)$ *be a mapping which assigns for each x in U its j-neighborhood in* $P(U)$*. The triple* (U, R, ξ_j) *is called a j-neighborhood space (in briefly, j-NS).*

Theorem 2.1. *[1] Let* (U, R, ξ_j) *be a j-NS, and* $A \subseteq U$*. Then,* $\forall j \in \{r, l, < r >, < l >, i, u, < i >, < u >\}$*, the collection* $\tau_j = \{A \subseteq U : \forall p \in A, N_j(p) \subseteq A\}$ *is a topology on U.*

Definition 2.4. *[1] Let* (U, R, ξ_j) *be a j-NS. A subset* $A \subseteq U$ *is called j-open set if* $A \in \tau_j$*, and the complement of j-open set is called j-closed set. The family* Γ_j *of all j-closed sets of a j-neighborhood space is defined by* $\Gamma_j = \{F \subseteq U : F' \in \tau_j\}$*, where* F' *is the complement of F.*

Definition 2.5. *[1] Let* (U, R, ξ_j) *be a j-NS,* $A \subseteq U$ *and* $\forall j \in \{r, l, < r >, < l >, i, u, < i >, < u >\}$*. The j-lower, j-upper approximations, j-boundary regions and j-accuracy of A are defined respectively by:*
$\underline{R}_j(A) = \cup\{G \in \tau_j : G \subseteq A\} = int_j(A)$, *where* $int_j(A)$ *represents j-interior of A.*
$\overline{R}_j(A) = \cap\{H \in \Gamma_j : A \subseteq H\} = cl_j(A)$, *where* $cl_j(A)$ *represents j-closure of A.*
$B_j(A) = \overline{R}_j(A) - \underline{R}_j(A)$.
$\sigma_j(A) = \frac{|\underline{R}_j(A)|}{|\overline{R}_j(A)|}$, *where* $|\overline{R}_j(A)| \neq 0$.

Definition 2.6. *[1] Let* (U, R, ξ_j) *be a j-NS, and* $\forall j \in \{r, l, < r >, < l >, i, u, < i >, < u >\}$ *A subset* $A \subseteq U$ *is called j-exact set if* $\overline{R}_j(A) = \underline{R}_j(A)$*. Otherwise, A is called j-rough set.*

3. Generalized topology based on different neighbourhoods by using ideal

Abd El-Monsef et al. [1] introduced eight different topologies based on different neighbourhoods. In this section, I generalize these topologies by using ideal. The relationships among these topologies are presented.

Theorem 3.1. *Let* (U, R, ξ_j) *be a j-NS,* $A \subseteq U$ *and* $\mathcal{I}$ *be an ideal on U. Then,* $\forall j \in \{r, l, < r >, < l >, i, u, < i >, < u >\}$*, the collection* $\tau_j^{\mathcal{I}} = \{A \subseteq U : \forall p \in A, N_j(p) \cap A' \in \mathcal{I}\}$ *is a topology on U.*

Proof.

1. Clearly U and ϕ belong to $\tau_j^{\mathcal{I}}$.
2. Let $A_i \in \tau_j^{\mathcal{I}}$ $(\forall i \in I)$ and $a \in \cup_{i \in I} A_i$. Then,
   $\exists\ i_0 \in I$ such that $a \in A_{i_0}$
   $\Rightarrow N_j(a) \cap A'_{i_0} \in \mathcal{I}$
   $\Rightarrow N_j(a) \cap (\cup_{i \in I} A_i)' \in \mathcal{I}$
   $\Rightarrow \cup_{i \in I} A_i \in \tau_j^{\mathcal{I}}$.

3. Let $A, B \in \tau_j^{\mathcal{I}}$, and $a \in A \cap B$.
   $\Rightarrow N_j(a) \cap A' \in \mathcal{I}$ *and* $N_j(a) \cap B' \in \mathcal{I}$
   $\Rightarrow (N_j(a) \cap A') \cup (N_j(a) \cap B') \in \mathcal{I}$
   $\Rightarrow N_j(a) \cap (A' \cup B') \in \mathcal{I}$
   $\Rightarrow (N_j(a) \cap (A \cap B)') \in \mathcal{I}$
   $\Rightarrow A \cap B \in \tau_j^{\mathcal{I}}$.
   From 1, 2 and 3 $\tau_j^{\mathcal{I}}$ is a topology on U.

The current type of this topologies are finner than the previous one [1] as it is shown in the following theorem.

Theorem 3.2. *Let (U, R, ξ_j) be a j-NS, $A \subseteq U$ and $\mathcal{I}$ be an ideal on U. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$, $\tau_j \subseteq \tau_j^{\mathcal{I}}$.*

Proof. Let $A \in \tau_j$. Then, $N_j(p) \subseteq A$ $\forall p \in A$ and consequently $N_j(p) \cap A' = \phi \in \mathcal{I}$ $\forall p \in A$. Therefore, $A \in \tau_j^{\mathcal{I}}$. Hence, $\tau_j \subseteq \tau_j^{\mathcal{I}}$.

Remark 3.1. *It should be noted that*

1. *if $\mathcal{I} = \{\phi\}$ in Theorem 3.2, then the present generated topologies coincide with the previous one in Theorem 2.1 [1]. So, the current work is consider as a generalization of Abd El-Monsef et al.'s work [1].*
2. *$\tau_j \subsetneq \tau_j^{\mathcal{I}}$ as it is shown in the following example.*

Example 3.1. *Let $U = \{a, b, c, d\}$, $R = \{(a, a), (a, b), (a, c), (b, c), (c, d)\}$ and $\mathcal{I} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. It's clear that*

1. $\tau_r = \{U, \phi, \{d\}, \{c, d\}\{a, d\}, \{b, c, d\}\}$ *and* $\tau_r^{\mathcal{I}} = \{U, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. *Thus,* $\tau_r \subsetneq \tau_r^{\mathcal{I}}$.
2. $\tau_l = \{U, \phi, \{a\}, \{a, b\}, \{a, b, c\}\}$ *and* $\tau_l^{\mathcal{I}} = \{U, \phi, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. *Thus,* $\tau_l \subsetneq \tau_l^{\mathcal{I}}$.
3. $\tau_u = \{U, \phi, \}$ *and* $\tau_u^{\mathcal{I}} = \{U, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$. *Thus,* $\tau_u \subsetneq \tau_u^{\mathcal{I}}$.
4. $\tau_{<r>} = \{U, \phi, \{c\}, \{d\}, \{c, d\}, \{1, 2, c\}\}$ *and* $\tau_{<r>}^{\mathcal{I}} = \{U, \phi, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. *Thus,* $\tau_{<r>} \subsetneq \tau_{<r>}^{\mathcal{I}}$.
5. $\tau_{<l>} = \{U, \phi, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$ *and* $\tau_{<l>}^{\mathcal{I}} = \{U, \phi, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. *Thus,* $\tau_{<l>} \subsetneq \tau_{<l>}^{\mathcal{I}}$.
6. $\tau_{<u>} = \{U, \phi, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}\}$ *and* $\tau_{<u>}^{\mathcal{I}} = \{U, \phi, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$. *Thus,* $\tau_{<u>} \subsetneq \tau_{<u>}^{\mathcal{I}}$.
7. $\tau_{<i>} = \{U, \phi, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$ *and* $\tau_{<i>}^{\mathcal{I}} = \{U, \phi, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. *Thus,* $\tau_{<i>} \subsetneq \tau_{<i>}^{\mathcal{I}}$.

Similarly, we can add example to show that $\tau_i \subsetneq \tau_i^{\mathcal{I}}$.

Proposition 3.1. *Let (U, R, ξ_j) be a j-NS and $\mathcal{I}$ be an ideal on U. Then*

1. $\tau_u^{\mathcal{I}} \subseteq \tau_r^{\mathcal{I}}$ *and* $\tau_u^{\mathcal{I}} \subseteq \tau_l^{\mathcal{I}}$.
2. $\tau_r^{\mathcal{I}} \subseteq \tau_i^{\mathcal{I}}$ *and* $\tau_l^{\mathcal{I}} \subseteq \tau_i^{\mathcal{I}}$.
3. $\tau_{<u>}^{\mathcal{I}} \subseteq \tau_{<r>}^{\mathcal{I}}$ *and* $\tau_{<u>}^{\mathcal{I}} \subseteq \tau_{<l>}^{\mathcal{I}}$.
4. $\tau_{<r>}^{\mathcal{I}} \subseteq \tau_{<i>}^{\mathcal{I}}$ *and* $\tau_{<l>}^{\mathcal{I}} \subseteq \tau_{<i>}^{\mathcal{I}}$.

Proof.

(1) Let $A \in \tau_u^{\mathcal{I}}$. Then, $N_u(p) \cap A' \in \mathcal{I}\ \forall p \in A$. Thus, $(N_r(p) \cup N_l(p)) \cap A' \in \mathcal{I}\ \forall p \in A$. Hence, $N_r(p) \cap A' \in \mathcal{I}\ \forall p \in A$ and $N_l(p) \cap A' \in \mathcal{I}\ \forall p \in A$. Therefore, $A \in \tau_r^{\mathcal{I}}$ and $A \in \tau_l^{\mathcal{I}}$. Hence, $\tau_u^{\mathcal{I}} \subseteq \tau_r^{\mathcal{I}}$ and $\tau_u^{\mathcal{I}} \subseteq \tau_l^{\mathcal{I}}$. Similarly, I can prove 3.

(2) Let $A \in \tau_r^{\mathcal{I}}$. Then, $N_r(p) \cap A' \in \mathcal{I}\ \forall p \in A$. Thus, $(N_r(p) \cap N_l(p)) \cap A' \in \mathcal{I}\ \forall p \in A$. Hence, $N_i(p) \cap A' \in \mathcal{I}\ \forall p \in A$. Therefore, $A \in \tau_i^{\mathcal{I}}$. Hence, $\tau_r^{\mathcal{I}} \subseteq \tau_i^{\mathcal{I}}$. Similarly, I can prove 4.

Corollary 3.1. *Let (U, R, ξ_j) be a j-NS and $\mathcal{I}$ be an ideal on U. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$.*

1. $\tau_u^{\mathcal{I}} \subseteq \tau_r^{\mathcal{I}} \subseteq \tau_i^{\mathcal{I}}$.
2. $\tau_u^{\mathcal{I}} \subseteq \tau_l^{\mathcal{I}} \subseteq \tau_i^{\mathcal{I}}$.
3. $\tau_{<u>}^{\mathcal{I}} \subseteq \tau_{<r>}^{\mathcal{I}} \subseteq \tau_{<i>}^{\mathcal{I}}$.
4. $\tau_{<u>}^{\mathcal{I}} \subseteq \tau_{<l>}^{\mathcal{I}} \subseteq \tau_{<i>}^{\mathcal{I}}$.

Remark 3.2. *Example 3.1 shows that the inclusion in Proposition 3.1 and Corollary 3.1 can not be replaced by equality relation, as*

1. $\tau_r^{\mathcal{I}} \nsubseteq \tau_u^{\mathcal{I}}$ *and* $\tau_i^{\mathcal{I}} \nsubseteq \tau_r^{\mathcal{I}}$.
2. $\tau_l^{\mathcal{I}} \nsubseteq \tau_u^{\mathcal{I}}$ *and* $\tau_i^{\mathcal{I}} \nsubseteq \tau_l^{\mathcal{I}}$.
3. $\tau_{<r>}^{\mathcal{I}} \nsubseteq \tau_{<u>}^{\mathcal{I}}$ *and* $\tau_{<i>}^{\mathcal{I}} \nsubseteq \tau_{<r>}^{\mathcal{I}}$.
4. $\tau_{<l>}^{\mathcal{I}} \nsubseteq \tau_{<u>}^{\mathcal{I}}$ *and* $\tau_{<i>}^{\mathcal{I}} \nsubseteq \tau_{<l>}^{\mathcal{I}}$.

Remark 3.3. *Let (U, R, ξ_j) be a j-NS and $\mathcal{I}$ be an ideal on U. Then, it should be noted that*

1. $\tau_r^{\mathcal{I}}$ *is not the dual of* $\tau_l^{\mathcal{I}}$ *(see Example 3.1). Although,* τ_r *is the dual of* τ_l *as it is proved in [1].*
2. $\tau_r^{\mathcal{I}}$ *and* $\tau_{<r>}^{\mathcal{I}}$ *are not necessarily to be comparable (see Example 3.1).*
3. $\tau_l^{\mathcal{I}}$ *and* $\tau_{<l>}^{\mathcal{I}}$ *are not necessarily to be comparable (see Example 3.1).*
4. $\tau_i^{\mathcal{I}}$ *and* $\tau_{<i>}^{\mathcal{I}}$ *are not necessarily to be comparable.*

4. $\mathcal{I}$-j-approximation spaces

In this section, new approximations namely $\mathcal{I}$-j-approximations are proposed to generalize j-approximations 2.5 [1]. The current approximations are based on the generated topologies which are introduced in the previous section. The properties of the new approximations are studied and compared to Abd El-Monsef et al.'s approximations 2.5 [1].

Definition 4.1. *Let (U, R, ξ_j) be a j-NS and $\mathcal{I}$ be an ideal on U. A subset $A \subseteq U$ is called $\mathcal{I}_j$-open set if $A \in \tau_j^{\mathcal{I}}$ and the complement of $\mathcal{I}_j$-open set is called $\mathcal{I}_j$-closed set. The family $\Gamma_j^{\mathcal{I}}$ of all $\mathcal{I}_j$-closed sets of a j-neighborhood space is defined by $\Gamma_j^{\mathcal{I}} = \{F \subseteq U : F' \in \tau_j^{\mathcal{I}}\}$.*

Definition 4.2. *Let (U, R, ξ_j) be a j-NS, $A \subseteq U, \mathcal{I}$ be an ideal on U and $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$. The $\mathcal{I}_j$-lower, $\mathcal{I}_j$-upper approximations, $\mathcal{I}_j$-boundary regions and $\mathcal{I}_j$-accuracy of the approximations of A are defined respectively by:*

$\underline{R}_j^{\mathcal{I}}(A) = \cup\{G \in \tau_j^{\mathcal{I}} : G \subseteq A\} = int_j^{\mathcal{I}}(A)$, *where $int_j^{\mathcal{I}}(A)$ represents $\mathcal{I}$-j-interior of A.*

$\overline{R}_j^{\mathcal{I}}(A) = \cap\{H \in \Gamma_j^{\mathcal{I}} : A \subseteq H\} = cl_j^{\mathcal{I}}(A)$, *where $cl_j^{\mathcal{I}}(A)$ represents $\mathcal{I}$-j-closure of A.*

$B_j^{\mathcal{I}}(A) = \overline{R}_j^{\mathcal{I}}(A) - \underline{R}_j^{\mathcal{I}}(A)$.

$\sigma_j^{\mathcal{I}}(A) = \frac{|\underline{R}_j^{\mathcal{I}}(A)|}{|\overline{R}_j^{\mathcal{I}}(A)|}$, *where $|\overline{R}_j^{\mathcal{I}}(A)| \neq 0$.*

The main properties of the current $\mathcal{I}$-j-lower and $\mathcal{I}$-j-upper approximations are studied in the following proposition.

Proposition 4.1. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A, B \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$,*

1. $\underline{R}_j^{\mathcal{I}}(A) \subseteq A \subseteq \overline{R}_j^{\mathcal{I}}(A)$ *equality hold if* $A = \phi$ *or* U.
2. $A \subseteq B \Rightarrow \overline{R}_j^{\mathcal{I}}(A) \subseteq \overline{R}_j^{\mathcal{I}}(B)$.
3. $A \subseteq B \Rightarrow \underline{R}_j^{\mathcal{I}}(A) \subseteq \underline{R}_j^{\mathcal{I}}(B)$.
4. $\overline{R}_j^{\mathcal{I}}(A \cap B) \subseteq \overline{R}_j^{\mathcal{I}}(A) \cap \overline{R}_j^{\mathcal{I}}(B)$.
5. $\underline{R}_j^{\mathcal{I}}(A \cup B) \supseteq \underline{R}_j^{\mathcal{I}}(A) \cup \underline{R}_j^{\mathcal{I}}(B)$.
6. $\overline{R}_j^{\mathcal{I}}(A \cup B) \supseteq \overline{R}_j^{\mathcal{I}}(A) \cup \overline{R}_j^{\mathcal{I}}(B)$.
7. $\underline{R}_j^{\mathcal{I}}(A \cap B) \subseteq \underline{R}_j^{\mathcal{I}}(A) \cap \underline{R}_j^{\mathcal{I}}(B)$.
8. $\underline{R}_j^{\mathcal{I}}(A) = (\overline{R}_j^{\mathcal{I}}(A'))'$, $\overline{R}_j^{\mathcal{I}}(A) = (\underline{R}_j^{\mathcal{I}}(A'))'$.
9. $\overline{R}_j^{\mathcal{I}}(\overline{R}_j^{\mathcal{I}}(A)) = \overline{R}_j^{\mathcal{I}}(A)$.
10. $\underline{R}_j^{\mathcal{I}}(\underline{R}_j^{\mathcal{I}}(A)) = \underline{R}_j^{\mathcal{I}}(A)$.
11. $\underline{R}_j^{\mathcal{I}}(\underline{R}_j^{\mathcal{I}}(A)) \subseteq \overline{R}_j^{\mathcal{I}}(\underline{R}_j^{\mathcal{I}}(A))$.
12. $\underline{R}_j^{\mathcal{I}}(\overline{R}_j^{\mathcal{I}}(A)) \subseteq \overline{R}_j^{\mathcal{I}}(\overline{R}_j^{\mathcal{I}}(A))$.

The proof of this proposition is simple using the properties of $\mathcal{I}$-j-interior and $\mathcal{I}$-j-closure, so I omit it.

Remark 4.1. *Example 3.1 shows that*

1. *the inclusion in Proposition 4.1 parts 1, 4, 5, 6, 7, 11 and 12 can not be replaced by equality relation:*
   - **(i)** *for part 1, if* $A = \{c\}, \underline{R}_r^{\mathcal{I}}(A) = \phi$, *then* $A \nsubseteq \underline{R}_r^{\mathcal{I}}(A)$. *If* $A = \{d\}, \overline{R}_r^{\mathcal{I}}(A) = \{b, d\}$, *then* $\overline{R}_r^{\mathcal{I}}(A) \nsubseteq A$
   - **(ii)** *for part 4, if* $A = \{b\}, B = \{d\}, A \cap B = \phi, \overline{R}_r^{\mathcal{I}}(A) = \{b\}, \overline{R}_r^{\mathcal{I}}(B) = \{b, d\}, \overline{R}_r^{\mathcal{I}}(A \cap B) = \phi$, *then* $\overline{R}_r^{\mathcal{I}}(A) \cap \overline{R}_r^{\mathcal{I}}(B) = \{b, d\} \nsubseteq \phi = \overline{R}_r^{\mathcal{I}}(A \cap B)$.
   - **(iii)** *for part 5, if* $A = \{c\}, B = \{d\}, A \cup B = \{c, d\}, \underline{R}_r^{\mathcal{I}}(A) = \phi, \underline{R}_r^{\mathcal{I}}(B) = \{d\}, \underline{R}_r^{\mathcal{I}}(A \cup B) = \{3, 4\}$, *then* $\underline{R}_r^{\mathcal{I}}(A \cup B) = \{c, d\} \nsubseteq \{4\} = \underline{R}_r^{\mathcal{I}}(A) \cup \underline{R}_r^{\mathcal{I}}(B)$.
   - **(iv)** *for part 11, if* $A = \{d\}, \underline{R}_r^{\mathcal{I}}(\underline{R}_r^{\mathcal{I}}(A)) = A, \overline{R}_r^{\mathcal{I}}(\underline{R}_r^{\mathcal{I}}(A)) = \{b, d\}$, *then* $\overline{R}_r^{\mathcal{I}}(\underline{R}_r^{\mathcal{I}}(A)) \nsubseteq \underline{R}_r^{\mathcal{I}}(\underline{R}_r^{\mathcal{I}}(A))$.
   - **(v)** *for part 12, if* $A = \{c\}, \overline{R}_r^{\mathcal{I}}(\overline{R}_r^{\mathcal{I}}(A)) = A, \underline{R}_r^{\mathcal{I}}(\overline{R}_r^{\mathcal{I}}(A)) = \phi$, *then* $\overline{R}_r^{\mathcal{I}}(\overline{R}_r^{\mathcal{I}}(A)) \nsubseteq \underline{R}_r^{\mathcal{I}}(\overline{R}_r^{\mathcal{I}}(A))$.
2. *the converse of parts 2 and 3 is not necessarily true:*
   - **(i)** *for part 2, if* $A = \{b\}, B = \{b, d\}$, *then* $\overline{R}_r^{\mathcal{I}}(A) = \{b\}, \overline{R}_r^{\mathcal{I}}(B) = \{b, d\}$. *Therefore,* $\overline{R}_r^{\mathcal{I}}(A) \subseteq \overline{R}_r^{\mathcal{I}}(B)$, *but* $A \nsubseteq B$.
   - **(ii)** *for part 3, if* $A = \{c\}, B = \{b\}$, *then* $\underline{R}_r^{\mathcal{I}}(A) = \phi, \underline{R}_r^{\mathcal{I}}(B) = \{b\}$. *Therefore,* $\underline{R}_r^{\mathcal{I}}(A) \subseteq \underline{R}_r^{\mathcal{I}}(B)$, *but* $A \nsubseteq B$.

Definition 4.3. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on $U, A \subseteq U, \forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$. A subset A is called $\mathcal{I}$-j-definable ($\mathcal{I}$-j-exact) set if $\overline{R}_j^{\mathcal{I}}(A) = \underline{R}_j^{\mathcal{I}}(A)$. Otherwise, A is called $\mathcal{I}$-j-rough set.*

In Example 3.1 $A = \{b\}$ is $\mathcal{I}$-r-exact, while $B = \{c\}$ is $\mathcal{I}$-r-rough.

Remark 4.2. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$, the intersection of two $\mathcal{I}$-j-rough sets need not to be $\mathcal{I}$-j-rough set as in Example 3.1 $\{a, c\}$ and $\{a, d\}$, are $\mathcal{I}$-r-rough sets, $\{a, c\} \cap \{a, d\} = \{a\}$ is not $\mathcal{I}$-r-rough set.*

The following theorem and corollary present the relationships between the current approximations in Definition 4.2 and the previous one in Definition 2.5 [1].

Theorem 4.1. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$.*

1. $\underline{R}_j(A) \subseteq \underline{R}_j^{\mathcal{I}}(A)$.
2. $\overline{R}_j^{\mathcal{I}}(A) \subseteq \overline{R}_j(A)$.

Proof.

(1) $\underline{R}_j(A) = \cup\{G \in \tau_j : G \subseteq A\} \subseteq \cup\{G \in \tau_j^{\mathcal{I}} : G \subseteq A\} = \underline{R}_j^{\mathcal{I}}(A)$. (by Theorem 3.2)

(2) Similar to (1).

Corollary 4.1. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$.*

1. $B_j^{\mathcal{I}}(A) \subseteq B_j(A)$.
2. $\sigma_j(A) \leqslant \sigma_j^{\mathcal{I}}(A)$.

Corollary 4.2. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$.*

1. *Every j-exact subset in U is $\mathcal{I}$-j-exact.*
2. *Every $\mathcal{I}$-j-rough subset in U is j-rough.*

Remark 4.3. *Example 3.1 shows that the converse of parts of Corollary 4.2 is not necessarily true.*

1. *if $A = \{b\}$, then it is $\mathcal{I}$-j-exact, but it is not j-exact.*
2. *if $A = \{a\}$, then it is r-rough, but it is not $\mathcal{I}$-r-rough.*

Table 1 shows that the converse of parts of Theorem 4.1 and Corollary 4.1 is not necessarily true. This table is calculated by using Example 3.1.

Table 1: Comparison between the boundary and accuracy by using the current approximations in Definition 4.2 and the previous one in Definition 2.5 [1] at $j = r$.

2*A	The previous one in Definition 2.5 [1]				The current method in Definition 4.2			
	$\underline{R}_r(A)$	$\overline{R}_r(A)$	$B_r(A)$	$\sigma_r(A)$	$\underline{R}_r^{\mathcal{I}}(A)$	$\overline{R}_r^{\mathcal{I}}(A)$	$B_r^{\mathcal{I}}(A)$	$\sigma_r^{\mathcal{I}}(A)$
$\{a\}$	ϕ	$\{a\}$	$\{a\}$	0	$\{a\}$	$\{a\}$	ϕ	1
$\{b\}$	ϕ	$\{a, b\}$	$\{a, b\}$	0	$\{b\}$	$\{b\}$	ϕ	1
$\{c\}$	ϕ	$\{a, b, c\}$	$\{a, b, c\}$	0	ϕ	$\{c\}$	$\{c\}$	0
$\{d\}$	$\{d\}$	U	$\{a, b, c\}$	$\frac{1}{4}$	$\{d\}$	$\{b, d\}$	$\{b\}$	$\frac{1}{2}$
$\{a, b\}$	ϕ	$\{a, b\}$	$\{a, b\}$	0	$\{a, b\}$	$\{a, b\}$	ϕ	0
$\{a, c\}$	ϕ	$\{a, b, c\}$	$\{a, b, c\}$	0	$\{a\}$	$\{a, c\}$	$\{c\}$	$\frac{1}{2}$
$\{a, d\}$	ϕ	U	U	0	$\{a, d\}$	$\{a, c, d\}$	$\{c\}$	$\frac{1}{3}$
$\{b, c\}$	ϕ	$\{a, b, c\}$	$\{a, b, c\}$	0	$\{b\}$	$\{b, c\}$	$\{c\}$	$\frac{1}{2}$
$\{b, d\}$	$\{d\}$	U	U	0	$\{b, d\}$	$\{b, c, d\}$	$\{c\}$	$\frac{2}{3}$
$\{c, d\}$	$\{c, d\}$	U	U	0	$\{c, d\}$	$\{c, d\}$	ϕ	1
$\{a, b, c\}$	ϕ	$\{a, b, c\}$	$\{a, b, c\}$	0	$\{a, b\}$	$\{a, b, c\}$	$\{c\}$	$\frac{2}{3}$
$\{a, b, d\}$	$\{d\}$	U	$\{a, b, c\}$	$\frac{1}{4}$	$\{a, b, d\}$	U	$\{c\}$	$\frac{3}{4}$
$\{a, c, d\}$	$\{c, d\}$	U	$\{a, b\}$	$\frac{1}{2}$	$\{a, c, d\}$	$\{a, c, d\}$	ϕ	1
$\{b, c, d\}$	$\{b, c, d\}$	U	U	$\frac{3}{4}$	$\{b, c, d\}$	$\{b, c, d\}$	ϕ	1
U	U	U	ϕ	1	U	U	ϕ	1

For example, take $A = \{b\}$, then the boundary and accuracy by the present method in Definition 4.2 are ϕ and 1 respectively. Whereas, the boundary and accuracy by using Abd El-Monsef et al.'s method 2.5 [1] are $\{a, b\}$ and 0 respectively.

The following propositions and corollaries are introduced the relationships among the $\mathcal{I}$-j-lower, $\mathcal{I}$-j-upper approximations, $\mathcal{I}$-j-boundary regions and $\mathcal{I}$-j-accuracy.

Proposition 4.2. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$. Then, the following statements are true in general.*

1. $\underline{R}^{\mathcal{I}}_u(A) \subseteq \underline{R}^{\mathcal{I}}_r(A) \subseteq \underline{R}^{\mathcal{I}}_i(A)$.
2. $\underline{R}^{\mathcal{I}}_u(A) \subseteq \underline{R}^{\mathcal{I}}_l(A) \subseteq \underline{R}^{\mathcal{I}}_i(A)$.
3. $\underline{R}^{\mathcal{I}}_{<u>}(A) \subseteq \underline{R}^{\mathcal{I}}_{<r>}(A) \subseteq \underline{R}^{\mathcal{I}}_{<i>}(A)$.
4. $\underline{R}^{\mathcal{I}}_{<u>}(A) \subseteq \underline{R}^{\mathcal{I}}_{<l>}(A) \subseteq \underline{R}^{\mathcal{I}}_{<i>}(A)$.

Proof. By using Proposition 3.1, the proof is obvious.

Proposition 4.3. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$. Then, the following statements are true in general.*

1. $\overline{R}^{\mathcal{I}}_i(A) \subseteq \overline{R}^{\mathcal{I}}_r(A) \subseteq \overline{R}^{\mathcal{I}}_u(A)$.
2. $\overline{R}^{\mathcal{I}}_i(A) \subseteq \overline{R}^{\mathcal{I}}_l(A) \subseteq \overline{R}^{\mathcal{I}}_u(A)$.
3. $\overline{R}^{\mathcal{I}}_{<i>}(A) \subseteq \overline{R}^{\mathcal{I}}_{<r>}(A) \subseteq \overline{R}^{\mathcal{I}}_{<u>}(A)$.
4. $\overline{R}^{\mathcal{I}}_{<i>}(A) \subseteq \overline{R}^{\mathcal{I}}_{<l>}(A) \subseteq \overline{R}^{\mathcal{I}}_{<u>}(A)$.

Proof. By using Proposition 3.1, the proof is obvious.

Corollary 4.3. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$.*

1. $B^{\mathcal{I}}_i(A) \subseteq B^{\mathcal{I}}_r(A) \subseteq B^{\mathcal{I}}_u(A)$.
2. $B^{\mathcal{I}}_i(A) \subseteq B^{\mathcal{I}}_l(A) \subseteq B^{\mathcal{I}}_u(A)$.
3. $B^{\mathcal{I}}_{<i>}(A) \subseteq B^{\mathcal{I}}_{<r>}(A) \subseteq B^{\mathcal{I}}_{<u>}(A)$.
4. $B^{\mathcal{I}}_{<i>}(A) \subseteq B^{\mathcal{I}}_{<l>}(A) \subseteq B^{\mathcal{I}}_{<u>}(A)$.

Corollary 4.4. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$.*

1. $\sigma^{\mathcal{I}}_u(A) \leqslant \sigma^{\mathcal{I}}_r(A) \leqslant \sigma^{\mathcal{I}}_i(A)$.
2. $\sigma^{\mathcal{I}}_u(A) \leqslant \sigma^{\mathcal{I}}_l(A) \leqslant \sigma^{\mathcal{I}}_i(A)$.
3. $\sigma^{\mathcal{I}}_{<u>}(A) \leqslant \sigma^{\mathcal{I}}_{<r>}(A) \leqslant \sigma^{\mathcal{I}}_{<i>}(A)$.
4. $\sigma^{\mathcal{I}}_{<u>}(A) \leqslant \sigma^{\mathcal{I}}_{<l>}(A) \leqslant \sigma^{\mathcal{I}}_{<i>}(A)$.

Remark 4.4. *Let (U, R, ξ_j) be a j-NS and $\mathcal{I}$ be an ideal on U. Then, it should be noted that*

1. *$\sigma^{\mathcal{I}}_r(A)$ and $\sigma^{\mathcal{I}}_{<r>}(A)$ are not necessarily to be comparable.*
2. *$\sigma^{\mathcal{I}}_l(A)$ and $\sigma^{\mathcal{I}}_{<l>}(A)$ are not necessarily to be comparable.*
3. *$\sigma^{\mathcal{I}}_i(A)$ and $\sigma^{\mathcal{I}}_{<i>}(A)$ are not necessarily to be comparable.*
4. *$\sigma^{\mathcal{I}}_u(A)$ and $\sigma^{\mathcal{I}}_{<u>}(A)$ are not necessarily to be comparable.*

Remark 4.5. *Table 2 shows that*

1. *the comparison among the $\mathcal{I}$-j-lower, $\mathcal{I}$-j-upper approximations, $\mathcal{I}$-j-boundary regions and $\mathcal{I}$-j-accuracy in Definition 4.2 by using Example 3.1, for $j \in \{r, l, i, u\}$.*
2. *there are different methods to approximate the sets by using $\tau^{\mathcal{I}}_i$ in constructing the approximations of sets, $\tau^{\mathcal{I}}_r$, $\tau^{\mathcal{I}}_l$ and $\tau^{\mathcal{I}}_u$. The best of these methods is there are given by using $\tau^{\mathcal{I}}_i$ in constructing the approximations of sets, since the boundary regions in this case are decreased (or canceled) by increasing the lower approximation and decreasing the upper approximation. Moreover, the $\mathcal{I}$-i-accuracy is more accurate than the other types since $\sigma^{\mathcal{I}}_u(A) \leqslant \sigma^{\mathcal{I}}_r(A) \leqslant \sigma^{\mathcal{I}}_i(A)$ and $\sigma^{\mathcal{I}}_u(A) \leqslant \sigma^{\mathcal{I}}_l(A) \leqslant \sigma^{\mathcal{I}}_i(A)$.*

Table 2: Comparison between the $\mathcal{I}$-j-lower, $\mathcal{I}$-j-upper approximations, $\mathcal{I}$-j-boundary regions and $\mathcal{I}$-j-accuracy are calculated by using the current approximations in Definition 4.2.

2*A	The current approximations in Definition 4.2 for $j = r$				The current approximations in Definition 4.2 for $j = l$				The current approximations in Definition 4.2 for $j = u$				The current approximations in Definition 4.2 for $j = i$			
	$\underline{R}^{\mathcal{I}}_r(A)$	$\overline{R}^{\mathcal{I}}_r(A)$	$B^{\mathcal{I}}_r(A)$	$\sigma^{\mathcal{I}}_r(A)$	$\underline{R}^{\mathcal{I}}_l(A)$	$\overline{R}^{\mathcal{I}}_l(A)$	$B^{\mathcal{I}}_l(A)$	$\sigma^{\mathcal{I}}_l(A)$	$\underline{R}^{\mathcal{I}}_u(A)$	$\overline{R}^{\mathcal{I}}_u(A)$	$B^{\mathcal{I}}_u(A)$	$\sigma^{\mathcal{I}}_u(A)$	$\underline{R}^{\mathcal{I}}_i(A)$	$\overline{R}^{\mathcal{I}}_i(A)$	$B^{\mathcal{I}}_i(A)$	$\sigma^{\mathcal{I}}_i(A)$
$\{a\}$	$\{a\}$	$\{a\}$	ϕ	1	$\{a\}$	$\{a,b,c\}$	$\{b,c\}$	1/3	$\{a\}$	$\{a,b,c\}$	$\{b,c\}$	1/3	$\{a\}$	$\{a\}$	ϕ	1
$\{b\}$	$\{b\}$	$\{b\}$	ϕ	1	ϕ	$\{b\}$	$\{b\}$	0	ϕ	$\{b\}$	$\{b\}$	0	$\{b\}$	$\{b\}$	ϕ	1
$\{c\}$	ϕ	$\{c\}$	$\{c\}$	0	ϕ	$\{c\}$	$\{c\}$	0	ϕ	$\{c\}$	$\{c\}$	0	$\{c\}$	$\{c\}$	ϕ	1
$\{d\}$	$\{d\}$	$\{b,d\}$	$\{b\}$	1/2	$\{d\}$	$\{d\}$	ϕ	1	$\{d\}$	$\{b,c,d\}$	$\{b,c\}$	1/3	$\{d\}$	$\{d\}$	ϕ	1
$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	ϕ	1	$\{a,b\}$	$\{a,b,c\}$	$\{c\}$	2/3	$\{a,b\}$	$\{a,b,c\}$	$\{c\}$	2/3	$\{a,b\}$	$\{a,b\}$	ϕ	1
$\{a,c\}$	$\{a\}$	$\{a,c\}$	$\{c\}$	1/2	$\{a,c\}$	$\{a,b,c\}$	$\{b\}$	2/3	$\{a\}$	$\{a,b,c\}$	$\{b,c\}$	1/3	$\{a,c\}$	$\{a,c\}$	ϕ	1
$\{a,d\}$	$\{a,d\}$	$\{a,c,d\}$	$\{c\}$	2/3	$\{a,d\}$	U	$\{b,c\}$	2/4	$\{a,d\}$	U	$\{b,c\}$	2/4	$\{a,d\}$	$U\{a,d\}$	ϕ	1
$\{b,c\}$	$\{b\}$	$\{b,c\}$	$\{c\}$	1/2	ϕ	$\{b,c\}$	$\{b,c\}$	0	$\{b\}$	$\{b,c\}$	$\{c\}$	1/2	$\{b,c\}$	$\{b,c\}$	ϕ	1
$\{b,d\}$	$\{b,d\}$	$\{b,c,d\}$	$\{c\}$	2/3	$\{d\}$	$\{b,d\}$	$\{b\}$	1/2	$\{d\}$	$\{b,c,d\}$	$\{b,c\}$	1/3	$\{b,d\}$	$\{b,d\}$	ϕ	1
$\{c,d\}$	$\{c,d\}$	$\{c,d\}$	ϕ	1	$\{d\}$	$\{c,d\}$	$\{c\}$	1/2	$\{d\}$	$\{c,d\}$	$\{c\}$	1/2	$\{c,d\}$	$\{c,d\}$	ϕ	1
$\{a,b,c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{c\}$	2/3	$\{a,b,c\}$	$\{a,b,c\}$	ϕ	1	$\{a,b\}$	$\{a,b,c\}$	$\{c\}$	2/3	$\{a,b,c\}$	$\{a,b,c\}$	ϕ	1
$\{a,b,d\}$	$\{a,b,d\}$	U	$\{c\}$	3/4	$\{a,b,d\}$	U	$\{c\}$	3/4	$\{a,b,d\}$	U	$\{c\}$	3/4	$\{a,b,d\}$	$\{a,b,d\}$	ϕ	1
$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	ϕ	1	$\{a,c,d\}$	U	$\{b\}$	3/4	$\{a,c,d\}$	U	$\{b\}$	3/4	$\{a,c,d\}$	$\{a,c,d\}$	ϕ	1
$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	ϕ	1	$\{d\}$	$\{b,c,d\}$	$\{b,c\}$	1/3	$\{d\}$	$\{b,c,d\}$	$\{b,c\}$	1/3	$\{b,c,d\}$	$\{b,c,d\}$	ϕ	1
U	U	U	ϕ	1	U	U	ϕ	1	U	U	ϕ	1	U	U	ϕ	1

5. Generalized $\mathcal{I}$-j-approximation spaces

The current method is presented to redefine j-approximation spaces via ideals. The properties of suggested method are studied. The relationships between the current approximations in this section, Sections 4 and the previous one in [1] are introduced.

Definition 5.1. *Let (U, R, ξ_j) be a j-NS, $A \subseteq U$, $\mathcal{I}$ be an ideal on U and $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$. The generalized $\mathcal{I}_j$-lower, $\mathcal{I}_j$-upper approximations, $\mathcal{I}_j$-boundary regions and $\mathcal{I}_j$-accuracy of the approximations of A are defined respectively by:*

$\underline{\underline{\underline{R}}}{}^{\mathcal{I}}_j(A) = \underline{\underline{R}}{}^{\mathcal{I}}_j(A) \cap A$, *where* $\underline{\underline{R}}{}^{\mathcal{I}}_j(A) = \cup\{G \in \tau^{\mathcal{I}}_j : G \cap A' \in \mathcal{I}\}$.

$\overline{\overline{\overline{R}}}{}^{\mathcal{I}}_j(A) = \overline{\overline{R}}{}^{\mathcal{I}}_j(A) \cup A$, *where* $\overline{\overline{R}}{}^{\mathcal{I}}_j(A) = \cap\{H \in \Gamma^{\mathcal{I}}_j : A \cap H' \in \mathcal{I}\}$.

$B^{**\mathcal{I}}_j(A) = \overline{\overline{\overline{R}}}{}^{\mathcal{I}}_j(A) - \underline{\underline{\underline{R}}}{}^{\mathcal{I}}_j(A)$.

$\sigma^{**\mathcal{I}}_j(A) = \dfrac{|\underline{\underline{\underline{R}}}{}^{\mathcal{I}}_j(A)|}{|\overline{\overline{\overline{R}}}{}^{\mathcal{I}}_j(A)|}$, *where* $|\overline{\overline{\overline{R}}}{}^{\mathcal{I}}_j(A)| \neq 0$.

Before studying the main properties of $\underline{\underline{\underline{R}}}{}^{\mathcal{I}}_j(A)$ and $\overline{\overline{\overline{R}}}{}^{\mathcal{I}}_j(A)$. I must study the properties of $\underline{\underline{R}}{}^{\mathcal{I}}_j(A)$ and $\overline{\overline{R}}{}^{\mathcal{I}}_j(A)$ as it is presented in the following propositions.

Proposition 5.1. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}, \mathcal{I}$ be two ideals on U and $A, B \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$,*

1. $\underline{\underline{R}}{}^{\mathcal{I}}_j(U) = U$.
2. $A \subseteq B \Rightarrow \underline{\underline{R}}{}^{\mathcal{I}}_j(A) \subseteq \underline{\underline{R}}{}^{\mathcal{I}}_j(B)$.
3. $\underline{\underline{R}}{}^{\mathcal{I}}_j(A \cup B) \supseteq \underline{\underline{R}}{}^{\mathcal{I}}_j(A) \cup \underline{\underline{R}}{}^{\mathcal{I}}_j(B)$.
4. $\underline{\underline{R}}{}^{\mathcal{I}}_j(A \cap B) = \underline{\underline{R}}{}^{\mathcal{I}}_j(A) \cap \underline{\underline{R}}{}^{\mathcal{I}}_j(B)$.
5. *if* $A' \in \mathcal{I} \Rightarrow \underline{\underline{R}}{}^{\mathcal{I}}_j(A) = U$.
6. $\mathcal{I} \subset \mathcal{J} \Rightarrow \underline{\underline{R}}{}^{\mathcal{I}}_j(A) \subseteq \underline{\underline{R}}{}^{\mathcal{J}}_j(A)$.

Proof.

1. $\underline{\underline{R}}{}^{\mathcal{I}}_j(U) = \cup\{G \in \tau^{\mathcal{I}}_j : G \cap \phi = \phi \in \mathcal{I}\} = U$.
2. Let $G \in \underline{\underline{R}}{}^{\mathcal{I}}_j(A)$. Then, $G \cap A' = \phi \in \mathcal{I}$. Thus, $G \cap B' = \phi \in \mathcal{I}$ (as $A \subseteq B$ and from the properties of $\mathcal{I}$). Hence, $G \in \underline{\underline{R}}{}^{\mathcal{I}}_j(B)$.
3. Immediately by (2).
4. By (2), $\underline{\underline{R}}{}^{\mathcal{I}}_j(A \cap B) \subseteq \underline{\underline{R}}{}^{\mathcal{I}}_j(A) \cap \underline{\underline{R}}{}^{\mathcal{I}}_j(B)$. To prove, $\underline{\underline{R}}{}^{\mathcal{I}}_j(A) \cap \underline{\underline{R}}{}^{\mathcal{I}}_j(B) \subseteq \underline{\underline{R}}{}^{\mathcal{I}}_j(A \cap B)$. Let $G \in (\underline{\underline{R}}{}^{\mathcal{I}}_j(A) \cap \underline{\underline{R}}{}^{\mathcal{I}}_j(B))$. Then, $G \in (\underline{\underline{R}}{}^{\mathcal{I}}_j(A)$ and $G \in \underline{\underline{R}}{}^{\mathcal{I}}_j(B))$. Thus, $G \cap A' \in \mathcal{I}$ and $G \cap B' \in \mathcal{I}$. Hence, $(G \cap A') \cup (G \cap B') \in \mathcal{I}$ (from the properties of $\mathcal{I}$). So, $(G \cap (A' \cup B') \in \mathcal{I}$. Therefore, $G \in \underline{\underline{R}}{}^{\mathcal{I}}_j(A \cap B)$.
5. and 6 are Straightforward.

Remark 5.1. *Example 3.1 shows that*

1. *the inclusion in Proposition 5.1 part 2 can not be replaced by equality relation. If $A = \{a, b\}, B = \{a, c\}, A \cup B = \{a, b, c\}, \underline{\underline{R}}{}^{\mathcal{I}}_r(A) = \{a, b\}, \underline{\underline{R}}{}^{\mathcal{I}}_r(B) = \{a, b\}, \underline{\underline{R}}{}^{\mathcal{I}}_r(A \cup B) = \{a, b, d\}$, then $\underline{\underline{R}}{}^{\mathcal{I}}_r(A \cup B) = \{a, b, d\} \not\subseteq \{a, b\} = \underline{\underline{R}}{}^{\mathcal{I}}_r(A) \cup \underline{\underline{R}}{}^{\mathcal{I}}_r(B)$.*

2. *the converse of parts 2,5 and 6 is not necessarily true:*

   (i) *for part 2, if* $A = \{b\}, B = \{a\}$, *then* $\underline{\underline{R}}_r^{\mathcal{I}}(A) = \{b\}, \underline{\underline{R}}_r^{\mathcal{I}}(B) = \{a, b\}$. *Therefore,* $\underline{\underline{R}}_r^{\mathcal{I}}(A) \subseteq \underline{\underline{R}}_r^{\mathcal{I}}(B)$, *but* $A \nsubseteq B$.

   (ii) *for part 6, if* $A = \{b\}, \mathcal{I} = \{\phi, \{d\}\}, \mathcal{J} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$, *then* $\underline{\underline{R}}_l^{\mathcal{I}}(A) = \phi, \underline{\underline{R}}_l^{\mathcal{J}}(A) = \{d\}$, *Therefore,* $\underline{\underline{R}}_l^{\mathcal{J}}(A) = \phi \subseteq \{4\} = \underline{\underline{R}}_l^{\mathcal{I}}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

   (iii) *for part 5. Similarly, I can add example to prove* $\underline{\underline{R}}_j^{\mathcal{I}}(A) = U \not\Longleftarrow A' \in \mathcal{I}$.

Proposition 5.2. *Let* (U, R, ξ_j) *be a j-NS,* $\mathcal{I}, \mathcal{I}$ *be two ideals on* U *and* $A, B \subseteq U$. *Then,* $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$,

1. $\overline{\overline{R}}_j^{\mathcal{I}}(\phi) = \phi$.
2. $A \subseteq B \Rightarrow \overline{\overline{R}}_j^{\mathcal{I}}(B) \subseteq \overline{\overline{R}}_j^{\mathcal{I}}(A)$.
3. $\overline{\overline{R}}_j^{\mathcal{I}}(A \cap B) \supseteq \overline{\overline{R}}_j^{\mathcal{I}}(A) \cap \overline{\overline{R}}_j^{\mathcal{I}}(B)$.
4. $\overline{\overline{R}}_j^{\mathcal{I}}(A \cup B) \subseteq \overline{\overline{R}}_j^{\mathcal{I}}(A) \cup \overline{\overline{R}}_j^{\mathcal{I}}(B)$.
5. *if* $A \in \mathcal{I} \Rightarrow \overline{\overline{R}}_j^{\mathcal{I}}(A) = \phi$.
6. $\mathcal{I} \subset \mathcal{J} \Rightarrow \overline{\overline{R}}_j^{\mathcal{I}}(A) \subseteq \overline{\overline{R}}_j^{\mathcal{J}}(A)$.

Proof. Similar to Proposition 5.1.

Remark 5.2. *Example 3.1 shows that*

1. *the inclusion in Proposition 5.2 parts 3 and 4 can not be replaced by equality relation:*

   (i) *for part 3, if* $A = \{d\}, B = \{a, b\}, A \cap B = \phi, \overline{\overline{R}}_r^{\mathcal{I}}(A) = \{c, d\}, \overline{\overline{R}}_r^{\mathcal{I}}(B) = \{a, b, c\}, \overline{\overline{R}}_r^{\mathcal{I}}(A \cap B) = \phi$, *then* $\overline{\overline{R}}_r^{\mathcal{I}}(A \cap B) = \phi \nsupseteq \{c\} = \overline{\overline{R}}_r^{\mathcal{I}}(A) \cap \overline{\overline{R}}_r^{\mathcal{I}}(B)$.

   (ii) *for part 4, if* $A = \{d\}, B = \{a, b\}, A \cup B = \{a, b, d\}, \overline{\overline{R}}_l^{\mathcal{I}}(A) = \{d\}, \overline{\overline{R}}_l^{\mathcal{I}}(B) = \{a, b, c\}, \overline{\overline{R}}_l^{\mathcal{I}}(A \cup B) = \phi$, *then* $\overline{\overline{R}}_l^{\mathcal{I}}(A) \cup \overline{\overline{R}}_l^{\mathcal{I}}(B) = U \nsubseteq \phi = \overline{\overline{R}}_l^{\mathcal{I}}(A \cup B)$.

2. *the converse of parts 2,5 and 6 is not necessarily true:*

   (i) *for part 2, if* $A = \{a\}, B = \{b\}$, *then* $\overline{\overline{R}}_r^{\mathcal{I}}(A) = \{a\}, \overline{\overline{R}}_r^{\mathcal{I}}(B) = \{b\}$. *Therefore,* $\overline{\overline{R}}_r^{\mathcal{I}}(B) \subseteq \overline{\overline{R}}_r^{\mathcal{I}}(A)$, *but* $A \nsubseteq B$.

   (ii) *for part 5, if* $A = \{a, d\}$, *then* $\overline{\overline{R}}_l^{\mathcal{I}}(A) = \phi$, *but* $\{a, d\} \notin \mathcal{I}$.

   (iii) *for part 6, if* $A = \{b\}, \mathcal{I} = \{\phi, \{b\}, \{c\}, \{b, c\}\}, \mathcal{J} = \{\phi, \{d\}\}$, *then* $\overline{\overline{R}}_l^{\mathcal{I}}(A) = \phi, \overline{\overline{R}}_l^{\mathcal{J}}(A) = A$. *Therefore,* $\overline{\overline{R}}_l^{\mathcal{I}}(A) = \phi \subseteq \{b\} = \overline{\overline{R}}_l^{\mathcal{J}}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

The deviations between the current approximations $\underline{\underline{R}}_j^{\mathcal{I}}(A)$ and $\overline{\overline{R}}_j^{\mathcal{I}}(A)$ and the previous one [1] are presented in the following remark.

Remark 5.3. *Let* (U, R, ξ_j) *be a j-NS,* $\mathcal{I}, \mathcal{I}$ *be two ideals on* U *and* $A, B \subseteq U$. *Then,* $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$, *Example 3.1 shows that in general:*

1. $\underline{\underline{R}}_j^{\mathcal{I}}(A) \nsubseteq A \nsubseteq \overline{\overline{R}}_j^{\mathcal{I}}(A)$. *As if* $j = r, A = \{a\}$, *then* $\underline{\underline{R}}_r^{\mathcal{I}}(A) = \{a, b\} \nsubseteq \{a\} = A$, *and if* $A = \{b\}$, *then* $\overline{\overline{R}}_r^{\mathcal{I}}(A) = \phi$, *then* $A = \{b\} \nsubseteq \phi = \overline{\overline{R}}_r^{\mathcal{I}}(A)$.

2. $\underline{\underline{R}}_j^{\mathcal{I}}(\phi) \neq \phi, \overline{\overline{R}}_j^{\mathcal{I}}(U) \neq U$. *As if* $j = r$, *then* $\underline{\underline{R}}_r^{\mathcal{I}}(\phi) = \{b\} \neq \phi, \overline{\overline{R}}_r^{\mathcal{I}}(U) = \{a, c, d\} \neq U$.
3. $\underline{\underline{R}}_j^{\mathcal{I}}(\underline{\underline{R}}_j^{\mathcal{I}}(A)) \neq \underline{\underline{R}}_j^{\mathcal{I}}(A)$. *If* $j = r, A = \{a, b, c\}$, *then* $\underline{\underline{R}}_r^{\mathcal{I}}(A) = \{a, b, d\}, \underline{\underline{R}}_r^{\mathcal{I}}(\underline{\underline{R}}_r^{\mathcal{I}}(A)) = U$. *Therefore,* $\underline{\underline{R}}_r^{\mathcal{I}}(A) = \{a, b, d\} \neq U = \underline{\underline{R}}_r^{\mathcal{I}}(\underline{\underline{R}}_r^{\mathcal{I}}(A))$. *Similarly, I can add example to show that,* $\overline{\overline{R}}_j^{\mathcal{I}}(\overline{\overline{R}}_j^{\mathcal{I}}(A)) \neq \overline{\overline{R}}_j^{\mathcal{I}}(A)$.
4. $\underline{\underline{R}}_j^{\mathcal{I}}(A) \neq (\overline{\overline{R}}_j^{\mathcal{I}}(A'))'$. *As if* $j = l, A = \{b, c\}$, *then* $\underline{\underline{R}}_l^{\mathcal{I}}(A) = \phi \neq U = (\overline{\overline{R}}_l^{\mathcal{I}}(A'))'$. *Similarly, I can add example to show that,* $\overline{\overline{R}}_j^{\mathcal{I}}(A) \neq (\underline{\underline{R}}_j^{\mathcal{I}}(A'))'$.
5. $A \subseteq B \nRightarrow \overline{\overline{R}}_j^{\mathcal{I}}(A) \subseteq \overline{\overline{R}}_j^{\mathcal{I}}(B)$. *If* $j = l, A = \{b, d\}, B = \{a, b, d\}$, *then* $\overline{\overline{R}}_l^{\mathcal{I}}(A) = \{d\}, \overline{\overline{R}}_l^{\mathcal{I}}(B) = \phi$. *Therefore,* $A \subseteq B$, *but* $\overline{\overline{R}}_l^{\mathcal{I}}(A) \nsubseteq \overline{\overline{R}}_l^{\mathcal{I}}(B)$.
6. $\overline{\overline{R}}_j^{\mathcal{I}}(A \cap B) \nsubseteq \overline{\overline{R}}_j^{\mathcal{I}}(A) \cap \overline{\overline{R}}_j^{\mathcal{I}}(B)$. *If* $j = l, A = \{b, d\}, B = \{a, b, d\}, A \cap B = \{b, d\}$, *then* $\overline{\overline{R}}_l^{\mathcal{I}}(A) = \{d\}, \overline{\overline{R}}_l^{\mathcal{I}}(B) = \phi, \overline{\overline{R}}_l^{\mathcal{I}}(A \cap B) = \{d\}$. *Therefore,* $\overline{\overline{R}}_l^{\mathcal{I}}(A \cap B) = \{d\} \nsubseteq \phi = \overline{\overline{R}}_l^{\mathcal{I}}(A) \cap \overline{\overline{R}}_l^{\mathcal{I}}(B)$.
7. $\overline{\overline{R}}_j^{\mathcal{I}}(A \cup B) \neq \overline{\overline{R}}_j^{\mathcal{I}}(A) \cup \overline{\overline{R}}_j^{\mathcal{I}}(B)$. *As if* $j = l, A = \{d\}, B = \{a, b\}, A \cup B = \{a, b, d\}, \overline{\overline{R}}_l^{\mathcal{I}}(A) = \{d\}, \overline{\overline{R}}_l^{\mathcal{I}}(B) = \{a, b, c\}, \overline{\overline{R}}_l^{\mathcal{I}}(A \cup B) = \phi$, *then* $\overline{\overline{R}}_l^{\mathcal{I}}(A \cup B) = \phi \neq U = \overline{\overline{R}}_l^{\mathcal{I}}(A) \cup \overline{\overline{R}}_l^{\mathcal{I}}(B)$.

Theorem 5.1. *Let* (U, R, ξ_j) *be a j-NS,* $\mathcal{I}, \mathcal{J}$ *be two ideal on* U *and* $A \subseteq U$. *Then,* $\forall j \in \{r, l, < r >, < l >, i, u, < i >, < u >\}$.

1. $\overline{\overline{R}}_j^{\mathcal{I} \cap \mathcal{J}}(A) = \overline{\overline{R}}_j^{\mathcal{I}}(A) \cap \overline{\overline{R}}_j^{\mathcal{J}}(A)$.
2. $\overline{\overline{R}}_j^{\mathcal{I} \cup \mathcal{J}}(A) = \overline{\overline{R}}_j^{\mathcal{I}}(A) \cup \overline{\overline{R}}_j^{\mathcal{J}}(A)$.

Proof.

(1)

$$\begin{aligned}
\overline{R}_j^{\mathcal{I} \cap \mathcal{J}}(A) &= \cap\{H \in \Gamma_j^{\mathcal{I}} : A \cap H' \in \mathcal{I} \cap \mathcal{J}\} \\
&= \cap\{H \in \Gamma_j^{\mathcal{I}} : A \cap H' \in \mathcal{I} \text{ and } A \cap H' \in \mathcal{J}\} \\
&= (\cap\{H \in \Gamma_j^{\mathcal{I}} : A \cap H' \in \mathcal{I}\}) \text{ and } (\cap\{H \in \Gamma_j^{\mathcal{I}} : A \cap H' \in \mathcal{I}\}) \\
&= \overline{\overline{R}}_j^{\mathcal{I}}(A) \text{ and } \overline{\overline{R}}_j^{\mathcal{J}}(A) \\
&= \overline{\overline{R}}_j^{\mathcal{I}}(A) \cap \overline{\overline{R}}_j^{\mathcal{J}}(A).
\end{aligned} \tag{1}$$

(2) Similar to (1).

The following theorem and corollary present the relationships between the approximations $\underline{\underline{R}}_j^{\mathcal{I}}(A)$ and $\overline{\overline{R}}_j^{\mathcal{I}}(A)$, the current approximations in Definition 4.2, and the previous one in Definition 2.5 [1].

Theorem 5.2. *Let* (U, R, ξ_j) *be a j-NS,* $\mathcal{I}$ *be an ideal on* U *and* $A \subseteq U$. *Then,* $\forall j \in \{r, l, < r >, < l >, i, u, < i >, < u >\}$.

1. $\underline{R}_j(A) \subseteq \underline{R}_j^{\mathcal{I}}(A) \subseteq \underline{\underline{R}}_j^{\mathcal{I}}(A)$.
2. $\overline{\overline{R}}_j^{\mathcal{I}}(A) \subseteq \overline{R}_j^{\mathcal{I}}(A) \subseteq \overline{R}_j(A)$.

Proof.

(1) $\underline{R}_j(A) \subseteq \underline{R}_j^{\mathcal{I}}(A)$. (by Theorem 4.1) Let $\phi \neq G \in \underline{R}_j^{\mathcal{I}}(A)$. Then, $G \subseteq A$. Thus, $G \cap A' = \phi$. Hence, $G \cap A' \in \mathcal{I}$. Therefore, $\underline{R}_j^{\mathcal{I}}(A) \subseteq \underline{\underline{R}}_j^{\mathcal{I}}(A)$.

(2) Similar to (1).

Corollary 5.1. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on $U, A \subseteq U$ and $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$. Then $B^{*\mathcal{I}}_j(A) \subseteq B^{\mathcal{I}}_j(A) \subseteq B_j(A), B^{*\mathcal{I}}_j(A) = \overline{\overline{R}}_j^{\mathcal{I}}(A) - \underline{\underline{R}}_j^{\mathcal{I}}(A)$.*

Remark 5.4. *Example 3.1 shows that the converse of parts of Theorem 5.2 and Corollary 5.1 is not necessarily true as*

1. *if $A = \{a\}$, then $\underline{\underline{R}}_r^{\mathcal{I}}(A) = \{a, b\} \nsubseteq \underline{R}_r^{\mathcal{I}}(A) = \{b\} \nsubseteq \phi = \underline{R}_r(A)$.*
2. *if $A = \{b\}$, then $\overline{R}_r(A) = \{a, b\} \nsubseteq \{b\} = \overline{R}_r^{\mathcal{I}}(A) \nsubseteq \phi = \overline{\overline{R}}_r^{\mathcal{I}}(A)$.*

Proposition 5.3. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$. Then, the following statements are true in general.*

1. $\underline{\underline{R}}_u^{\mathcal{I}}(A) \subseteq \underline{\underline{R}}_r^{\mathcal{I}}(A) \subseteq \underline{\underline{R}}_i^{\mathcal{I}}(A)$.
2. $\underline{\underline{R}}_u^{\mathcal{I}}(A) \subseteq \underline{\underline{R}}_l^{\mathcal{I}}(A) \subseteq \underline{\underline{R}}_i^{\mathcal{I}}(A)$.
3. $\underline{\underline{R}}_{<u>}^{\mathcal{I}}(A) \subseteq \underline{\underline{R}}_{<r>}^{\mathcal{I}}(A) \subseteq \underline{\underline{R}}_{<i>}^{\mathcal{I}}(A)$.
4. $\underline{\underline{R}}_{<u>}^{\mathcal{I}}(A) \subseteq \underline{\underline{R}}_{<l>}^{\mathcal{I}}(A) \subseteq \underline{\underline{R}}_{<i>}^{\mathcal{I}}(A)$.

Proof. By using Proposition 3.1, the proof is obvious.

Proposition 5.4. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$. Then, the following statements are true in general.*

1. $\overline{\overline{R}}_i^{\mathcal{I}}(A) \subseteq \overline{\overline{R}}_r^{\mathcal{I}}(A) \subseteq \overline{\overline{R}}_u^{\mathcal{I}}(A)$.
2. $\overline{\overline{R}}_i^{\mathcal{I}}(A) \subseteq \overline{\overline{R}}_l^{\mathcal{I}}(A) \subseteq \overline{\overline{R}}_u^{\mathcal{I}}(A)$.
3. $\overline{\overline{R}}_{<i>}^{\mathcal{I}}(A) \subseteq \overline{\overline{R}}_{<r>}^{\mathcal{I}}(A) \subseteq \overline{\overline{R}}_{<u>}^{\mathcal{I}}(A)$.
4. $\overline{\overline{R}}_{<i>}^{\mathcal{I}}(A) \subseteq \overline{\overline{R}}_{<l>}^{\mathcal{I}}(A) \subseteq \overline{\overline{R}}_{<u>}^{\mathcal{I}}(A)$.

Proof. By using Proposition 3.1, the proof is obvious.

Corollary 5.2. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$.*

1. *$B^{*\mathcal{I}}_i(A) \subseteq B^{*\mathcal{I}}_r(A) \subseteq B^{*\mathcal{I}}_u(A)$, where $B^{*\mathcal{I}}_j(A) = \overline{\overline{R}}_j^{\mathcal{I}}(A) - \underline{\underline{R}}_j^{\mathcal{I}}(A)$.*
2. $B^{*\mathcal{I}}_i(A) \subseteq B^{*\mathcal{I}}_l(A) \subseteq B^{*\mathcal{I}}_u(A)$.
3. $B^{*\mathcal{I}}_{<i>}(A) \subseteq B^{*\mathcal{I}}_{<r>}(A) \subseteq B^{*\mathcal{I}}_{<u>}(A)$.
4. $B^{*\mathcal{I}}_{<i>}(A) \subseteq B^{*\mathcal{I}}_{<l>}(A) \subseteq B^{*\mathcal{I}}_{<u>}(A)$.

Corollary 5.3. *Let (U, R, ξ_j) be a j-NS, $\mathcal{I}$ be an ideal on U and $A \subseteq U$. Then, $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$.*

1. *$\sigma^{*\mathcal{I}}_u(A) \leqslant \sigma^{*\mathcal{I}}_r(A) \leqslant \sigma^{*\mathcal{I}}_i(A)$, where $\sigma^{*\mathcal{I}}_j(A) = \frac{|\underline{\underline{R}}_j^{\mathcal{I}}(A)|}{|\overline{\overline{R}}_j^{\mathcal{I}}(A)|}, |\overline{\overline{R}}_j^{\mathcal{I}}(A)| \neq 0$.*
2. $\sigma^{*\mathcal{I}}_u(A) \leqslant \sigma^{*\mathcal{I}}_l(A) \leqslant \sigma^{*\mathcal{I}}_i(A)$.
3. $\sigma^{*\mathcal{I}}_{<u>}(A) \leqslant \sigma^{*\mathcal{I}}_{<r>}(A) \leqslant \sigma^{*\mathcal{I}}_{<i>}(A)$.

4. $\sigma^{*\mathcal{I}}_{<u>}(A) \leqslant \sigma^{*\mathcal{I}}_{<l>}(A) \leqslant \sigma^{*\mathcal{I}}_{<i>}(A)$.

Remark 5.5. *Example 3.1 shows that the the inclusion in Propositions 5.3, 5.4 and Corollary 5.2, 5.3 can not be replaced by equality relation as*

1. *if* $A = \{a\}$, *then* $\underline{\underline{\underline{R}}}^{\mathcal{I}}_{u}(A) = \{a\} \neq \{a, b\} = \underline{\underline{\underline{R}}}^{\mathcal{I}}_{r}(A) \neq \{a, b, c\} = \underline{\underline{\underline{R}}}^{\mathcal{I}}_{i}(A)$.
2. *if* $A = \{a\}$, *then* $\underline{\underline{\underline{R}}}^{\mathcal{I}}_{u}(A) = \{a\} \neq \{a, b, c\} = \underline{\underline{\underline{R}}}^{\mathcal{I}}_{l}(A)$ *and if* $A = \{d\}$, *then* $\underline{\underline{\underline{R}}}^{\mathcal{I}}_{l}(A) = \{d\} \neq \{b, c, d\} = \underline{\underline{\underline{R}}}^{\mathcal{I}}_{i}(A)$.
3. *if* $A = \{b\}$, *then* $B^{*\mathcal{I}}_{i}(A) = \{a, d\} \neq \phi = B^{*\mathcal{I}}_{r}(A)$ *and if* $A = \{d\}$, *then* $B^{*\mathcal{I}}_{r}(A) = \phi \neq \{c\} = B^{*\mathcal{I}}_{u}(A)$.
4. *if* $A = \{b\}$, *then* $B^{*\mathcal{I}}_{i}(A) = \{a, d\} \neq \phi = B^{\mathcal{I}}_{l}(A)$ *and if* $A = \{b\}$, *then* $B^{*\mathcal{I}}_{l}(A) = \phi \neq \{b, c\} = B^{*\mathcal{I}}_{u}(A)$.

Similarly, I can add examples for the other parts.

Remark 5.6. *It should be noted that the the current approximations in Definition 5.1 has the same properties of the current approximations* $\underline{\underline{R}}^{\mathcal{I}}_{j}(A)$ *and* $\overline{\overline{R}}^{\mathcal{I}}_{j}(A)$, *which are stated in Propositions 5.1, 5.2 and Theorem 5.1. Additionally, it satisfies the following properties:*

1. $\underline{\underline{\underline{R}}}^{\mathcal{I}}_{j}(A) \subseteq A \subseteq \overline{\overline{\overline{R}}}^{\mathcal{I}}_{j}(A)$.
2. $\underline{\underline{\underline{R}}}^{\mathcal{I}}_{j}(\phi) = \phi, \overline{\overline{\overline{R}}}^{\mathcal{I}}_{j}(U) = U$.

The following theorem and corollary present the relationships between the current approximations in Definition 4.2, 5.1 and the previous one in Definition 2.5 [1].

Theorem 5.3. *Let* (U, R, ξ_j) *be a j-NS,* $\mathcal{I}$ *be an ideal on* U *and* $A \subseteq U$. *Then,* $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$.

1. $\underline{R}_j(A) \subseteq \underline{R}^{\mathcal{I}}_{j}(A) \subseteq \underline{\underline{\underline{R}}}^{\mathcal{I}}_{j}(A) \subseteq \underline{\underline{R}}^{\mathcal{I}}_{j}(A)$.
2. $\overline{\overline{R}}^{\mathcal{I}}_{j}(A) \subseteq \overline{\overline{\overline{R}}}^{\mathcal{I}}_{j}(A) \subseteq \overline{R}^{\mathcal{I}}_{j}(A) \subseteq \overline{R}_j(A)$.

Proof.

1. $\underline{R}_j(A) \subseteq \underline{R}^{\mathcal{I}}_{j}(A)$ from Theorem 4.1. To prove $\underline{R}^{\mathcal{I}}_{j}(A) \subseteq \underline{\underline{\underline{R}}}^{\mathcal{I}}_{j}(A)$. Let $x \in \underline{R}^{\mathcal{I}}_{j}(A)$. Then, $x \in A$ by Proposition 4.1 and $x \in \underline{\underline{R}}^{\mathcal{I}}_{j}(A)$ by Theorem 5.2. Hence, $x \in \underline{\underline{\underline{R}}}^{\mathcal{I}}_{j}(A)$. Thus, $\underline{R}^{\mathcal{I}}_{j}(A) \subseteq \underline{\underline{\underline{R}}}^{\mathcal{I}}_{j}(A)$ and $\underline{\underline{\underline{R}}}^{\mathcal{I}}_{j}(A) \subseteq \underline{\underline{R}}^{\mathcal{I}}_{j}(A)$ is straightforward. This complete the proof.
2. Similar to 1.

Corollary 5.4. *Let* (U, R, ξ_j) *be a j-NS,* $\mathcal{I}$ *be an ideal on* U, $A \subseteq U$ *and* $\forall j \in \{r, l, <r>, <l>, i, u, <i>, <u>\}$. *Then*

1. $B^{*\mathcal{I}}_{j}(A) \subseteq B^{**\mathcal{I}}_{j}(A) \subseteq B^{\mathcal{I}}_{j}(A) \subseteq B_j(A)$.
2. $\sigma_j(A) \leqslant \sigma^{\mathcal{I}}_{j}(A) \leqslant \sigma^{**\mathcal{I}}_{j}(A) \leqslant \sigma^{*\mathcal{I}}_{j}(A)$.

Remark 5.7. *Example 3.1 shows that the converse of parts of Theorem 5.3 and Corollary 5.4 is not necessarily true as if* $A = \{a, b, d\}$, *then*

1. $\underline{R}_j(A) = \{a, b\} \neq \{a, b, d\} = \underline{R}^{\mathcal{I}}_{j}(A), \underline{\underline{\underline{R}}}^{\mathcal{I}}_{j}(A) = \{a, b, d\} \neq U = \underline{\underline{R}}^{\mathcal{I}}_{j}(A)$.
2. $\overline{\overline{R}}^{\mathcal{I}}_{j}(A) = \phi \neq \{a, b, d\} = \overline{\overline{\overline{R}}}^{\mathcal{I}}_{j}(A) = \{a, b, d\} \neq U = \overline{R}^{\mathcal{I}}_{j}(A)$.

6. Conclusions

The recent generalization of rough set theory has led to the introduction of topological rough set approaches. Ideal is a fundamental concept in topological spaces and plays an important role in the study of topological problems. This motivated us to use it in rough sets. The present paper depended on generating different topologies by using ideals. These topologies were used to generalize the basic concepts of rough set. The main aim of rough set is to reduce the boundary region by increasing the lower approximation and decreasing the upper approximation. So, in this paper different methods were proposed to achieve this aim. The properties of suggested methods were studied. It was showed that these methods are more accurate of the other methods.

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