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ON A TYPE OF SEMI-SYMMETRIC METRIC CONNECTION ON AN ALMOST CONTACT METRIC MANIFOLD

U.C. De and Joydeep Sengupta

(*Dedicated to Exp. Mileva Prvanović)*

Abstract

We find[1] the expression for the curvature tensor of an almost contact metric manifold that admits a type of semi-symmetric metric connection. Also, we study the properties of the curvature tensor, the Weyl conformal curvature tensor and the projective curvature tensor.

1 Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold of class C^∞ with metric tensor g and let ∇ be the Levi-Civita connection on M^n. A linear connection $\bar{\nabla}$ on (M^n, g) is said to be semi-symmetric [1] if the torsion tensor T of the connection $\bar{\nabla}$ satisfies

$$T(X,Y) = \pi(Y)X - \pi(X)Y \tag{1}$$

where π is a 1-form on M^n with ρ as associated vector-field, i.e.,

$$\pi(X) = g(X, \rho) \tag{2}$$

for any differentiable vector field X on M^n.

A semi-symmetric connection $\bar{\nabla}$ is called semi-symmetric metric connection [2] if it further satisfies

$$\bar{\nabla}_g = 0. \tag{3}$$

Let M^n be an n-dimensional C^∞ manifold and let there exists in M^n a vector valued linear function ϕ, a vector field ξ and an 1-form η such that

$$\phi^2 X = -X + \eta(X)\xi, \tag{4}$$

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$$\bar{X} \overset{\text{defn}}{=} \phi X \tag{5}$$

for any vector field X. Then M^n is called an almost contact manifold.

From (4) the following relations hold [3],

$$\phi\xi = 0, \tag{6}$$

$$\eta(\phi X) = 0, \tag{7}$$

and

$$\eta(\xi) = 1. \tag{8}$$

In addition, if in M^n , there exists a metric tensor g satisfying

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{9}$$

and

$$g(X,\xi) = \eta(X), \tag{10}$$

then M^n is called an almost contact metric manifold.

In [4] , Sharfuddin and Hussain defined a semi-symmetric metric connection in an almost contact manifold by identifying the 1-form π of (1) with the contact 1-form η, i.e. by setting

$$T(X,Y) = \eta(Y)X - \eta(X)Y. \tag{11}$$

In 1995, Mileva Prvanović [5] studied a semi-symmetric metric connection in a locally decomposable Riemannian space whose torsion tensor T satisfies the condition

$$(\bar{\nabla}_X T)(Y,Z) = A(X)T(Y,Z) + A(FX)F(T(Y,Z)), \tag{12}$$

where A is a 1-form and F is a tensor field of type(1,1).

In this paper we study a semi-symmetric metric connection on an almost contact metric manifold satisfying the condition (11) and

$$(\bar{\nabla}_X T)(Y,Z) = A(X)T(Y,Z) + A(\phi X)\phi(T(Y,Z)), \tag{13}$$

where ϕ is the tensor field of type (1,1) of the almost contact metric manifold. In Section 3, we find the expression for curvature tensor of $\bar{\nabla}$ and deduce some properties of the curvature tensor. It is proved that if the curvature tensor of $\bar{\nabla}$ vanishes then the manifold is of quasi-constant curvature [6]. Next we prove that if the Ricci tensor of $\bar{\nabla}$ vanishes, then the manifold becomes an η-Einstein manifold. In section 4 , we prove that the Weyl conformal curvature tensor of $\bar{\nabla}$ is equal to the Weyl conformal curvature tensor of the manifold. In the last section , we obtain a necessary condition under which the projective curvature tensor of $\bar{\nabla}$ becomes equal to the projective curvature tensor of the manifold.

2 Preliminaries

The relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ of (M^n, g) has been obtained by K.Yano [7], which is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)\rho. \tag{14}$$

Further, a relation between the curvature tensors R and $\bar{R}$ of type (1,3) of the connections ∇ and $\bar{\nabla}$ respectively are given by [7],

$$\bar{R}(X,Y)Z = R(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X - g(Y,Z)LX + g(X,Z)LY \tag{15}$$

where

$$\alpha(Y,Z) = g(LY,Z) = (\nabla_Y \pi)(Z) - \pi(Y)\pi(Z) + \frac{1}{2}\pi(\rho)g(Y,Z). \tag{16}$$

The Weyl conformal curvature tensor of type (1,3) of the manifold is defined by

$$C(X,Y)Z = R(X,Y)Z + \lambda(Y,Z)X - \lambda(X,Z)Y + g(Y,Z)QX - g(X,Z)QY, \tag{17}$$

where

$$\lambda(Y,Z) = g(QY,Z) = -\frac{1}{n-2}S(Y,Z) + \frac{r}{2(n-1)(n-2)}g(Y,Z), \tag{18}$$

S and r denote respectively the (0,2) Ricci tensor and scalar curvature of the manifold.

The projective curvature tensor of the manifold is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{19}$$

3 Curvature tensor of the semi-symmetric metric connection

We have

$$T(Y,Z) = \eta(Z)Y - \eta(Y)Z, \tag{20}$$

where

$$\eta(Z) = g(Z,\xi). \tag{21}$$

From (20) we get by contracting Y,

$$(C_1^1 T)(Z) = (n-1)\eta(Z). \tag{22}$$

Now,

$$(\bar{\nabla}_X C_1^1 T)(Z) = (n-1)(\bar{\nabla}_X \eta)(Z). \tag{23}$$

Let,

$$(\bar{\nabla}_X T)(Y,Z) = A(X)T(Y,Z) + A(\phi X)\phi(T(Y,Z)) \tag{24}$$

where A is a 1-form and ϕ is a tensor field of type (1,1).

From (24) we get by contracting Y,

$$(\bar{\nabla}_X C_1^1 T)(Z) = (n-1)A(X)\eta(Z) + aA(\phi X)\eta(Z), \tag{25}$$

where

$$A = (C_1^1\phi)(Y). \tag{26}$$

Combining (23) and (25) we get

$$(\bar{\nabla}_X \eta)(Z) = A(X)\eta(Z) + bA(\phi X)\eta(Z) \tag{27}$$

where

$$b = \frac{a}{n-1}. \tag{28}$$

Using (8) we get,

$$(\bar{\nabla}_X \eta)(Z) = (\nabla_X \eta)(Z) - \eta(X)\eta(Z) + g(X,Z). \tag{29}$$

Combining (27) and (29) we get

$$(\nabla_X \eta)(Z) = A(X)\eta(Z) + bA(\phi X)\eta(Z) + \eta(X)\eta(Z) - g(X,Z). \tag{30}$$

Then , from (16) and (30),it follows

$$\alpha(X,Z) = A(X)\eta(Z) + bA(\phi X)\eta(Z) - \frac{1}{2}g(X,Z). \tag{31}$$

From (16) and (31) we can say,

$$LX = A(X)\xi + bA(\phi X)\xi - \frac{1}{2}X. \tag{32}$$

Therefore, the curvature tensor $\bar{R}$ of the manifold with respect to the semi-symmetric metric connection $\bar{\nabla}$ is given by

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + \{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + \{A(X) + bA(\phi X)\}\{\eta(Z)Y - g(Y,Z)\xi\} \\ &\quad - \{A(Y) + bA(\phi Y)\}\{\eta(Z)X - g(X,Z)\xi\}, \end{aligned} \tag{33}$$

where R denotes the curvature tensor of the manifold.

In view of the above, we can state the following :

Theorem 3.1 *The curvature tensor with respect to $\bar{\nabla}$ of an almost contact metric manifold admitting a semi-symmetric metric connection $\bar{\nabla}$ is of the form (33).*

From (33) it is obvious that

$$\bar{R}(Y,X)Z = -\bar{R}(X,Y)Z. \tag{34}$$

We now define a tensor $'\bar{R}$ of type (0,4) by

$$'\bar{R}(X,Y,Z,V) = g(\bar{R}(X,Y)Z,V). \tag{35}$$

From (33) and (35) it follows that

$$\bar{R}(X,Y,Z,V) = -\bar{R}(X,Y,V,Z). \tag{36}$$

Combining (36) and (34) one finds that

$$\bar{R}(X,Y,Z,V) = \bar{R}(Y,X,V,Z). \tag{37}$$

Again from (33) we get,

$$\begin{aligned} &\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y \\ &\quad = (A(X) + bA(\phi X))(\eta(Z)Y - \eta(Y)Z) \\ &\quad +(A(Y) + bA(\phi Y))(\eta(X)Z - \eta(Z)X) \\ &\quad +(A(Z) + bA(\phi Z))(\eta(Y)X - \eta(X)Y). \end{aligned} \tag{38}$$

This is the first Bianchi identity with respect to $\bar{\nabla}$.

Let $\bar{S}$ and S denote respectively the Ricci tensor of the manifold with respect to $\bar{\nabla}$ and ∇. From (33) we get by contracting X.

$$\bar{S}(Y,Z) = S(Y,Z)+(n-1)g(Y,Z)-(n-2)(A(Y)+bA(\phi Y))\eta(Z)-A(\xi)g(Y,Z), \tag{39}$$

since $\phi\xi = 0$.

In (39) we put $Y = Z = e_i$, $1 \leq i \leq n$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold.
Then summing over i we get

$$\bar{r} = r + n(n-1) - 2(n-1)A(\xi), \tag{40}$$

where $\bar{r}$ and r denote the scalar curvatures of the manifold with respect to $\bar{\nabla}$ and ∇ respectively.
From (39) it follows that $\bar{S}$ is symmetric if and only if

$$\eta(Y)(A(Z)) + bA(\phi Z)) = \eta(Z)(A(Y)) + bA(\phi Y)). \tag{41}$$

In particular, if $\bar{S} = 0$, then from (39) we have

$$S(Y,Z) = (n-2)(A(Y) + bA(\phi Y))\eta(Z) + A(\xi)g(Y,Z) - (n-1)g(Y,Z). \tag{42}$$

Since S is symmetric, we get from (42),

$$[A(Y) + bA(\phi Y)]\eta(Z) = [A(Z) + bA(\phi Z)]\eta(Y). \tag{43}$$

Putting $Z = \xi$, we get from the above relation

$$A(Y) + bA(\phi Y) = A(\xi)\eta(Y). \tag{44}$$

Now, if $\bar{R} = 0$, then $\bar{S} = 0$ and then from (33) and (44) we obtain

$$\begin{aligned} {}'R(X,Y,Z,V) \;=\; & -\eta(\xi)[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)] \\ & +A(\xi)[\eta(Y)\eta(Z)g(X,V) - \eta(X)\eta(Z)g(Y,V) \\ & -\eta(Y)\eta(V)g(X,Z) + \eta(X)\eta(V)g(Y,Z)] \end{aligned} \tag{45}$$

since $\eta(\xi) = 1$

where

$$'R(X,Y,Z,V) = g(R(X,Y)Z,V). \tag{46}$$

Hence we can state the following theorem.

Theorem 3.2 *If the curvature tensor of an almost contact metric manifold with respect to the semi-symmetric metric connection vanishes , then the manifold is of quasi-constant curvature.*

Next, let us assume that $\bar{S}$ is symmetric. Then (41) holds. Putting $Z = \xi$ in (41) we get

$$A(Y) + bA(\phi Y) = A(\xi)\eta(Y).$$

Using the result from (38) we get

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0. \tag{47}$$

Conversely, we assume that (47) holds, then in virtue of (38) we have

$$(48)\qquad \begin{aligned}&(A(X)+bA(\phi X))(\eta(Z)Y-\eta(Y)Z)\\&+(A(Y)+bA(\phi Y))(\eta(X)Z-\eta(Z)X)\\&+(A(Z)+bA(\phi Z))(\eta(Y)X-\eta(X)Y)=0.\end{aligned}$$

Contracting X, we get from (48)

$$\eta(Y)(A(Z)+bA(\phi Z))=\eta(Z)(A(Y)+bA(\phi Y)).$$

Hence by (41), $\bar{S}$ is symmetric.
Thus we can state:

Theorem 3.3 *A necessary and sufficient condition for the Ricci tensor of an almost contact metric manifold with respect to the semi-symmetric metric connection to be symmetric is*

$$\bar{R}(X,Y)Z+\bar{R}(Y,Z)X+\bar{R}(Z,X)Y=0.$$

Next, if $\bar{S}=0$, then $\bar{r}=0$ and so from (40) we get,

$$(49)\qquad A(\xi)=\frac{1}{2}\left\{\frac{r}{n-1}+n\right\}.$$

Putting this value of $A(\xi)$ we get from (49),

$$(50)\qquad S(Y,Z)=\mu g(Y,Z)+\nu\eta(Y)\eta(Z),$$

where

$$\mu=\frac{1}{2}\left\{\frac{r}{n-1}-n+2\right\}$$

and

$$\nu=\frac{1}{2}\left(\frac{n-2}{n-1}\right)(r+n^2-n).$$

So we can state:

Theorem 3.4 *If the Ricci tensor of an almost contact metric manifold with respect to the semi-symmetric metric connection vanishes,then the manifold becomes an η-Einstein manifold.*

4 Weyl conformal curvature tensor

The Weyl conformal curvature tensor of type (1,3) of the almost contact metric manifold with respect to the semi-symmetric metric connection $\bar{\nabla}$ is defined by

(51) $$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z + \bar{\lambda}(Y,Z)X - \bar{\lambda}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y$$

where

(52) $$\bar{\lambda}(Y,Z) = \bar{g}(QY,Z) = -\frac{1}{n-1}\bar{S}(Y,Z) + \frac{\bar{r}}{2(n-1)(n-2)}g(Y,Z).$$

Putting the values of $\bar{S}$ and $\bar{r}$ from (39) and (40) respectively in (52) we get ,

(53) $$\bar{\lambda}(Y,Z) = \lambda(Y,Z) - \frac{1}{2}g(Y,Z) + \eta(Z)(A(Y) + bA(\phi Y)).$$

Combining the results (51), (33) and (53) we get,

(54) $$\bar{C}(X,Y)Z = C(X,Y)Z.$$

So we can state :

Theorem 4.1 *The Weyl conformal curvature tensors of an almost contact metric manifold with respect to the Levi-Civita connection and the semi-symmetric metric connection are equal.*

Next, if in particular $\bar{S} = 0$,then $\bar{r} = 0$. So from (52) we get

(55) $$\bar{\lambda}(Y,Z) = 0.$$

Putting this result in (52) and using (54) we get

(56) $$C(X,Y)Z = \bar{R}(X,Y)Z.$$

Hence we can state :

Theorem 4.2 *If the Ricci tensor of an almost contact metric manifold with respect to the semi-symmetric metric connection vanishes, then the Weyl conformal curvature tensor of the manifold is equal to the curvature tensor of the manifold with respect to the semi-symmetric metric connection.*

5 Projective curvature tensor

The projective curvature tensor of type (1,3) of an almost contact metric manifold with respect to the semi-symmetric metric connection is defined by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-1}\{\bar{S}(Y,Z)X - \bar{S}(X,Z)Y\}. \tag{57}$$

Using (33) and (39) we get from (57),

$$\begin{aligned} \bar{P}(X,Y)Z &= P(X,Y)Z + \frac{1}{n-1}A(\xi)\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + \{A(X) + bA(\phi X)\}\left\{\frac{1}{n-1}\eta(Z)Y - g(Y,Z)\xi\right\} \\ &\quad - \{A(Y) + bA(\phi Y)\}\left\{\frac{1}{n-1}\eta(Z)X - g(X,Z)\xi\right\}. \end{aligned} \tag{58}$$

If, in particular, $\bar{S}$ is symmetric, then we already have,

$$A(Y) + bA(\phi Y) = A(\xi)\eta(Y).$$

Using the above result we get from (58)

$$\begin{aligned} \bar{P}(X,Y)Z &= P(X,Y)Z + \frac{1}{n-1}A(\xi)\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + A(\xi)\eta(X)\left\{\frac{1}{n-1}\eta(Z)Y - g(Y,Z)\xi\right\} \\ &\quad - A(\xi)\eta(Y)\left\{\frac{1}{n-1}\eta(Z)X - g(X,Z)\xi\right\}. \end{aligned} \tag{59}$$

From (59), it follows that $P = \bar{P}$ if $A(\xi) = 0$.
So,we have

Theorem 5.1 *If the Ricci tensor of an almost contact metric manifold with respect to the semi-symmetric metric connection is symmetric, then a necessary condition for the projective curvature tensors of the manifold with respect to the Levi-Civita connection and the semi-symmetric metric connection to be equal is that* $A(\xi) = 0$.

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Department of Mathematics
University of Kalyani
Kalyani, Nadia
West Bengal, India 741235
E-mail: ucde@klyuniv.ernet.in