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NEW SPECIAL GEODESIC MAPPINGS OF GENERAL AFFINE CONNECTION SPACES

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Abstract

In this work[1] we define $\underset{\theta}{R}$-projective geodesic mappings ($\theta = 1, ..., 5$) of two general affine connection spaces and obtain some invariant geometric objects of these mappings, generalizing Weyl's tensor. Also, we define $\underset{\theta}{R}$-projectively flat affine connection spaces $G\overline{A}_N$ and find necessary conditions for the space GA_N to be $\underset{\theta}{R}$-projectively flat.

Introduction

Consider two N-dimensional differentiable manifolds GA_N and $G\overline{A}_N$ and differentiable mapping $f : GA_N \to G\overline{A}_N$. We can consider these manifolds in the *common by this mapping system of local coordinates.* If the connection coefficients $L^i_{jk}(x)$ and $\overline{L}^i_{jk}(x)$, for the connection introduced in GA_N and $G\overline{A}_N$ respectively, are non-symmetric in lower indices, we call GA_N and $G\overline{A}_N$ *general affine connection spaces.*

One says that reciprocal one valued mapping $f : GA_N \to G\overline{A}_N$ is *geodesic,* [5,6] if geodesics of the space GA_N pass to geodesics of the space $G\overline{A}_N$. In the corresponding points $M(x)$ and $\overline{M}(x)$ we can put

$$\overline{L}^i_{jk}(x) = L^i_{jk}(x) + P^i_{jk}(x), \quad (i, j, k = 1, ..., N), \tag{0.1}$$

where $P^i_{jk}(x)$ is *the deformation tensor* of the connection L of GA_N according to the mapping $f : GA_N \to G\overline{A}_N$.

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A necessary and sufficient condition that the mapping f be geodesic [5] is that the deformaton tensor P^i_{jk} from (0.2) has the form

$$P^i_{jk}(x) = \delta^i_j\,\psi_k(x) + \delta^i_k\,\psi_j(x) + \xi^i_{jk}(x), \tag{0.2}$$

where

$$\psi_i = \frac{1}{N+1}(\overline{L}^p_{\underline{ip}} - L^p_{\underline{ip}}), \tag{0.3}$$

$$\xi^i_{jk} = \overline{L}^i_{\underset{\vee}{jk}} - L^i_{\underset{\vee}{jk}} \tag{0.4}$$

and $\underline{jk}$ denotes symmetrisation and $\underset{\vee}{jk}$ -antisymmetrisation with respect to j, k. In GA_N one can define four kinds of covariant derivatives [1,2,3]. For example, for a tensor a^i_j in GA_N we have

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + L^i_{pm}a^p_j - L^p_{jm}a^i_p, \quad a^i_{j\underset{2}{|}m} = a^i_{j,m} + L^i_{mp}a^p_j - L^p_{mj}a^i_p,$$
$$a^i_{j\underset{3}{|}m} = a^i_{j,m} + L^i_{pm}a^p_j - L^p_{mj}a^i_p, \quad a^i_{j\underset{4}{|}m} = a^i_{j,m} + L^i_{mp}a^p_j - L^p_{jm}a^i_p.$$

Denote by $\underset{\theta}{|}, \underset{\overline{\theta}}{|}$ a covariant derivative of the kind θ in GA_N and $G\overline{A}_N$ respectively.

In the case of the space GA_N we have five independent curvature tensors [4] (in [4] $\underset{5}{R}$ is denoted by $\underset{2}{\tilde{R}}$):

$$\underset{1}{R}{}^i{}_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{pm},$$
$$\underset{2}{R}{}^i{}_{jmn} = L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{mp},$$
$$\underset{3}{R}{}^i{}_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{nm}(L^i_{pj} - L^i_{jp}),$$
$$\underset{4}{R}{}^i{}_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{mn}(L^i_{pj} - L^i_{jp}),$$
$$\underset{5}{R}{}^i{}_{jmn} = \frac{1}{2}(L^i_{jm,n} + L^i_{mj,n} - L^i_{jn,m} - L^i_{nj,m} + L^p_{jm}L^i_{pn} + L^p_{mj}L^i_{np} - L^p_{jn}L^i_{mp} - L^p_{nj}L^i_{pm}).$$

By virtue of the geodesic mapping $f : GA_N \to G\overline{A}_N$ we obtain tensors $\underset{\theta}{\overline{R}}{}^i_{jmn}$ ($\theta = 1, \ldots, 5$), where for example

$$\underset{1}{\overline{R}}{}^i_{jmn} = \overline{L}^i_{jm,n} - \overline{L}^i_{jn,m} + \overline{L}^p_{jm}\overline{L}^i_{pn} - \overline{L}^p_{jn}\overline{L}^i_{pm}. \tag{0.5}$$

In the case of geodesic mapping $f : A_N \to \overline{A}_N$ of the symmetric affine connection spaces A_N and $\overline{A}_N$ we have an invariant geometric object

$$W^i{}_{jmn} = R^i{}_{jmn} + \frac{2}{N+1}\delta^i_j R_{[mn]} + \frac{1}{N^2-1}\left[\delta^i_m (NR_{jn} + R_{nj}) - \delta^i_n (NR_{jm} + R_{mj})\right], \tag{0.6}$$

where $R^i{}_{jmn}$ is Riemann-Cristoffel's curvature tensor of the space A_N, and R_{jm} Richi's tensor.

The object $W^i{}_{jmn}$ is called Weil's tensor, or a tensor of projective curvature [8]. Having a geodesic mapping of two general affine connection spaces, we can not find a generalization of Weil's tensor as an invariant of geodesic mapping in general case. For that reason we define a special geodesic mapping.

1. $\underset{1}{R}$-projective mappings

For the first kind curvature tensors of the spaces GA_N and $G\overline{A}_N$ respectively we find the relation

$$\begin{aligned}\underset{1}{\overline{R}}{}^i_{jmn} &= \underset{1}{R}{}^i{}_{jmn} + \delta^i_j (\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m \underset{1}{\psi}_{jn} - \delta^i_n \underset{1}{\psi}_{jm} \\ &- \delta^i_m \xi^p_{jn}\psi_p + \delta^i_n \xi^p_{jm}\psi_p + \xi^i_{jm\underset{1}{|}n} - \xi^i_{jn\underset{1}{|}m} + 2\psi_j\xi^i_{mn} \\ &+ \xi^p_{jm}\xi^i_{pn} - \xi^p_{jn}\xi^i_{pm} + 2L^i_{\underset{\vee}{mn}}\psi_j + 2L^p_{\underset{\vee}{mn}}\psi_p\delta^i_j + 2L^p_{\underset{\vee}{mn}}\xi^i_{jp},\end{aligned} \tag{1.1}$$

$$\underset{\theta}{\psi}_{mn} = \psi_{m\underset{\theta}{|}n} - \psi_m\psi_n, \quad (\theta = 1, 2).$$

Contracting with respect to i and n from (1.1) we get

$$\begin{aligned}\underset{1}{\overline{R}}_{jm} &= \underset{1}{R}_{jm} - \underset{1}{\psi}_{[jm]} - (N-1)\underset{1}{\psi}_{jm} + (N-1)\xi^p_{jm}\psi_p + \xi^p_{jm\underset{1}{|}p} \\ &- \xi^p_{jp\underset{1}{|}m} + \psi_j\overline{L}^p_{mp} + \xi^p_{jm}\xi^q_{pq} - \xi^p_{jq}\xi^q_{pm} + 2L^p_{\underset{\vee}{mj}}\psi_p + 2L^p_{\underset{\vee}{mq}}\xi^q_{jp}\end{aligned} \tag{1.2}$$

Here $\underset{1}{\overline{R}}_{jm}$ and $\underset{1}{R}_{jm}$ are the first kind Ricci tensors of the spaces $G\overline{A}_N$ and GA_N respectively and $[jm]$ denotes an alternation without a division. ¿From (1.2) we obtain

$$\begin{aligned}\underset{1}{\overline{R}}_{[jm]} &= \underset{1}{R}_{[jm]} - 2\underset{1}{\psi}_{[jm]} - (N-1)\underset{1}{\psi}_{[jm]} + 2(N-1)\xi^p_{jm}\psi_p \\ &+ \xi^p_{jm\underset{1}{|}p} - \xi^p_{jp\underset{1}{|}m} + \xi^p_{mp\underset{1}{|}j} + 2\psi_j\overline{L}^p_{\underset{\vee}{mp}} - 2\psi_m\overline{L}^p_{\underset{\vee}{jp}} + 2\xi^p_{jm}\xi^q_{pq} \\ &- \xi^p_{jq}\xi^q_{pm} + \xi^p_{mq}\xi^q_{pj} + 4L^p_{\underset{\vee}{mj}}\psi_p + 2L^p_{\underset{\vee}{mq}}\xi^q_{jp} - 2L^p_{\underset{\vee}{jq}}\xi^q_{mp},\end{aligned} \tag{1.3}$$

from where

$$
(1.4)\qquad
\begin{aligned}
(N+1)\underset{1}{\psi}_{[jm]} &= \underset{1}{R}_{[jm]} - \underset{1}{\overline{R}}_{[jm]} + 2(N-1)\xi^p_{jm}\psi_p + \xi^p_{jm\underset{1}{|}p} - \xi^p_{jp\underset{1}{|}m}\\
&+ \xi^p_{mp\underset{1}{|}j} + 2\psi_j\underset{\vee}{\overline{L}}{}^p_{mp} - 2\psi_m\underset{\vee}{\overline{L}}{}^p_{jp} + 2\xi^p_{jm}\xi^q_{pq} + 4\underset{\vee}{L}{}^p_{mj}\psi_p + 2\underset{\vee}{L}{}^p_{mq}\xi^q_{jp} - 2\underset{\vee}{L}{}^p_{jq}\xi^q_{mp}.
\end{aligned}
$$

Substituting (1.4) in (1.2), we get

$$
(1.5)\qquad
\begin{aligned}
(N-1)\underset{1}{\psi}_{jm} &= \underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm} - \frac{1}{N+1}\big[\underset{1}{R}_{[jm]} - \underset{1}{\overline{R}}_{[jm]} + 2(N-1)\xi^p_{jm}\psi_p\\
&+ 2\xi^p_{jm\underset{1}{|}p} - \xi^p_{jp\underset{1}{|}m} + \xi^p_{mp\underset{1}{|}j} + 2\psi_j\underset{\vee}{\overline{L}}{}^p_{mp} - 2\psi_m\underset{\vee}{\overline{L}}{}^p_{jp} + 2\xi^p_{jm}\xi^q_{pq}\\
&+ 4\underset{\vee}{L}{}^p_{mj}\psi_p + 2\underset{\vee}{L}{}^p_{mq}\xi^q_{jp} - 2\underset{\vee}{L}{}^p_{jq}\xi^q_{mp}\big] + (N-1)\xi^p_{jm}\psi_p + \xi^p_{jm\underset{1}{|}p} - \xi^p_{jp\underset{1}{|}m}\\
&+ 2\psi_j\underset{\vee}{\overline{L}}{}^p_{mp} + \xi^p_{jm}\xi^q_{pq} - \xi^p_{jq}\xi^q_{pm} + 2\underset{\vee}{L}{}^p_{mj}\psi_p + 2\underset{\vee}{L}{}^p_{mq}\xi^q_{jp}.
\end{aligned}
$$

Let us denote

$$
(1.6)\qquad
\begin{aligned}
\underset{1}{\mathcal{D}}_{jm} &= \frac{N-1}{N+1}\xi^p_{jm}\psi_p + \frac{2}{N^2-1}(N\psi_j\underset{\vee}{\overline{L}}{}^p_{mp} - \psi_m\underset{\vee}{\overline{L}}{}^p_{jp} + N\underset{\vee}{L}{}^p_{mq}\xi^q_{jp} + \underset{\vee}{L}{}^p_{jq}\xi^q_{mp})\\
&+ \frac{1}{N+1}\xi^p_{jm}\xi^q_{pq} - \frac{1}{N-1}\xi^p_{jq}\xi^q_{pm} + \frac{2}{N+1}\underset{\vee}{L}{}^p_{mj}\psi_p.
\end{aligned}
$$

Now, (1.5) we can expresed in the form

$$
(1.7)\qquad
\begin{aligned}
\underset{1}{\psi}_{jm} &= \frac{1}{N-1}\big\{\underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm} - \frac{1}{N+1}\big[\underset{1}{R}_{[jm]} - \underset{1}{\overline{R}}_{[jm]}\\
&+ 2\xi^p_{jm\underset{1}{|}p} - \xi^p_{jp\underset{1}{|}m} + \xi^p_{mp\underset{1}{|}j}\big] + \xi^p_{jm\underset{1}{|}p} - \xi^p_{jp\underset{1}{|}m}\big\} + \underset{1}{\mathcal{D}}_{jm},
\end{aligned}
$$

i.e.

$$
(1.8)\qquad
\begin{aligned}
\underset{1}{\psi}_{jm} &= \frac{1}{N-1}(\underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm}) - \frac{1}{N^2-1}(\underset{1}{R}_{[jm]} - \underset{1}{\overline{R}}_{[jm]})\\
&+ \frac{1}{N+1}(\underset{\vee}{\overline{L}}{}^p_{jm\underset{1}{|}p} - \underset{\vee}{L}{}^p_{lm\underset{1}{|}p}) - \frac{N}{N+1}(\underset{\vee}{\overline{L}}{}^p_{jp\underset{1}{|}m} - \underset{\vee}{L}{}^p_{jp\underset{1}{|}m})\\
&- \frac{1}{N^2-1}(\underset{\vee}{\overline{L}}{}^p_{mp\underset{1}{|}j} - \underset{\vee}{L}{}^p_{mp\underset{1}{|}j}) + \underset{1}{\mathcal{D}}_{jm}.
\end{aligned}
$$

Substituting (1.8) in (1.1) one obtains

$$
\begin{aligned}
&\underset{1}{\overline{R}}{}^{i}_{jmn} = \underset{1}{R}{}^{i}{}_{jmn} + \frac{1}{N-1}\delta^i_j\,(\underset{1}{R}_{[mn]} - \underset{1}{\overline{R}}_{[mn]})\\
&- \frac{1}{N^2-1}\delta^i_j\,(2\underset{1}{R}_{[mn]} - 2\underset{1}{\overline{R}}_{[mn]}) + \frac{2}{N+1}\delta^i_j\,(\overline{L}{}^p_{\underset{\vee}{mn}|\underset{1}{p}} - L^p_{\underset{\vee}{mn}|\underset{1}{p}})\\
(1.9)\quad &- \frac{N}{N+1}\delta^i_j\,(\overline{L}{}^p_{\underset{\vee}{mp}|\underset{1}{n}} - \overline{L}{}^p_{\underset{\vee}{np}|\underset{1}{m}} - L^p_{\underset{\vee}{mp}|\underset{1}{n}} + L^p_{\underset{\vee}{np}|\underset{1}{m}}) - \frac{1}{N^2-1}\delta^i_j\,(\overline{L}{}^p_{\underset{\vee}{np}|\underset{1}{m}} - \overline{L}{}^p_{\underset{\vee}{mp}|\underset{1}{n}}\\
&- L^p_{\underset{\vee}{np}|\underset{1}{m}} + L^p_{\underset{\vee}{mp}|\underset{1}{n}}) + \delta^i_j\,\underset{1}{\mathcal{D}}_{mn} + \frac{1}{N-1}\delta^i_m\,(\underset{1}{R}_{jn} - \underset{1}{\overline{R}}_{jn})\\
&- \frac{1}{N^2-1}\delta^i_m\,(\underset{1}{R}_{[jn]} - \underset{1}{\overline{R}}_{[jn]}) + \frac{1}{N+1}\delta^i_m\,(\overline{L}{}^p_{\underset{\vee}{jn}|\underset{1}{p}} - L^p_{\underset{\vee}{jn}|\underset{1}{p}})
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{N}{N+1}\delta^i_m\,(\overline{L}{}^p_{\underset{\vee}{jp}|\underset{1}{n}} - L^p_{\underset{\vee}{jp}|\underset{1}{n}}) - \frac{1}{N^2-1}\delta^i_m\,(\overline{L}{}^p_{\underset{\vee}{np}|\underset{1}{j}} - L^p_{\underset{\vee}{np}|\underset{1}{j}}) + \delta^i_m\,\underset{1}{\mathcal{D}}_{jn}\\
&- \frac{1}{N-1}\delta^i_n\,(\underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm}) + \frac{1}{N^2-1}\delta^i_n\,(\underset{1}{R}_{[jm]} - \underset{1}{\overline{R}}_{[jm]})\\
(1.9)\quad &- \frac{1}{N+1}\delta^i_n\,(\overline{L}{}^p_{\underset{\vee}{jm}|\underset{1}{p}} - L^p_{\underset{\vee}{jm}|\underset{1}{p}}) - \frac{N}{N+1}\delta^i_m\,(\overline{L}{}^p_{\underset{\vee}{jp}|\underset{1}{m}} - L^p_{\underset{\vee}{jp}|\underset{1}{m}})\\
&+ \frac{1}{N^2-1}\delta^i_n\,(\overline{L}{}^p_{\underset{\vee}{mp}|\underset{1}{j}} - L^p_{\underset{\vee}{mp}|\underset{1}{j}}) - \delta^i_n\,\underset{1}{\mathcal{D}}_{jm} - \delta^i_m\,\xi^p_{jn}\psi_p + \delta^i_n\,\xi^p_{jm}\psi_p + \xi^i_{jm|\underset{1}{n}}\\
&- \xi^i_{jn|\underset{1}{m}} + 2\psi_j\xi^i_{mn} + \xi^p_{jm}\xi^i_{pn} - \xi^p_{jn}\xi^i_{pm} + 2L^i_{\underset{\vee}{mn}}\psi_j + 2L^p_{\underset{\vee}{mn}}\psi_p\delta^i_j + 2L^p_{\underset{\vee}{mn}}\xi^i_{jp}.
\end{aligned}
$$

Introducing the condition

$$
\begin{aligned}
(1.10)\quad &\delta^i_j\,\underset{1}{\mathcal{D}}_{[mn]} + \delta^i_m\,\underset{1}{\mathcal{D}}_{jn} - \delta^i_n\,\underset{1}{\mathcal{D}}_{jm} - \delta^i_m\,\xi^p_{jn}\psi_p + \delta^i_n\,\xi^p_{jm}\psi_p + 2\psi_j\xi^i_{mn}\\
&+ \xi^p_{jm}\xi^i_{pn} - \xi^p_{jn}\xi^i_{pm} + 2L^i_{\underset{\vee}{mn}}\psi_j + 2L^p_{\underset{\vee}{mn}}\psi_p\delta^i_j + 2L^p_{\underset{\vee}{mn}}\xi^i_{jp} = 0,
\end{aligned}
$$

(1.9) can be expresed in the form

$$
(1.11)\qquad \overline{\mathcal{W}}(\underset{1}{R})^{i}{}_{jmn} = \mathcal{W}(\underset{1}{R})^{i}{}_{jmn}
$$

where

$$
\begin{aligned}
&\mathcal{W}(\underset{1}{R})^{i}{}_{jmn} = \underset{1}{R}{}^{i}{}_{jmn} + \frac{1}{N+1}\delta^i_j\,\underset{1}{R}_{[mn]} + \frac{1}{N^2-1}[(N\underset{1}{R}_{jn} + \underset{1}{R}_{nj})\delta^i_m\\
&- (N\underset{1}{R}_{jm} + \underset{1}{R}_{mj})\delta^i_n] - \frac{2}{N+1}\delta^i_j\,L^p_{\underset{\vee}{mn}|\underset{1}{p}} + \frac{N^2-N-1}{N^2-1}\delta^i_j\,(L^p_{\underset{\vee}{mp}|\underset{1}{n}} - L^p_{\underset{\vee}{np}|\underset{1}{m}})\\
(1.12)\quad &- \frac{1}{N+1}\delta^i_m\,L^p_{\underset{\vee}{jn}|\underset{1}{p}} + \frac{1}{N+1}\delta^i_m\,(NL^p_{\underset{\vee}{jp}|\underset{1}{n}} + \frac{1}{N-1}L^p_{\underset{\vee}{np}|\underset{1}{j}}) + \frac{1}{N+1}\delta^i_n\,L^p_{\underset{\vee}{jm}|\underset{1}{p}}\\
&- \frac{1}{N+1}\delta^i_n\,(NL^p_{\underset{\vee}{jp}|\underset{1}{m}} + \frac{1}{N-1}L^p_{\underset{\vee}{mp}|\underset{1}{j}}) - L^i_{\underset{\vee}{jm}|\underset{1}{n}} + L^i_{\underset{\vee}{jn}|\underset{1}{m}}.
\end{aligned}
$$

Definition 1.1. The geodesic mapping $f : GA_N \to G\overline{A}_N$ is $\underset{1}{R}$-projective if the condition (1.10) is satisfied.

Definition 1.2. The space GA_N is $\underset{1}{R}$-projectively flat if there exists an $\underset{1}{R}$-projective mapping of the space GA_N to a flat space (i.e. to a space, whose connection coeffitients in special coordinates are zero). So, we proved

Theorem 1.1. *The tensor* (1.12) *is an invariant of an* $\underset{1}{R}$*-projective mapping.*

Theorem 1.2. *If* GA_N *is* $\underset{1}{R}$*-projectively flat, then*

$$\mathcal{W}(\underset{1}{R})^i{}_{jmn} = 0. \tag{1.13}$$

Proof. $G\overline{A}_N$ is a flat space. Then by (1.12) we get $\overline{\mathcal{W}}(\underset{1}{R})^i{}_{jmn} = 0$, and using (1.11) we can see that (1.13) holds.

2. $\underset{2}{R}$-projective mappings

Definition 2.1. The geodesic mapping $f : GA_N \to G\overline{A}_N$ is $\underset{2}{R}$-projective if the following condition is satisfied

$$\begin{aligned}\delta^i_j \underset{2}{\mathcal{D}}_{[mn]} &+ \delta^i_m \underset{2}{\mathcal{D}}_{jn} - \delta^i_n \underset{2}{\mathcal{D}}_{jm} - \delta^i_m \xi^p_{nj}\psi_p + \delta^i_n \xi^p_{mj}\psi_p + 2\psi_j\xi^i_{nm} \\ &+ \xi^p_{mj}\xi^i_{np} - \xi^p_{nj}\xi^i_{mp} + 2L^i_{\underset{\vee}{nm}}\psi_j + 2L^p_{\underset{\vee}{nm}}\psi_p\delta^i_j + 2L^p_{\underset{\vee}{nm}}\xi^i_{pj} = 0.\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}\underset{2}{\mathcal{D}}_{jm} = \frac{N-1}{N+1}\xi^p_{mj}\psi_p &+ \frac{2}{N^2-1}(N\psi_j\overline{L}^p_{\underset{\vee}{pm}} - \psi_m\overline{L}^p_{\underset{\vee}{pj}} + NL^p_{\underset{\vee}{qm}}\xi^q_{pj} \\ &+ L^p_{\underset{\vee}{qj}}\xi^q_{pm}) + \frac{1}{N+1}\xi^p_{mj}\xi^q_{qp} - \frac{1}{N-1}\xi^p_{qj}\xi^q_{mp} + \frac{2}{N+1}L^p_{\underset{\vee}{jm}}\psi_p,\end{aligned} \tag{2.2}$$

For the curvature tensors $\underset{2}{R}$ and $\underset{2}{\overline{R}}$ of the space GA_N and $G\overline{A}_N$ we have the relation

$$\begin{aligned}\underset{2}{\overline{R}}{}^i_{jmn} &= \underset{2}{R}{}^i{}_{jmn} + \delta^i_j(\underset{2}{\psi}_{mn} - \underset{2}{\psi}_{nm}) + \delta^i_m \underset{2}{\psi}_{jn} - \delta^i_n \underset{2}{\psi}_{jm} \\ &- \delta^i_m \xi^p_{nj}\psi_p + \delta^i_n \xi^p_{mj}\psi_p + \xi^i_{mj\underset{2}{|}n} - \xi^i_{nj\underset{2}{|}m} + 2\psi_j\xi^i_{nm} \\ &+ \xi^p_{mj}\xi^i_{np} - \xi^p_{nj}\xi^i_{mp} + 2L^i_{\underset{\vee}{nm}}\psi_j + 2L^p_{\underset{\vee}{nm}}\psi_p\delta^i_j + 2L^p_{\underset{\vee}{nm}}\xi^i_{pj}.\end{aligned} \tag{2.3}$$

Analogously to the previous case, we get an invariant of an $\underset{2}{R}$-projective mapping $f : GA_N \to G\overline{A}_N$

$$
(2.4)\quad
\begin{aligned}
\mathcal{W}(\underset{2}{R})^i{}_{jmn} &= \underset{2}{R}^i{}_{jmn} + \frac{1}{N+1}\delta^i_j\,\underset{2}{R}_{[mn]} + \frac{1}{N^2-1}[(N\underset{2}{R}_{jn} + \underset{2}{R}_{nj})\delta^i_m \\
&- (N\underset{2}{R}_{jm} + \underset{2}{R}_{mj})\delta^i_n\,] - \frac{2}{N+1}\delta^i_j\,L^p_{\underset{\vee}{nm}\underset{2}{|}p} + \frac{N^2-N-1}{N^2-1}\delta^i_j\,(L^p_{\underset{\vee}{pm}\underset{2}{|}n} - L^p_{\underset{\vee}{pn}\underset{2}{|}m}) \\
&- \frac{1}{N+1}\delta^i_m\,L^p_{\underset{\vee}{nj}\underset{2}{|}p} + \frac{1}{N+1}\delta^i_m\,(NL^p_{\underset{\vee}{pj}\underset{2}{|}n} + \frac{1}{N-1}L^p_{\underset{\vee}{pn}\underset{2}{|}j}) + \frac{1}{N+1}\delta^i_n\,L^p_{\underset{\vee}{mj}\underset{2}{|}p} \\
&- \frac{1}{N+1}\delta^i_n\,(NL^p_{\underset{\vee}{pj}\underset{2}{|}m} + \frac{1}{N-1}L^p_{\underset{\vee}{pm}\underset{2}{|}j}) - L^i_{\underset{\vee}{mj}\underset{2}{|}n} + L^i_{\underset{\vee}{nj}\underset{2}{|}m}.
\end{aligned}
$$

Consequently, the next theorems are valid

Theorem 2.1. *The tensor* (2.4) *is an invariant of an $\underset{2}{R}$-projective mapping $f : GA_N \to G\overline{A}_N$.*

Theorem 2.2. *If GA_N is $\underset{2}{R}$-projectively flat then we have*

$$\mathcal{W}(\underset{2}{R})^i{}_{jmn} = 0.$$

3. $\underset{3}{R}$-projective mappings

Definition 3.1. The geodesic mapping $f : GA_N \to G\overline{A}_N$ is $\underset{3}{R}$-projective if the following condition holds

$$
(3.1)\quad
\begin{aligned}
&\delta^i_j\,\underset{3}{\mathcal{D}}_{[mn]} + \delta^i_m\,\underset{3}{\mathcal{D}}_{jn} - \delta^i_n\,\underset{3}{\mathcal{D}}_{jm} + 2\delta^i_j\,L^p_{\underset{\vee}{mn}}\psi_p + 2\delta^i_m\,L^p_{\underset{\vee}{jn}}\psi_p \\
&\quad + (\delta^i_n\,\xi^p_{jm} - \delta^i_m\,\xi^p_{nj})\psi_p + \xi^p_{jm}\xi^i_{np} - \xi^p_{jn}\xi^i_{pm} + 2\psi_n(L^i_{\underset{\vee}{mj}} + \xi^i_{mj}) \\
&\quad + 2\psi_m(L^i_{\underset{\vee}{nj}} + \xi^i_{nj}) + 2\xi^p_{nm}(L^i_{\underset{\vee}{pj}} + \xi^i_{pj}) = 0.
\end{aligned}
$$

where

$$
\begin{aligned}
\underset{3}{\mathcal{D}}_{jm} &= \xi^p_{jm}\psi_p + \frac{1}{N+1}\xi^p_{jm}\xi^q_{qp} - \frac{2}{N^2-1}\big[2\psi_p(L^p_{\underset{\vee}{mj}} + \xi^p_{mj}) \\
&+ \psi_m(L^p_{\underset{\vee}{pj}} + \xi^p_{pj}) - \psi_j(L^p_{\underset{\vee}{pm}} + \xi^p_{pm}) + \xi^p_{qm}L^q_{\underset{\vee}{pj}} - \xi^p_{qj}L^q_{\underset{\vee}{pm}}\big] \\
&+ \frac{1}{N-1}\big[2\psi_p(L^p_{\underset{\vee}{mj}} + \xi^p_{mj}) + 2\psi_m(L^p_{\underset{\vee}{pj}} + \xi^p_{pj}) + 2\xi^p_{qm}(L^q_{\underset{\vee}{pj}} + \xi^q_{pj}) - \xi^p_{jq}\xi^q_{pm}\big]
\end{aligned}
$$

Definition 3.2. The space GA_N is $\underset{3}{R}$-projectively flat if there exists an $\underset{3}{R}$-projective mapping of the space GA_N into a flat space.

In the case of curvature tensors of the third kind of the spaces GA_N and $G\overline{A}_N$ we get the relation

$$
\begin{aligned}
\overline{\underset{3}{R}}{}^i_{jmn} &= \underset{3}{R}{}^i{}_{jmn} + \delta^i_j\,(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m\,\underset{2}{\psi}_{jn} - \delta^i_n\,\underset{1}{\psi}_{jm} \\
&+ \psi_p(\delta^i_n\,\xi^p_{jm} - \delta^i_m\,\xi^p_{nj}) + \xi^i_{jm\underset{2}{|}n} - \xi^i_{nj\underset{1}{|}m} + \xi^p_{jm}\xi^i_{np} - \xi^p_{jn}\xi^i_{pm} \\
&+ 2\psi_n(\underset{\vee}{L}{}^i_{mj} + \xi^i_{mj}) + 2\psi_m(\underset{\vee}{L}{}^i_{nj} + \xi^i_{nj}) + 2\xi^p_{nm}(\underset{\vee}{L}{}^i_{pj} + \xi^i_{pj}).
\end{aligned}
\tag{3.2}
$$

Also, it holds

$$
\underset{2}{\psi}_{mn} = \underset{1}{\psi}_{mn} + 2\underset{\vee}{L}{}^p_{mn}\psi_p. \tag{3.3}
$$

¿From (3.2) and (3.3) we get

$$
\begin{aligned}
\overline{\underset{3}{R}}{}^i_{jmn} &= \underset{3}{R}{}^i{}_{jmn} + \delta^i_j\,(\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m\,\underset{1}{\psi}_{jn} - \delta^i_n\,\underset{1}{\psi}_{jm} + 2\delta^i_j\,\underset{\vee}{L}{}^p_{mn}\psi_p \\
&+ 2\delta^i_m\,\underset{\vee}{L}{}^p_{jn}\psi_p + (\delta^i_n\,\xi^p_{jm} - \delta^i_m\,\xi^p_{nj})\psi_p + \xi^i_{jm\underset{2}{|}n} - \xi^i_{nj\underset{1}{|}m} + \xi^p_{jm}\xi^i_{np} - \xi^p_{jn}\xi^i_{pm} \\
&+ 2\psi_n(\underset{\vee}{L}{}^i_{mj} + \xi^i_{mj}) + 2\psi_m(\underset{\vee}{L}{}^i_{nj} + \xi^i_{nj}) + 2\xi^p_{nm}(\underset{\vee}{L}{}^i_{pj} + \xi^i_{pj}).
\end{aligned}
\tag{3.4}
$$

Contracting (3.4) with respect to i and n we get

$$
\begin{aligned}
\overline{\underset{3}{R}}_{jm} &= \underset{3}{R}_{jm} - \underset{1}{\psi}_{[jm]} - (N-1)\underset{1}{\psi}_{jm} + (N+1)\xi^p_{jm}\psi_p + \xi^p_{jm\underset{2}{|}p} - \xi^p_{pj\underset{2}{|}m} \\
&+ \xi^p_{jm}\xi^q_{qp} - \xi^p_{jq}\xi^q_{pm} + 2\psi_p(\underset{\vee}{L}{}^p_{mj} + \xi^p_{mj}) + 2\psi_m(\underset{\vee}{L}{}^p_{pj} + \xi^p_{pj}) + 2\xi^p_{qm}(\underset{\vee}{L}{}^q_{pj} + \xi^q_{pj}),
\end{aligned}
\tag{3.5}
$$

hence

$$
\begin{aligned}
(N+1)\underset{1}{\psi}_{[jm]} &= \underset{3}{R}_{[jm]} - \overline{\underset{3}{R}}_{[jm]} + 2(N+1)\xi^p_{jm}\psi_p + 2\xi^p_{jm\underset{2}{|}p} - \xi^p_{pj\underset{1}{|}m} + \xi^p_{pm\underset{1}{|}j} \\
&+ 2\xi^p_{jm}\xi^q_{qp} - \xi^p_{jq}\xi^q_{pm} + \xi^p_{mq}\xi^q_{pj} + 4\psi_p(\underset{\vee}{L}{}^p_{mj} + \xi^p_{mj}) + 2\psi_m(\underset{\vee}{L}{}^p_{pj} + \xi^p_{pj}) \\
&- 2\psi_j(\underset{\vee}{L}{}^p_{pm} + \xi^p_{pm}) + 2\xi^p_{qm}(\underset{\vee}{L}{}^q_{pj} + \xi^q_{pj}) - 2\xi^p_{qj}(\underset{\vee}{L}{}^q_{pm} + \xi^q_{pm}).
\end{aligned}
\tag{3.6}
$$

Substituting (3.6) in (3.5) we get

$$
\begin{aligned}
(N-1)\underset{1}{\psi}_{jm} &= \underset{3}{R}_{jm} - \overline{\underset{3}{R}}_{jm} - \frac{1}{N+1}\big[\underset{3}{R}_{[jm]} - \overline{\underset{3}{R}}_{[jm]} + 2\xi^p_{jm\underset{2}{|}p} \\
&- \xi^p_{pj\underset{1}{|}m} + \xi^p_{pm\underset{1}{|}j}\big] + \xi^p_{jm\underset{2}{|}p} - \xi^p_{pj\underset{1}{|}m} + (N-1)\underset{3}{\mathcal{D}}_{jm},
\end{aligned}
\tag{3.7}
$$

Now, from (3.4,6,7) by condition (3.1) we get

$$\overline{\mathcal{W}}(\underset{3}{R})^i{}_{jmn} = \mathcal{W}(\underset{3}{R})^i{}_{jmn} \tag{3.8}$$

where

$$\begin{aligned}\mathcal{W}(\underset{3}{R})^i{}_{jmn} &= \underset{3}{R}^i{}_{jmn} + \frac{1}{N+1}\delta^i_j\,\underset{3}{R}_{[mn]} + \frac{1}{N^2-1}[(N\underset{3}{R}_{jn} + \underset{3}{R}_{nj})\delta^i_m \\ &- (N\underset{3}{R}_{jm} + \underset{3}{R}_{mj})\delta^i_n] + \frac{2}{N^2-1}\delta^i_j\,(2L^p_{\underset{\vee}{mn}|p \atop 2} - L^p_{\underset{\vee}{pm}|n \atop 1} + L^p_{\underset{\vee}{pn}|m \atop 1} - L^p_{\underset{\vee}{mn}|p \atop 2}) \\ &+ \frac{1}{N-1}\delta^i_j\,(L^p_{\underset{\vee}{pm}|n \atop 1} - L^p_{\underset{\vee}{pn}|m \atop 1}) + \frac{1}{N^2-1}\delta^i_m\,(2L^p_{\underset{\vee}{jn}|p \atop 2} - L^p_{\underset{\vee}{pj}|n \atop 1} + L^p_{\underset{\vee}{pn}|j \atop 1}) \\ &- \frac{1}{N-1}\delta^i_m\,(L^p_{\underset{\vee}{jn}|p \atop 2} - L^p_{\underset{\vee}{pj}|n \atop 1}) - \frac{1}{N^2-1}\delta^i_n\,(2L^p_{\underset{\vee}{jm}|p \atop 2} - L^p_{\underset{\vee}{pj}|m \atop 1} + L^p_{\underset{\vee}{pm}|j \atop 1}) \\ &+ \frac{1}{N-1}\delta^i_n\,(L^p_{\underset{\vee}{jm}|p \atop 2} - L^p_{\underset{\vee}{pj}|m \atop 1}) - L^i_{\underset{\vee}{jm}|n \atop 2} + L^i_{\underset{\vee}{nj}|m \atop 1}\end{aligned}$$

Consequently, the next theorems hold:

Theorem 3.1. *The tensor* $\mathcal{W}(\underset{3}{R})^i{}_{jmn}$ *is an invariant of an* $\underset{3}{R}$*-projective mapping.*

Theorem 3.2. *If* GA_N *is* $\underset{3}{R}$*-projectively flat then*

$$\mathcal{W}(\underset{3}{R})^i{}_{jmn} = 0.$$

4. $\underset{4}{R}$-projective mappings

Definition 4.1. A geodesic mapping $f : GA_N \to G\overline{A}_N$ is $\underset{4}{R}$-projective if the following condition is satisfied

$$\begin{aligned}&\delta^i_j\,\underset{4}{\mathcal{D}}_{[mn]} + \delta^i_m\,\underset{4}{\mathcal{D}}_{jn} - \delta^i_n\,\underset{4}{\mathcal{D}}_{jm} + 2\delta^i_j\,L^p_{\underset{\vee}{mn}}\psi_p + 2\delta^i_m\,L^p_{\underset{\vee}{jn}}\psi_p \\ &\quad + (\delta^i_n\,\xi^p_{jm} - \delta^i_m\,\xi^p_{nj})\psi_p + \xi^p_{jm}\xi^i_{np} - \xi^p_{jn}\xi^i_{pm} + 2\psi_n(L^i_{\underset{\vee}{mj}} + \xi^i_{mj}) \\ &\quad + 2\psi_m(L^i_{\underset{\vee}{nj}} + \xi^i_{nj}) + 2\xi^p_{mn}(L^i_{\underset{\vee}{pj}} + \xi^i_{pj}) = 0.\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}\underset{4}{\mathcal{D}}_{jm} &= \xi^p_{jm}\psi_p + \frac{1}{N+1}\xi^p_{jm}\xi^q_{qp} - \frac{2}{N^2-1}\big[2\psi_p(L^p_{\underset{\vee}{mj}} + \xi^p_{mj}) \\ &+ \psi_m(L^p_{\underset{\vee}{pj}} + \xi^p_{pj}) - \psi_j(L^p_{\underset{\vee}{pm}} + \xi^p_{pm}) + \xi^p_{mq}L^q_{\underset{\vee}{pj}} - \xi^p_{jq}L^q_{\underset{\vee}{pm}}\big] \\ &+ \frac{1}{N-1}\big[2\psi_p(L^p_{\underset{\vee}{mj}} + \xi^p_{mj}) + 2\psi_m(L^p_{\underset{\vee}{pj}} + \xi^p_{pj}) + 2\xi^p_{mq}(L^q_{\underset{\vee}{pj}} + \xi^q_{pj}) - \xi^p_{jq}\xi^q_{pm}\big]\end{aligned}$$

Definition 4.2. The space GA_N is $\underset{4}{R}$-projectively flat if there exists an $\underset{4}{R}$-projective mapping of the space GA_N into a flat space.

For curvature tensors of the fourth kind we get

$$
\begin{aligned}
\overline{\underset{4}{R}}{}^i_{jmn} &= \underset{4}{R}{}^i_{jmn} + \delta^i_j(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m \underset{2}{\psi}_{jn} - \delta^i_n \underset{1}{\psi}_{jm} \\
&+ \psi_p(\delta^i_n \xi^p_{jm} - \delta^i_m \xi^p_{nj}) + \xi^i_{jm\underset{2}{|}n} - \xi^i_{nj\underset{1}{|}m} + \xi^p_{jm}\xi^i_{np} - \xi^p_{jn}\xi^i_{pm} \\
&+ 2\psi_n(L^i_{\underset{\vee}{mj}} + \xi^i_{mj}) + 2\psi_m(L^i_{\underset{\vee}{nj}} + \xi^i_{nj}) + 2\xi^p_{mn}(L^i_{\underset{\vee}{pj}} + \xi^i_{pj}).
\end{aligned}
\tag{4.2}
$$

Now analogously to the previous cases, we get an invariant of an $\underset{4}{R}$-projective mapping in the form

$$
\begin{aligned}
\mathcal{W}(\underset{4}{R})^i{}_{jmn} &= \underset{4}{R}{}^i{}_{jmn} + \frac{1}{N+1}\delta^i_j \underset{4}{R}_{[mn]} + \frac{1}{N^2-1}[(N\underset{4}{R}_{jn} + \underset{4}{R}_{nj})\delta^i_m \\
&- (N\underset{4}{R}_{jm} + \underset{4}{R}_{mj})\delta^i_n] + \frac{2}{N^2-1}\delta^i_j(2L^p_{\underset{\vee}{mn}\underset{2}{|}p} - L^p_{\underset{\vee}{pm}\underset{1}{|}n} + L^p_{\underset{\vee}{pn}\underset{1}{|}m} \\
&- L^p_{\underset{\vee}{mn}\underset{2}{|}p}) + \frac{1}{N-1}\delta^i_j(L^p_{\underset{\vee}{pm}\underset{1}{|}n} - L^p_{\underset{\vee}{pn}\underset{1}{|}m}) + \frac{1}{N^2-1}\delta^i_m(2L^p_{\underset{\vee}{jn}\underset{2}{|}p} \\
&- L^p_{\underset{\vee}{pj}\underset{1}{|}n} + L^p_{\underset{\vee}{pn}\underset{1}{|}j}) - \frac{1}{N-1}\delta^i_m(L^p_{\underset{\vee}{jn}\underset{2}{|}p} - L^p_{\underset{\vee}{pj}\underset{1}{|}n}) - \frac{1}{N^2-1}\delta^i_n(2L^p_{\underset{\vee}{jm}\underset{2}{|}p} \\
&- L^p_{\underset{\vee}{pj}\underset{1}{|}m} + L^p_{\underset{\vee}{pm}\underset{1}{|}j}) + \frac{1}{N-1}\delta^i_n(L^p_{\underset{\vee}{jm}\underset{2}{|}p} - L^p_{\underset{\vee}{pj}\underset{1}{|}m}) - L^i_{\underset{\vee}{jm}\underset{2}{|}n} + L^i_{\underset{\vee}{nj}\underset{1}{|}m}
\end{aligned}
$$

i.e. the next theorems hold:

Theorem 4.1. *The tensor $\mathcal{W}(\underset{4}{R})^i{}_{jmn}$ is an invariant of an $\underset{4}{R}$-projective mapping.*

Theorem 4.2. *If GA_N is $\underset{4}{R}$-projectively flat then*

$$\mathcal{W}(\underset{4}{R})^i{}_{jmn} = 0.$$

5. $\underset{5}{R}$-projective mappings

Definition 5.1. A geodesic mapping $f : GA_N \to G\overline{A}_N$ is $\underset{5}{R}$-projective if the following condition is satisfied

$$
\begin{aligned}
&\frac{2}{N+1}\delta^i_j \xi^p_{mn}\xi^q_{pq} + \frac{1}{N+1}\delta^i_m \xi^p_{jn}\xi^q_{pq} - \frac{1}{N+1}\delta^i_n \xi^p_{jm}\xi^q_{pq} \\
&- \frac{1}{N+1}\delta^i_m \xi^p_{jq}\xi^q_{np} + \frac{1}{N-1}\delta^i_n \xi^p_{jq}\xi^q_{mp} + \xi^p_{jm}\xi^i_{pn} - \xi^p_{jn}\xi^i_{mp} = 0
\end{aligned}
\tag{5.1}
$$

Definition 5.2. The space GA_N is $\underset{5}{R}$-projectively flat if there exists an $\underset{5}{R}$-projective mapping of the space GA_N into a flat space.

For curvature tensors of the fifth kind (0.9) of the spaces GA_N and $G\overline{A}_N$ we find the relation

$$
\begin{aligned}
\overline{\underset{5}{R}}{}^i_{jmn} = {} & \underset{5}{R}{}^i{}_{jmn} + \frac{1}{2}(\underset{1}{\psi}_{mn} - \underset{2}{\psi}_{nm} + \underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) \\
& + \frac{1}{2}\delta^i_m(\underset{1}{\psi}_{jn} + \underset{2}{\psi}_{jn}) - \frac{1}{2}\delta^i_n(\underset{2}{\psi}_{jm} + \underset{1}{\psi}_{jm}) + \frac{1}{2}(\xi^i_{jm\underset{3}{|}n} - \xi^i_{jn\underset{4}{|}m} \\
& + \xi^i_{mj\underset{4}{|}n} - \xi^i_{nj\underset{3}{|}m} + \xi^p_{jm}\xi^i_{pn} - \xi^p_{jn}\xi^i_{mp} + \xi^p_{mj}\xi^i_{np} - \xi^p_{nj}\xi^i_{pm}).
\end{aligned} \tag{5.2}
$$

Putting

$$\underset{12}{\psi}_{mn} = \frac{1}{2}(\underset{1}{\psi}_{mn} + \underset{2}{\psi}_{mn})$$

we get from (5.2)

$$
\begin{aligned}
\overline{\underset{5}{R}}{}^i_{jmn} = {} & \underset{5}{R}{}^i{}_{jmn} + (\underset{12}{\psi}_{mn} - \underset{12}{\psi}) + \delta^i_m\underset{12}{\psi}_{jn} - \delta^i_n\underset{12}{\psi}_{jm} + \frac{1}{2}(\xi^i_{jm\underset{3}{|}n} - \xi^i_{jn\underset{4}{|}m} \\
& + \xi^i_{mj\underset{4}{|}n} - \xi^i_{nj\underset{3}{|}m} + \xi^p_{jm}\xi^i_{pn} - \xi^p_{jn}\xi^i_{mp} + \xi^p_{mj}\xi^i_{np} - \xi^p_{nj}\xi^i_{pm}),
\end{aligned} \tag{5.3}
$$

Eliminating $\underset{12}{\psi}_{mn}$ from (5.3), analogously to the previous cases, we get

$$\overline{\mathcal{W}}(\underset{5}{R})^i{}_{jmn} = \mathcal{W}(\underset{5}{R})^i{}_{jmn} \tag{5.4}$$

where

$$
\begin{aligned}
\mathcal{W}(\underset{5}{R})^i{}_{jmn} = {} & \underset{5}{R}{}^i{}_{jmn} + \frac{1}{N+1}\delta^i_j\underset{5}{R}_{[mn]} + \frac{1}{N^2-1}[(N\underset{5}{R}_{jn} + \underset{5}{R}_{nj})\delta^i_m \\
& - (N\underset{5}{R}_{jm} + \underset{5}{R}_{mj})\delta^i_n] - \frac{1}{N+1}\delta^i_j(L^p_{mn\underset{\vee 3}{|}p} + L^p_{nm\underset{\vee 4}{|}p}) \\
& - \frac{1}{2(N+1)}\delta^i_j(L^p_{np\underset{\vee 4}{|}m} - L^p_{mp\underset{\vee 4}{|}n} + L^p_{pn\underset{\vee 3}{|}m} - L^p_{pm\underset{\vee 3}{|}n}) \\
& - \frac{1}{2(N+1)}\delta^i_m(L^p_{jn\underset{\vee 3}{|}p} + L^p_{nj\underset{\vee 4}{|}p}) + \frac{1}{2(N^2-1)}\delta^i_m(L^p_{np\underset{\vee 4}{|}j} + L^p_{pn\underset{\vee 3}{|}j}) \\
& + \frac{N}{2(N^2-1)}\delta^i_m(L^p_{jp\underset{\vee 4}{|}n} + L^p_{pj\underset{\vee 3}{|}n}) + \frac{1}{2(N+1)}\delta^i_n(L^p_{jm\underset{\vee 3}{|}p} + L^p_{mj\underset{\vee 4}{|}p}) \\
& - \frac{1}{2(N^2-1)}\delta^i_n(L^p_{mp\underset{\vee 4}{|}j} + L^p_{pm\underset{\vee 3}{|}j}) - \frac{N}{2(N^2-1)}\delta^i_n(L^p_{jp\underset{\vee 4}{|}m} + L^p_{pj\underset{\vee 3}{|}m}) \\
& - \frac{1}{2}(L^i_{jm\underset{\vee 3}{|}n} - L^i_{jn\underset{\vee 4}{|}m} + L^i_{mj\underset{\vee 4}{|}n} - L^i_{nj\underset{\vee 3}{|}m}).
\end{aligned}
$$

Hence:

Theorem 5.1. *The tensor* $\mathcal{W}(\underset{5}{R})^i{}_{jmn}$ *is an invariant of an* $\underset{5}{R}$*-projective mapping.*

Theorem 5.2. *If* GA_N *is* $\underset{5}{R}$*-projectively flat then*

$$\mathcal{W}(\underset{5}{R})^i{}_{jmn} = 0.$$

In the case of generalized Riemannian space (GR_N) the connection coeffitiens are defined by means of a non-symmetric basic tensor [1]-[3], [7] and they are non-symmetric too. The tensors $W(\underset{\theta}{R})^i_{jmn}$ [9] obtained as invariants of a map $f : GR_N \to G\overline{R}_N$ are particular cases of obtained here tensors $\mathcal{W}(\underset{\theta}{R})^i{}_{jmn}$ $(\theta = 1, \cdots 5)$. For example

$$\begin{aligned} W(\underset{1}{R})^i_{jmn} &= \underset{1}{R}^i{}_{jmn} + \frac{1}{N+1}\delta^i_j \underset{1}{R}_{[mn]} + \frac{1}{N^2-1}[(N\underset{1}{R}_{jn} + \underset{1}{R}_{nj})\delta^i_m \\ &\quad - (N\underset{1}{R}_{jm} + \underset{1}{R}_{mj})\delta^i_n] - \frac{2}{N+1}\delta^i_j L^p_{\underset{\vee}{mn}\underset{1}{|}p} - \frac{1}{N+1}\delta^i_m L^p_{\underset{\vee}{jn}\underset{1}{|}p} \\ &\quad + \frac{1}{N+1}\delta^i_n L^p_{\underset{\vee}{jm}\underset{1}{|}p} - L^i_{\underset{\vee}{jm}\underset{1}{|}n} + L^i_{\underset{\vee}{jn}\underset{1}{|}m}. \end{aligned}$$

When GA_N (GR_N) reduces to the Riemannian space, the magnitudes $\mathcal{W}(\underset{\theta}{R})$ $(W(\underset{\theta}{R}))$ $(\theta = 1, \cdots 5)$ reduce to the Weil's tensor [8]

$$W^i{}_{jmn} = R^i{}_{jmn} + \frac{1}{N-1}(\delta^i_m R_{jn} - \delta^i_n R_{jm}).$$

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