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An Operation on Intuitionistic Fuzzy Matrices

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Abstract. In this paper, we define an operation on the intuitionistic fuzzy matrices called the Gödel implication operator as an extension to the definition of this operator in the case of ordinary fuzzy matrices due to Sanchez and Hashimoto. Using this operator, we prove several important results for intuitionistic fuzzy matrices. Particularly, some properties concerning pre-orders, sub-inverses, and regularity . We concentrate our discussion on the reflexive and transitive matrices. This studying enables us to give a largest sub-inverse and a largest generalized inverse for a reflexive and transitive intuitionistic fuzzy matrix. Also, we obtain an idempotent intuitionistic fuzzy matrix from any given one .

1. Introduction

In 1965, Zadeh [15] has introduced the concept of fuzzy sets as an extension to the theory of ordinary sets by assigning to each element in the universe X a number in the unit interval $[0, 1]$ called the degree of membership. In 1977, Thomason [14] has introduced the concept of fuzzy matrices (matrices having values any where in the closed interval $[0, 1]$). In 1980, Kim and Roush have developed a theory for fuzzy matrices analogous to that for Boolean matrices [7]. After that a lot of works have been done on fuzzy matrices and its variants $[4 - 6]$. In 1986, Atanasove [1] has introduced the concept of intuitionistic fuzzy sets as an extension to the theory of ordinary fuzzy sets by assigning to each element in the universe not only a membership degree but also a non-membership degree. In 2002, Khan, Shymal and Pal [10] have introduced the concept of intuitionistic fuzzy matrices as an extension to the theory of ordinary fuzzy matrices. So, the concept of intuitionistic fuzzy matrices (or finite intuitionistic fuzzy relations) is an extension of the concept of ordinary fuzzy matrices (finite fuzzy relations).

In this paper, we define the Gödel implication operator $\rhd$ as an extension of the Sanchez α operator on fuzzy relations [12]. Sanchez used this operator to solve some kinds of fuzzy relational equations. Also, Hashimoto used this operator for presenting many properties of fuzzy matrices [4, 5, 6]. By extending this operator to intuitionistic fuzzy matrices we also obtain several results as extension to the results obtained on the ordinary fuzzy matrices and finite binary relations or Boolean matrices due to Schein [13]. However, we concentrate our studying to some kinds of intuitionistic fuzzy matrices, namely, reflexive and transitive intuitionistic fuzzy matrices which are well known representing pre-orders. We obtain more than pre-order from any intuitionistic fuzzy matrix. Moreover, we can construct an idempotent intuitionistic fuzzy matrix

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from any given one through the operator $\triangleright$ as we shall see in Section 3. Also, this operator is useful in studying the sub-inverses and generalized inverses of intuitionistic fuzzy matrices as we also shall see in Section 4.

2. Preliminaries

In this section we recall the notion of an intuitionistic fuzzy matrix and we define some operations on intuitionistic fuzzy matrices. As it is well known, a fuzzy matrix A is a function from the Cartesian product $X \times Y$ to the unit interval $[0,1]$, where X and Y are finite. If $|X| = m, |Y| = n$, then the number $A(x_i, y_j) = a_{ij}$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ is called the degree of membership of the element $A(x_i, y_j)$ in the fuzzy matrix A. Thus in briefly, a fuzzy matrix takes its elements from the interval $[0,1]$ and we denote it by $A = \left[a_{ij}\right]_{m\times n}$. Now, we extend this definition to intuitionistic fuzzy matrices as follows.

Definition 2.1. (intuitionistic fuzzy matrices [2, 3, 10]) Let $A^{'} = [a^{'}_{ij}]_{m\times n}$ and $A^{''} = \left[a^{''}_{ij}\right]_{m\times n}$ be two fuzzy matrices such that $a^{'}_{ij} + a''_{ij} \leq 1$ for every $i \leq m$, and $j \leq n$. The pair $\left\langle A^{'}, A^{''}\right\rangle$ is called an intuitionistic fuzzy matrix and is denoted by A and then we may write $A = \left[a_{ij} = \left\langle a'_{ij}, a''_{ij}\right\rangle\right]_{m\times n}$.

As an example of an intuitionistic fuzzy matrix, we put the identity intuitionistic fuzzy matrix $I_n = \left[\delta_{ij} = \left\langle \delta'_{ij}, \delta''_{ij}\right\rangle\right]$ in the form

$$I_n = \begin{bmatrix} \langle 1,0\rangle \, \langle 0,1\rangle \ldots \langle 0,1\rangle \\ \langle 0,1\rangle \, \langle 1,0\rangle \ldots \langle 0,1\rangle \\ . \\ . \\ . \\ \langle 0,1\rangle \, \langle 0,1\rangle \ldots \langle 1,0\rangle \end{bmatrix}_{n\times n}$$

i.e.,

$$\delta'_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}, \delta''_{ij} = \begin{cases} 0 & \text{if} \quad i = j, \\ 1 & \text{if} \quad i \neq j. \end{cases}.$$

We see in Definition 2.1 that the intuitionistic fuzzy matrix is a pair of fuzzy matrices which represent a membership and a non-membership function, respectively. Thus, an intuitionistic fuzzy matrix takes its elements from the set $F = \{a =< a', a'' >: a', a'' \in [0,1], a' + a'' \leq 1\}$. When $a^{'}_{ij} + a^{''}_{ij} = 1$ for every $i \leq m$ and $j \leq n$, the intuitionistic fuzzy matrix A is reduced to be an ordinary fuzzy matrix. The existence of the membership degree interval $[a^{'}_{ij}, a''_{ij} = 1 - a^{'}_{ij}]$ is always possible thanks to the condition $a^{'}_{ij} + a^{''}_{ij} \leq 1$ which an intuitionistic fuzzy matrix $A = \left[a_{ij} = \left\langle a'_{ij}, a''_{ij}\right\rangle\right]$ should fulfill. The number $\pi_{ij} = 1 - a'_{ij} - a^{''}_{ij}$ is called an index of an element a_{ij} in the intuitionistic fuzzy matrix A. It is also described as an index or degree of hesitation whether a_{ij} is in the intuitionistic fuzzy matrix A or not. The larger of the intuitionistic indices π_{ij} the higher is the value of non-determinancy or uncertainty.

Now, we define some operations on the set F. For $a, b \in F$, we have:
$a \vee b =< a', a'' > \vee < b', b'' >=< a' \vee b', a'' \wedge b'' >$,
$a \wedge b =< a', a'' > \wedge < b', b'' >=< a' \wedge b', a'' \vee b'' >$,
$a \leq b$ if and only if $a' \leq b'$ and $a'' \geq b''$ where $a^{'} \vee b^{'} = \max(a^{'}, b^{'})$ and $a^{'} \wedge b^{'} = \min(a^{'}, b^{'})$.

We may write $\mathbf{0}$ instead of the element $< 0, 1 >\in F$ and $\mathbf{1}$ instead of the element $< 1, 0 >$. It is noted that $a \vee \mathbf{0} = \mathbf{0} \vee a = a$ and $a \wedge \mathbf{1} = \mathbf{1} \wedge a = a$, for any $a \in F$.

Basic operations on intuitionistic fuzzy matrices are extensions of the respective operations on fuzzy matrices. As a result, operations on fuzzy matrices are particular cases of the ones on intuitionistic fuzzy matrices which are defined in the following way.

Definition 2.2. [2, 3, 10] Let $A = \left[a_{ij} = \left\langle a'_{ij}, a''_{ij}\right\rangle\right]_{m\times n}$, $B = \left[b_{ij} = \left\langle b'_{ij}, b''_{ij}\right\rangle\right]_{m\times n}$ and $C = \left[c_{ij} = \left\langle c'_{ij}, c''_{ij}\right\rangle\right]_{n\times l}$ be three intuitionistic fuzzy matrices. We define the following operations:
$A \vee B = \left[a_{ij} \vee b_{ij}\right]_{m\times n}$,
$A \wedge B = \left[a_{ij} \wedge b_{ij}\right]_{m\times n}$,
$A^t = \left[a_{ji} = \left\langle a'_{ji}, a''_{ji}\right\rangle\right]$ (the transpose of A),
$D = AC = \left[d_{ij} = \left\langle d'_{ij}, d''_{ij}\right\rangle = \left\langle \bigvee\limits_{k=1}^{n} (a'_{ik} \wedge c'_{kj}), \bigwedge\limits_{k=1}^{n} (a''_{ik} \vee c''_{kj})\right\rangle\right]_{m\times l}$,
$A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for every $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

Definition 2.3. For $a = \langle a', a''\rangle, b = \langle b', b''\rangle \in F$, we define $a \rhd b$ as:

$$a \rhd b = \begin{cases} \langle 1, 0\rangle & \text{if } a' \leq b', \\ \langle b', 0\rangle & \text{if } a' > b', a'' \geq b'', \\ \langle b', b''\rangle & \text{if } a' > b', a'' < b''. \end{cases}$$

This definition is an extension of the definition of Hashimoto [4] for the ordinary fuzzy matrices which corresponds to Sanchez α operator [8]. We recall the definition of this operation in the ordinary fuzzy case which is as follows:

$$a \rhd b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a > b \end{cases}$$

for every $a, b \in [0, 1]$.

However, the operator $\rhd$ is the Gödel implication operator which is well known in many branches of fuzzy mathematics. Its properties in ordinary fuzzy case were examined by some authors [5, 6, 9, 11]. From the definition of the operation $\rhd$ on the set F, it is noted that $a \rhd b \geq b, a \rhd \mathbf{1} = \mathbf{1}$ and $\mathbf{0} \rhd a = \mathbf{1}$ for every $a, b \in F$. Moreover, $a \rhd b = b \lhd a$.

The $min-\rhd$ composition of two intuitionistic fuzzy matrices $A = \left[a_{ij} = \left\langle a'_{ij}, a''_{ij}\right\rangle\right]_{m\times n}$ and $C = \left[c_{ij} = \left\langle c'_{ij}, c''_{ij}\right\rangle\right]_{n\times l}$ is denoted by $D = A \rhd C$ and is defined as:

$$d_{ij} = \bigwedge_{k=1}^{n} (a_{ik} \rhd c_{kj}).$$

3. Reflexivity and Transitivity of Intuitionistic Fuzzy Matrices

In this section, we examine some properties of the operations defined above. Also, we examine in briefly, some properties of intuitionistic fuzzy matrices representing intuitionistic fuzzy pre-orders using the operations $\rhd$ ($\lhd$). We concentrate our discussions on the reflexive and transitive intuitionistic fuzzy matrices. Now, let us point out some useful properties of the operations $\lhd$ ($\rhd$).

Lemma 3.1. For $a = \langle a', a''\rangle, b = \langle b', b''\rangle, c = \langle c', c''\rangle \in F$, we have: $(a \rhd b) \lhd c = a \rhd (b \lhd c) = b \lhd (a \wedge c)$.

Proof. Based on the definition of the operation $\rhd$ on the set F, we have the following three cases, each of them has also three subcases. The cases are:

Case (1). If $a' \leq b'$.
Case (2). If $a' > b'$ and $a'' \geq b''$.
Case (3). If $a' > b'$ and $a'' < b''$.

The subcases of each case are:
(i) If $c' \leq b'$.
(ii) If $c' > b'$ and $c'' \geq b''$.
(iii) If $c' > b'$ and $c'' < b''$.

We prove one case, namely, Case (2) and the proofs of the other two cases are similar. To do that, suppose $a' > b'$ and $a'' \geq b''$.

(i) If $c' \leq b'$, then
$(\langle a', a''\rangle \rhd \langle b', b''\rangle) \lhd \langle c', c''\rangle = \langle b', 0\rangle \lhd \langle c', c''\rangle = \langle 1, 0\rangle$
and
$\langle a', a''\rangle \rhd (\langle b', b''\rangle \lhd \langle c', c''\rangle) = \langle a', a''\rangle \rhd \langle 1, 0\rangle = \langle 1, 0\rangle$.
Since $a' \wedge c' \leq b'$, we get
$\langle b', b''\rangle \lhd (\langle a', a''\rangle \wedge \langle c', c''\rangle) = \langle 1, 0\rangle$.

(ii) If $c' > b'$ and $c'' < b''$, then
$(\langle a', a''\rangle \rhd \langle b', b''\rangle) \lhd \langle c', c''\rangle = \langle b', 0\rangle \lhd \langle c', c''\rangle = \langle b', 0\rangle$
and
$\langle a', a''\rangle \rhd (\langle b', b''\rangle \lhd \langle c', c''\rangle) = \langle a', a''\rangle \rhd \langle b', b''\rangle = \langle b', 0\rangle$.
Since $a' \wedge c' > b'$ and $a'' \vee c'' \geq b''$ we get
$\langle b', b''\rangle \lhd (\langle a', a''\rangle \wedge \langle c', c''\rangle) = \langle b', 0\rangle$.

(iii) If $c' > b'$ and $c'' \geq b''$, then
$(\langle a', a''\rangle \rhd \langle b', b''\rangle) \lhd \langle c', c''\rangle = \langle b', 0\rangle \lhd \langle c', c''\rangle = \langle b', 0\rangle$
and
$\langle a', a''\rangle \rhd (\langle b', b''\rangle \lhd \langle c', c''\rangle) = \langle a', a''\rangle \rhd \langle b', 0\rangle = \langle b', 0\rangle$.
Last, $a' \wedge c' > b'$ and $a'' \vee c'' \geq b''$ imply
$\langle b', b''\rangle \lhd (\langle a', a''\rangle \wedge \langle c', c''\rangle) = \langle b', 0\rangle$.

Hence from all the above cases we conclude
$(a \rhd b) \lhd c = a \rhd (b \lhd c) = b \lhd (a \wedge c)$ for every $a, b, c \in F$. □

From this lemma we can write $a \rhd b \lhd c$ instead of $(a \rhd b) \lhd c$ or $a \rhd (b \lhd c)$. That is, we may remove parentheses. Also, we note that this lemma is equivalent to the relationship
$(a \wedge c) \rhd b = a \rhd (c \rhd b) = c \rhd (a \rhd b)$.

Lemma 3.2. For $a, b, c, d \in F$, we have $b \rhd a \lhd c < d$ implies $a < b, c, d$.

Proof. Let $b \rhd a \lhd c < d$. Then we have $a \leq b \rhd a \leq (b \rhd a) \lhd c = b \rhd a \lhd c < d$. Thus $a < d$ and $b \rhd a < d$ which yields $a < b$ and so $(b \rhd a) \lhd c = a \lhd c < d$. But this means $a < c$. □

Lemma 3.3. For $a, b, c \in F$, if we have $a' \wedge c' \leq b'$, then $a \rhd b \lhd c = \mathbf{1}$.

Proof. By Lemma 3.1. □

Proposition 3.4. Let $A = \left[a_{ij}\right]_{m\times p}$, $B = \left[b_{ij}\right]_{p\times g}$ and $C = \left[c_{ij}\right]_{g\times n}$ be three intuitionistic fuzzy matrices. Then $(A \rhd B) \lhd C = A \rhd (B \lhd C)$.

Proof. Let $D = (A \rhd B) \lhd C$ and $R = A \rhd (B \lhd C)$. Then by Lemma 3.1,
$d_{ij} = \bigwedge_{l=1}^{g} \left[\bigwedge_{k=1}^{p} (a_{ik} \rhd b_{kl}) \lhd c_{lj} \right] = \bigwedge_{l=1}^{g} \bigwedge_{k=1}^{p} (a_{ik} \rhd b_{kl} \lhd c_{lj})$.
Also,

$r_{ij} = \bigwedge_{k=1}^{p}\left[a_{ik} \rhd \bigwedge_{l=1}^{g}(b_{kl} \lhd c_{lj})\right] = \bigwedge_{k=1}^{p}\bigwedge_{l=1}^{g}(a_{ik} \rhd b_{kl} \lhd c_{lj}).$
Hence $d_{ij} = r_{ij}$. □

By this proposition we denote $(A \rhd B) \lhd C$ or $A \rhd (B \lhd C)$ by $A \rhd B \lhd C$.

Proposition 3.5. For any $m \times n$ intuitionistic fuzzy matrix A, we have:
(1) $A \lhd I_n = A$,
(2) $I_m \rhd A = A$.

Proof. (1). Let $B = A \lhd I_n$. Then $b_{ij} = \bigwedge_{k=1}^{n}(a_{ik} \lhd \delta_{kj}) = a_{ij} \lhd \delta_{jj} = a_{ij} \lhd \mathbf{1} = a_{ij}$.Thus $B = A$
(2). Similar to (1). □

Definition 3.6. [2, 3, 10] An $n \times n$ intuitionistic fuzzy matrix A is called reflexive (irreflexive) if and only if $a_{ii} = \mathbf{1}$ ($\mathbf{0}$) for every $i \leq n$.

Definition 3.7. [2, 3, 10] An $n \times n$ intuitionistic fuzzy matrix A is called transitive if and only if $A^2 \leq A$ and it is called idempotent if and only if $A^2 = A$.

From this definition, it is noted that the idempotent intuitionistic fuzzy matrix is transitive and not vice-versa.

Proposition 3.8. If an intuitionistic fuzzy matrix A is reflexive and transitive, then A is idempotent.

Proof. Since A is transitive, it is enough to show that $A^2 \geq A$. Now, let A be of order $n \times n$ and let $S = A^2$. Then

$$\begin{aligned} s_{ij} = \left\langle s'_{ij}, s''_{ij}\right\rangle &= \left\langle \bigvee_{k=1}^{n}(a'_{ik} \wedge a'_{kj}), \bigwedge_{k=1}^{n}(a''_{ik} \vee a''_{kj})\right\rangle \\ &\geq \left\langle a'_{ij} \wedge a'_{jj}, a''_{ij} \vee a''_{jj}\right\rangle \\ &= \left\langle a'_{ij} \wedge 1, a''_{ij} \vee 0\right\rangle \\ &= \left\langle a'_{ij}, a''_{ij}\right\rangle \\ &= a_{ij} \text{ (since } A \text{ is reflexive).} \end{aligned}$$

That is $A^2 \geq A$ and A is so idempotent. □

Theorem 3.9. Let A and B be two $m \times n$ intuitionistic fuzzy matrices such that $A \leq B$. Then $A \lhd B^t$ and $B^t \rhd A$ are transitive.

Proof. Let $T = A \lhd B^t$. To prove that T is transitive, we must show that $t_{ij} \geq t_{il} \wedge t_{lj}$ for every $l \leq n$. Suppose that $t_{il} \wedge t_{lj} = c > \mathbf{0}$.Then

$t_{il} = \bigwedge_{k=1}^{n}(a_{ik} \lhd b_{lk}) \geq c$ and $t_{lj} = \bigwedge_{k=1}^{n}(a_{lk} \lhd b_{jk}) \geq c$.

If $t_{ij} = \left\langle t'_{ij}, t''_{ij}\right\rangle$ be such that $t'_{ij} < c'$ and $t''_{ij} > c''$ where $c = \langle c', c''\rangle$, that is, if $\left\langle a'_{ih}, a''_{ih}\right\rangle \lhd \left\langle b'_{jh}, b''_{jh}\right\rangle < \langle c', c''\rangle$, then

$\left\langle a'_{ih}, a''_{ih}\right\rangle \lhd \left\langle b'_{jh}, b''_{jh}\right\rangle \neq \langle 1, 0\rangle, \left\langle a'_{ih}, 0\right\rangle$ (since $0 \leq c''$).

So that,

$\left\langle a'_{ih}, a''_{ih}\right\rangle \lhd b'_{jh}, b''_{jh} = \left\langle a'_{ih}, a''_{ih}\right\rangle < \langle c', c''\rangle$.

Thus, $a'_{ih} < c'$ and $a''_{ih} > c''$ and in this case, $a'_{ih} < b'_{jh}$ and $a''_{ih} > b''_{jh}$ by the definition of $\lhd$.

Since $t_{il} \geq c$, we have $\left\langle a'_{ih}, a''_{ih}\right\rangle \lhd \left\langle b'_{lh}, b''_{lh}\right\rangle \geq \langle c', c''\rangle$ and so $\left\langle a'_{ih}, a''_{ih}\right\rangle \lhd \left\langle b'_{lh}, b''_{lh}\right\rangle \neq \left\langle a'_{ih}, 0\right\rangle, \left\langle a'_{ih}, a''_{ih}\right\rangle$.

Therefore,
$\left\langle a'_{ih}, a''_{ih} \right\rangle \lhd \left\langle b'_{lh}, b''_{lh} \right\rangle = \langle 1, 0 \rangle$.
But
$a'_{ih} < c'$ and $a''_{ih} > c''$.
Then
$a'_{ih} \geq b'_{lh}$ and $a''_{ih} \leq b''_{lh}$.

Also, since $t_{lj} \geq c$, we have that $\left\langle a'_{lh}, a''_{lh} \right\rangle \lhd \left\langle b'_{jh}, b''_{jh} \right\rangle \geq \langle c', c'' \rangle$ and so that $a'_{lh} \geq b'_{jh}$ and $a''_{lh} \leq b''_{jh}$.
Since $A \leq B$, we have $c' > a'_{ih} \geq b'_{lh} \geq a'_{lh} \geq b'_{jh}$. However, this contradicts that $a'_{ih} < b'_{jh}$. So that $t'_{ij} \geq c'$.

Also, Since $A \leq B$, we have $c'' < a''_{ih} \leq b''_{lh} \leq a''_{lh} \leq b''_{jh}$. However, this contradicts that $a''_{ih} > b''_{jh}$ .So that $t''_{ij} \leq c''$. Therefore, $t_{ij} = \left\langle t'_{ij}, t''_{ij} \right\rangle \geq \langle c', c'' \rangle = c$ and T is thus transitive.
The transitivity of the matrix $B^t \rhd A$ can be obtaied by a similar manner. □

Corollary 3.10. For any $m \times n$ intuitionistic fuzzy matrix A, the intuitionistic fuzzy matrices $A \lhd A^t$ and $A^t \rhd A$ are idempotent.

Proof. It is easy to see that $A \lhd A^t$ and $A^t \rhd A$ are reflexive. Then by Theorem 3.9 and Proposition 3.8 they are idempotent. □

As it is well known, if the intuitionistic fuzzy relation R is reflexive and transitive, then R is a pre-order. Since, $A^t \rhd A$ and $A \lhd A^t$ are idempotent, we can obtain two pre-orders from a given intuitionistic fuzzy matrix A.

Theorem 3.11. Let A be any $n \times n$ intuitionistic fuzzy matrix. Then A is reflexive and transitive if and only if $A \lhd A^t = A$ (also if and only if $A^t \rhd A = A$).

Proof. Suppose A is reflexive and transitive and let $T = A \lhd A^t$. i.e;
$t_{ij} = \bigwedge_{k=1}^{n} \left(\left\langle a'_{ik}, a''_{ik} \right\rangle \lhd \left\langle a'_{jk}, a''_{jk} \right\rangle \right)$.
Suppose $t_{ij} = \left\langle t'_{ij}, t''_{ij} \right\rangle = \langle c', c'' \rangle > \langle 0, 1 \rangle$. Then $\left\langle a'_{ij}, a''_{ij} \right\rangle \lhd \left\langle a'_{jj}, a''_{jj} \right\rangle \geq \langle c', c'' \rangle$. That is $\left\langle a'_{ij}, a''_{ij} \right\rangle \lhd \langle 1, 0 \rangle \geq \langle c', c'' \rangle$ (by the reflexivity of A) which yielding $a_{ij} = \left\langle a'_{ij}, a''_{ij} \right\rangle \geq \langle c', c'' \rangle = t_{ij}$ and so $A \geq A \lhd A^t$.

On the other hand we will show that $A \leq A \lhd A^t$ using the transitivity of A. Suppose that $a_{ij} = \left\langle a'_{ij}, a''_{ij} \right\rangle = \langle c', c'' \rangle > \langle 0, 1 \rangle$ and $t_{ij} = \left\langle t'_{ij}, t''_{ij} \right\rangle = \left\langle a'_{il}, a''_{il} \right\rangle \lhd \left\langle a'_{jl}, a''_{jl} \right\rangle$ for some $l \leq n$. Based on the definition of the operation $\lhd$, we have the following three cases:

Case (1). If $\left\langle a'_{il}, a''_{il} \right\rangle \lhd \left\langle a'_{jl}, a''_{jl} \right\rangle = \langle 1, 0 \rangle$, then it is clear that $t_{ij} \geq c = a_{ij}$. Thus $T \geq A$.

Case (2). If $\left\langle a'_{il}, a''_{il} \right\rangle \lhd \left\langle a'_{jl}, a''_{jl} \right\rangle = \left\langle a'_{il}, a''_{il} \right\rangle < \langle c', c'' \rangle$, then $a'_{il} < c', a'_{jl}$, and $a''_{il} > c'', a''_{jl}$.i.e;
$a'_{il} < c' \wedge a'_{jl}$ and $a''_{il} > c'' \vee a''_{jl}$.
Now since we have that A is transitive, we get $\left\langle a'_{ij}, a''_{ij} \right\rangle \wedge \left\langle a'_{jl}, a''_{jl} \right\rangle \leq \left\langle a'_{il}, a''_{il} \right\rangle$. Thus. $\langle c', c'' \rangle \wedge \left\langle a'_{jl}, a''_{jl} \right\rangle \leq \left\langle a'_{il}, a''_{il} \right\rangle$. Therefore, $a'_{il} < c' \wedge a'_{jl} \leq a'_{il}$ and $a''_{il} > c'' \vee a''_{jl} \geq a''_{il}$. However, these are contradictions and so $t'_{ij} \geq c'$ and $t''_{ij} \leq c''$. Thus $t_{ij} \geq c = a_{ij}$ and so $T \geq A$.

Case (3). If $t_{ij} = \left\langle a'_{il}, 0 \right\rangle$, then $a'_{il} < a'_{jl}$. Since we have that $0 < c''$, it is enough to show that $a'_{il} \geq c'$. Suppose that $a'_{il} < c'$. Then

$a'_{il} < c' \wedge a'_{jl} = a^{'}_{ij} \wedge a^{'}_{jl} \leq a^{'}_{il}$ (Again by the transitivity of A) and also we have a contradiction and so $a'_{il} \geq c'$ and $t_{ij} \geq c$. Thus $t_{ij} \geq c = a_{ij}$. i.,e. $T \geq A$.

Conversely, if $A \lhd A^t = A$, then by the proof of Corollary 3.10, A is reflexive and transitive so it is idempotent.

Similarly, we can show that $A^t \rhd A = A$. □

Corollary 3.12. If A is a reflexive and transitive intuitionistic fuzzy matrix, then $(A \lhd A^t)A = (A^t \rhd A)A = A$.

Proof. By Theorem 3.11 and Corollary 3.10. □

Corollary 3.13. If A is a reflexive and transitive intuitionistic fuzzy matrix, then $(A^t \rhd A) \lhd A^t = A$.

Theorem 3.11 shows interesting properties of preorders. Thus A is a matrix representing a preorder if and only if $A \lhd A^t = A$ (or $A^t \rhd A = A$). However, since $A \lhd A^t$ is obtained by using A if we multiply $A \lhd A^t$ by A, any information is not added to A. That is, the product $(A \lhd A^t)A$ is equal to A.

Example 3.14. Let

$$A = \begin{bmatrix} \langle 0.5, 0.4\rangle & \langle 0.7, 0.3\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.8, 0.2\rangle \\ \langle 0.9, 0\rangle & \langle 0.4, 0.6\rangle \end{bmatrix}.$$

Then

$$A \lhd A^t = \begin{bmatrix} \langle 0.5, 0.4\rangle & \langle 0.7, 0.3\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.8, 0.2\rangle \\ \langle 0.9, 0\rangle & \langle 0.4, 0.6\rangle \end{bmatrix} \lhd \begin{bmatrix} \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.9, 0\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.8, 0.2\rangle & \langle 0.4, 0.6\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 1, 0\rangle & \langle 0.5, 0.4\rangle & \langle 0.5, 0.4\rangle \\ \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.4, 0.6\rangle & \langle 1, 0\rangle \end{bmatrix}.$$

It is clear that $A \lhd A^t$ is reflexive and

$$(A \lhd A^t)^2 = \begin{bmatrix} \langle 1, 0\rangle & \langle 0.5, 0.4\rangle & \langle 0.5, 0.4\rangle \\ \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.4, 0.6\rangle & \langle 1, 0\rangle \end{bmatrix} \begin{bmatrix} \langle 1, 0\rangle & \langle 0.5, 0.4\rangle & \langle 0.5, 0.4\rangle \\ \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.4, 0.6\rangle & \langle 1, 0\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 1, 0\rangle & \langle 0.5, 0.4\rangle & \langle 0.5, 0.4\rangle \\ \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.4, 0.6\rangle & \langle 1, 0\rangle \end{bmatrix} = A \lhd A^t.$$

That is $A \lhd A^t$ is reflexive and transitive and so idempotent. Moreover, if we let $B = A \lhd A^t$. Then

$$B \lhd B^t = \begin{bmatrix} \langle 1, 0\rangle & \langle 0.5, 0.4\rangle & \langle 0.5, 0.4\rangle \\ \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.4, 0.6\rangle & \langle 1, 0\rangle \end{bmatrix} \lhd \begin{bmatrix} \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 0.4, 0.6\rangle \\ \langle 0.5, 0.4\rangle & \langle 1, 0\rangle & \langle 0.4, 0.6\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 1, 0\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 1, 0\rangle & \langle 0.5, 0.4\rangle & \langle 0.5, 0.4\rangle \\ \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.4, 0.6\rangle & \langle 1, 0\rangle \end{bmatrix} = B,$$

$$(B \lhd B^t)B = \begin{bmatrix} \langle 1,0\rangle & \langle 0.5,0.4\rangle & \langle 0.5,0.4\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle & \langle 0.6,0.3\rangle \\ \langle 0.4,0.6\rangle & \langle 0.4,0.6\rangle & \langle 1,0\rangle \end{bmatrix}\begin{bmatrix} \langle 1,0\rangle & \langle 0.5,0.4\rangle & \langle 0.5,0.4\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle & \langle 0.6,0.3\rangle \\ \langle 0.4,0.6\rangle & \langle 0.4,0.6\rangle & \langle 1,0\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 1,0\rangle & \langle 0.5,0.4\rangle & \langle 0.5,0.4\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle & \langle 0.6,0.3\rangle \\ \langle 0.4,0.6\rangle & \langle 0.4,0.6\rangle & \langle 1,0\rangle \end{bmatrix} = B.$$

Proposition 3.15. Let $A = \left[a_{ij}\right]_{m\times n}$, $B = \left[b_{ij}\right]_{m\times p}$ and $C = \left[c_{ij}\right]_{p\times n}$ be three intuitionistic fuzzy matrices. If $BC \leq A$, then $B^t \rhd A \lhd C^t$ is reflexive.

Proof. Suppose that $BC \leq A$ and let $D = B^t \rhd A \lhd C^t$. Then $d_{ii} = \bigwedge_{k=1}^{m}\bigwedge_{l=1}^{n}(b_{ki} \rhd a_{kl} \lhd c_{il})$. Since $b_{ki} \wedge c_{il} \leq a_{kl}$, by Lemma 3.3 we have $d_{ii} = \mathbf{1}$. □

Corollary 3.16. If an intuitionistic fuzzy matrix A is transitive, then $A^t \rhd A \lhd A^t$ is reflexive.

Proof. By Proposition 3.15. □

4. Inverses and Sub-Inverses of Intuitionistic Fuzzy Matrices

In this section we establish interesting matrix inequalities which we use in discussing sub-inverses and generalized inverses of intuitionistic fuzzy matrices. This discussion is an extension of that on the ordinary fuzzy matrices and binary relations or Boolean matrices

Definition 4.1. [5, 9, 13] Let A be any $m \times n$ intuitionistic fuzzy matrix. If $ABA \leq A$ for some intuitionistic fuzzy matrix B, then B is called a sub-inverse of A.

From this definition it is noted that the set of sub-inverses of an intuitionistic fuzzy matrix A is closed under the operation $\vee$. That is if we have B_1 and B_2 are two sub-inverses to an intuitionistic fuzzy matrix A, then $B_1 \vee B_2$ is also a sub-inverse of A.

Theorem 4.2. For intuitionistic fuzzy matrices $A = \left[a_{ij}\right]_{m\times n}$, $B = \left[b_{ij}\right]_{m\times p}$, $C = \left[c_{ij}\right]_{g\times n}$ and $D = \left[d_{ij}\right]_{p\times g}$, if $BDC \leq A$, then $D \leq B^t \rhd A \lhd C^t$.

Proof. Let $W = B^t \rhd A \lhd C^t$. Then $w_{ij} = \bigwedge_{k=1}^{m}\bigwedge_{l=1}^{n}\left(\left\langle b'_{ki}, b''_{ki}\right\rangle \rhd \left\langle a'_{kl}, a''_{kl}\right\rangle \lhd \left\langle c'_{jl}, c''_{jl}\right\rangle\right)$.
Suppose $d_{ij} = \left\langle d'_{ij}, d''_{ij}\right\rangle = \langle e', e''\rangle > \langle 0,1\rangle$. If $w_{ij} = \left\langle w'_{ij}, w''_{ij}\right\rangle < \langle e', e''\rangle$, then $\left\langle b'_{ui}, b''_{ui}\right\rangle \rhd \langle a'_{uv}, a''_{uv}\rangle \lhd \left\langle c'_{jv}, c''_{jv}\right\rangle < \langle e', e''\rangle$ for some $u \leq m, v \leq n$.

By Lemma 3.2, we have $\langle a'_{uv}, a''_{uv}\rangle < \left\langle b'_{ui}, b''_{ui}\right\rangle$, $\langle a'_{uv}, a''_{uv}\rangle < \langle e', e''\rangle$ and $\langle a'_{uv}, a''_{uv}\rangle < \left\langle c'_{jv}, c''_{jv}\right\rangle$.
Thus, $b'_{ui} \wedge d'_{ij} \wedge c'_{jv} > a'_{uv}$ which contradicts $BDC \leq A$. Hence $w'_{ij} \geq e'$.

Similarly, since $a''_{uv} > b''_{ui}, a''_{uv} > c''_{jv}$ and $a''_{uv} > e''$. Thus $b''_{ui} \vee d''_{ij} \vee c''_{jv} < a''_{uv}$ which is also a contradiction and so $w''_{ij} \leq e''$. Therefore, $w_{ij} \geq e = d_{ij}$. □

Corollary 4.3. If A is transitive intuitionistic fuzzy matrix, then $A \leq A^t \rhd A \lhd A^t$.

Proof. By Theorem 4.2. □

Proposition 4.4 [3, 10]. For intuitionistic fuzzy matrices $A = \left[a_{ij}\right]_{m\times n}$, $B = \left[b_{ij}\right]_{m\times n}$, $C = \left[c_{ij}\right]_{n\times p}$ and $D = \left[d_{ij}\right]_{p\times m}$, if $A \leq B$, then $AC \leq BC$ and $DA \leq DB$.

Proposition 4.5. For intuitionistic fuzzy matrices $A = \left[a_{ij}\right]_{m\times n}$, $B = \left[b_{ij}\right]_{m\times p}$, $C = \left[c_{ij}\right]_{q\times n}$ and $D = \left[d_{ij}\right]_{p\times q}$, if $BDC = A$, then $A \leq B(B^t \rhd A \lhd C^t)C$.

Proof. Suppose that $A = BDC$. Then by Theorem 4.2, $D \leq B^t \rhd A \lhd C^t$ and so by Proposition 4.4, $BDC \leq B(B^t \rhd A \lhd C^t)C$. Thus
$A \leq B(B^t \rhd A \lhd C^t)C$. □

Proposition 4.6. If B is a sub-inverse of A, then $B \leq A^t \rhd A \lhd A^t$.

Proof. By Theorem 4.2. □

Proposition 4.7. If B is a sub-inverse of transitive intuitionistic fuzzy matrix A, then AB and BA are also sub-inverses of A.

Proof. Since we have that B is a sub-inverse of A, we get $AABA \leq ABA \leq A$ and $ABAA \leq ABA \leq A$. Hence the proof. □

Clearly I_n and A itself are sub-inverses of any $n \times n$ transitive intuitionistic fuzzy matrix A. In fact, if $B \leq A$, then B is a sub-inverse of A.

Remark. Let S be the set of such subinverses constructed as in Proposition 4.7. Then it is clear that the pair $(S, \vee)$ forms a commutative monoid with $AO = O$ (the zero matrix) as the unit element of the operation $\vee$. Also, the pair $(S, \circ)$ forms a semigroup with the composition of intuitionistic fuzzy matrices $(\circ)$. Here in this paper,we write AB instead of $A \circ B$ - when the composition is suitable- with I_n as a unit element. Moreover, the triple $(S, \vee, \circ)$ forms a semiring of sub-inverses of the transitive intuitionistic fuzzy matrix A. Note that if AB_1, AB_2 and AB_3 are sub-inverses of A, then $AB_1(AB_2 \vee AB_3) = AB_1AB_2 \vee AB_1AB_3$ and $(AB_2 \vee AB_3)AB_1 = AB_2AB_1 \vee AB_3AB_1$

Definition 4.8. [5, 8, 9] An intuitionistic fuzzy matrix A of order $m \times n$ is said to be regular if there exists an intuitionistic fuzzy matrix G of order $n \times m$ such that $AGA = A$ and then G is called a generalized inverse (g-inverse) of A .

Example 4.9. From

$$\begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 0.4,0,5\rangle & \langle 0,1\rangle \end{bmatrix}\begin{bmatrix} \langle 0,1\rangle & \langle 0.4,0.5\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle \end{bmatrix}\begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 0.4,0,5\rangle & \langle 0,1\rangle \end{bmatrix} = \begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 0.4,0,5\rangle & \langle 0,1\rangle \end{bmatrix}.$$

We see that the intuitionistic fuzzy matrix on the right is regular.

Proposition 4.10. If A is regular intuitionistic fuzzy matrix and G is a g-inverse of A, then
(i) $G \leq A^t \rhd A \lhd A^t$,
(ii) $A \leq A(A^t \rhd A \lhd A^t)A$.

Proof. (i) By Theorem 4.2.
(ii) By Proposition 4.5. □

In the case when an intuitionistic fuzzy matrix A is reflexive and transitive, the matrix A itself is a sub-inverse of A. It is also a g-inverse of A. But in this case $A = A^t \rhd A \lhd A^t$ by Corollary 3.13. Thus

$A^t \rhd A \lhd A^t$ is a sub-inverse of A and it is also a g-inverse of A. In fact, this matrix is the largest one as we seen in Propositions 4.6 and 4.10.

5. Conclusions

In this paper we have shown some properties of the pre- orders and subinverses of intuitionistic fuzzy matrices. They are useful in discussion of regularity of intuitionistic fuzzy matrices. The results we have obtained on subinverses are generalizations of Schein's results [13] and Hashimoto's results [5]. Of course these results hold for Boolean matrices and also for ordinary fuzzy matrices. However, the operation $\rhd$ plays an important role in our discussions. We may use this operation in future for decomposition of rectangular intuitionistic fuzzy matrices which is useful for decomposition of intuitionistic fuzzy databases.

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