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Relations between L-Fuzzy Topogenous Orders and L-Fuzzy Pre-Uniformities

Ahmed Ramadan, Enas Elkordy

Maths and Comp. Sci. Dept., Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

Abstract. In this paper, we introduce the notions of L-fuzzy topogenous orders and pre-uniformities as a continuation of previous work. The continuity notion and the Galois correspondence are also discussed.

1. Introduction.

Ward et al. [13] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure for many valued logic. Considering the concepts of topological structures, information systems and decision rules are investigated in complete residuated lattices [1,3, 4-6,11,12]. Höhle [5,6] introduced L-fuzzy topologies with algebraic structure L(cqm, quantales, MV-algebra).

Katsaras [8,9] introduced the concepts of fuzzy topogenous order and fuzzy topogenous structures in completely distributive lattice which are a unified approach to the three spaces: Chang's fuzzy topologies [2], Katsaras's fuzzy proximities [8] and Hutton's fuzzy uniformities [7]. As an extension of Katsaras's definition, El-Dardery [10] introduced L-fuzzy topogenous order in view points of Sostak's fuzzy topology [3] and Kim's L-fuzzy proximities [10] on strictly two-sided, commutative quantales and studied their topological properties.

In this paper, we introduce a slightly different definition for L-fuzzy topogenous order and its relations with pre-uniformities as a continuation of previous work. The continuity notion and the Galois correspondence are also discussed.

2. Preliminaries.

Definition 2.1 [2,4-6]. An algebra $(L, \wedge, \vee, \odot, \rightarrow, \bot, \top)$ is called a complete residuated lattice if
(L1) $(L, \leq, \vee, \wedge, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$,
(L2) $(L, \odot, \top)$ is a commutative monoid,
(L3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

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Email addresses: aramadan58@hotmail.com (Ahmed Ramadan), enas.elkordi@science.bsu.edu.eg (Enas Elkordy)

In this paper, we assume that $(L, \leq, \odot, \rightarrow, ^*)$ is
(1) a complete residuated lattice with an order reversing involution * which is defined by
$x \oplus y = (x^* \odot y^*)^*$, $x^* = x \rightarrow \bot$ unless otherwise specified,
(2) an idempotence if $x \odot x = x$ for each $x \in L$.
For $\alpha \in L, f \in L^X$, we denote $(\alpha \rightarrow f)$, $(\alpha \odot f)$, $\alpha_X \in L^X$ as

$$(\alpha \rightarrow f)(x) = \alpha \rightarrow f(x), \ (\alpha \odot f)(x) = \alpha \odot f(x), \ \alpha_X(x) = \alpha,$$

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \bot, & \text{otherwise,} \end{cases} \quad \top_x^*(y) = \begin{cases} \bot, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

Lemma 2.2 [2,4-6]. For each $x, y, z, x_i, y_i, w \in L$, the following hold.
(1) $\top \rightarrow x = x$, $\bot \odot x = \bot$,
(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$,
(3) $x \leq y$ iff $x \rightarrow y = \top$,
(4) $x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i)$,
(5) $(\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)$,
(6) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,
(7) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(8) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$, $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,
(9) $x \rightarrow y = y^* \rightarrow x^*$, $x \odot y = (x \rightarrow y^*)^*$,
(10) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,
(11) $z \rightarrow x \leq (x \rightarrow y) \rightarrow (z \rightarrow y)$, $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,
(12) $\bigvee_{i \in I} x_i \rightarrow \bigvee_{i \in I} y_i \geq \bigwedge_{i \in I}(x_i \rightarrow y_i)$, $\bigwedge_{i \in I} x_i \rightarrow \bigwedge_{i \in I} y_i \geq \bigwedge_{i \in I}(x_i \rightarrow y_i)$.

Definition 2.3 [1]. Let X be a set. A mapping $R : X \times X \rightarrow L$ is called a L-fuzzy relation on X, then for all $x, y, z \in X$ the relation R is said to be
(1) reflexive if $R(x, x) = \top$,
(2) symmetric if $R(x, y) = R(y, x)$,
(3) transitive if $R(x, y) \odot R(y, z) \leq R(x, z)$.
A L-fuzzy relation on X is called a L-fuzzy pre-order if it is reflexive and transitive and called a L-fuzzy equivalence relation if it is reflexive, symmetric and transitive.

Lemma 2.4 [5]. For a given set X, define a map $S : L^X \times L^X \rightarrow L$ by

$$S(f, g) = \bigwedge_{x \in X}(f(x) \rightarrow g(x)).$$

Then, for each $f, g \in L^X$ and for all $\alpha \in L$ the following hold.
(1) S is a L-partial order on L^X,
(2) $f \leq g$ iff $S(f, g) \geq \top$,
(3) If $f \leq g$, then $S(h, f) \leq S(h, g)$ and $S(f, h) \geq S(g, h)$ $\forall\, h \in L^X$,
(4) $S(f, g) \odot S(k, h) \leq S(f \odot k, g \odot h)$ and $S(f, g) \odot S(k, h) \leq S(f \oplus k, g \oplus h)$,
(5) $\bigwedge_i S(f_i, g_i) \leq S(\bigwedge_i f_i, \bigwedge_i g_i)$,
(6) $S(f, g) = \bigvee_{h \in L^X} S(f, h) \odot S(h, g)$,
(7) If $\phi : X \rightarrow Y$ is a map, then for $f, g \in L^X$ and $h, k \in L^Y$,

$$S(f, g) \leq S(\phi^{\rightarrow}(f), \phi^{\rightarrow}(g)), \ \ S(h, k) \leq S(\phi^{\leftarrow}(h), \phi^{\leftarrow}(k))$$

and the equalities hold if ϕ is bijective.

Lemma 2.5 [5]. For each $f, g \in L^X$, define two maps $u_{f,g}$, $u^{-1}_{f,g} : X \times X \rightarrow L$ by

$$u_{f,\,g}(x, y) = f(x) \rightarrow g(y) \quad \text{and} \quad u^{-1}_{f,\,g}(x, y) = u_{f,\,g}(y, x).$$

Then, the following hold
(1) $\top_{X\times X} = u_{\bot_X,\bot_X} = u_{\top_X,\top_X}$,
(2) If $f_2 \le f_1$ and $g_1 \le g_2$, then $u_{f_1,\,g_1} \le u_{f_2,\,g_2}$,
(3) For any $u_{f,\,g} \in L^{X\times X}$ and $h \in L^X$, it holds that $u_{f,\,h} \circ u_{h,\,g} \le u_{f,\,g}$ where

$$u_{f,\,h}(x,y) \circ u_{h,\,g}(y,z) = \bigvee_{y\in X}(f(x) \to h(y)) \odot (h(y) \to g(z)),$$

(4) $u_{\bigvee_{i\in I} f_i,\,g} = \bigwedge_{i\in I} u_{f_i,\,g}$ and $u_{f,\bigwedge_{i\in I} g_i} = \bigwedge_{i\in I} u_{f,\,g_i}$,
(5) $u_{f_1,g_1} \odot u_{f_2,g_2} \le u_{f_1\odot g_1,\, f_2\odot g_2}$,
(6) $u_{f_1,g_1} \odot u_{f_2,g_2} \le u_{f_1\oplus g_1,\, f_2\oplus g_2}$,
(7) $u_{\alpha\odot f,\,g} = \alpha \to u_{f,\,g}$ and $u_{f,\,\alpha\to g} = \alpha \to u_{f,\,g}$,
(8) $u_{\alpha\odot f,\,\alpha\odot g} \ge u_{f,\,g}$ and $u_{\alpha\to f,\,\alpha\to g} \ge u_{f,\,g}$,
(9) $u_{f,\,g} = \bigwedge_{z\in X}(f(z) \to u_{\top_z,\,g})$ and $u_{f,\,g} = \bigwedge_{z\in X}(g^*(z) \to u_{f,\top_z^*})$,
(10) $u_{f,\,g} = \bigwedge_{y,z\in X}(f(y) \to (g(z) \to u_{\top_y,\top_y^*}))$,
(11) $u^{-1}_{f,\,g} = u_{g^*,f^*}$.

Definition 2.6 [12]. A map $\mathcal{U} : L^{X\times X} \to L$ is called a L-fuzzy pre-uniformity on X if
(U1) $\mathcal{U}(u_{f,\,g}) \ge \bigvee_{x\in X} f(x) \to \bigwedge_{x\in X} g(x)$,
(U2) If $v \le u$, then $\mathcal{U}(v) \le \mathcal{U}(u)$,
(U3) For every $u, v \in L^{X\times X}$, $\mathcal{U}(u \odot v) \ge \mathcal{U}(u) \odot \mathcal{U}(v)$,
(U4) $\mathcal{U}(u_{f,\,g}) \le S(f, g)$ for each $f, g \in L^X$.
A L-fuzzy pre-uniformity is called
(QU) a L-fuzzy quasi-uniformity on X if $\mathcal{U}(u) \le \bigvee\{\mathcal{U}(v) \odot \mathcal{U}(w) \mid v \circ w \le u\}$ where

$$v \circ w(x,z) = \bigvee_{y\in X} v(x,y) \odot w(y,z),$$

(St) stratified if $\mathcal{U}(\alpha \to u) \ge \alpha \to \mathcal{U}(u)$ for each $\alpha \in L$,
(SE) separated if $\mathcal{U}(u_{\top_x,\top_x}) = \top$ for each $x \in X$,
(P) perfect if $\mathcal{U}(\bigwedge_{i\in I} u_i) = \bigwedge_{i\in I} \mathcal{U}(u_i)$.
Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two L-fuzzy pre-uniform spaces and $\phi : X \to Y$ ba a mapping. Then, ϕ is said to be LF-uniformly continuous if

$$\mathcal{V}(v) \le \mathcal{U}((\phi \times \phi)^{\leftarrow}(v)) \ \forall\, v \in L^{Y\times Y}.$$

The category of L-fuzzy pre-uniform spaces and LF-uniformly continuous mappings for morphisms is denoted by **L-FPUNS**.

Remark 2.7 [12].
(1) Let $\mathcal{U}$ be a L-fuzzy pre-uniformity on X, then by (U1) we have

$$\mathcal{U}(u_{\top_X,\top_X}) \ge \bigvee_{x\in X} \top_X(x) \to \bigwedge_{x\in X} \top_X(x) = \top \to \top = \top,$$

(2) Define a map $\mathcal{U}^s : L^{X\times X} \to L$ as $\mathcal{U}^s(u) = \mathcal{U}(u^{-1})$. Then, $\mathcal{U}^s$ is a L-fuzzy pre-uniformity on X.

Definition 2.8 [1]. Suppose that $F : \mathcal{D} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$ are concrete functors, then
(1) $\mathcal{C}$ and $\mathcal{D}$ are said to be *isomorphic* if $F \circ G = id_{\mathcal{C}}$ and $G \circ F = id_{\mathcal{D}}$,
(2) The pair (F, G) is called a *Galois correspondence* between $\mathcal{C}$ and $\mathcal{D}$ if for each $Y \in \mathcal{C}$, $id_Y : F \circ G(Y) \to Y$ is a $\mathcal{C}$-morphism, and for each $X \in \mathcal{D}$, $id_X : X \to G \circ F(X)$ is a $\mathcal{D}$-morphism.

If (F, G) is a Galois correspondence, then it is easy to check that F is a left adjoint of G, or equivalently that G is a right adjoint of F.

3. L-fuzzy Topogenous Orders and L-fuzzy Pre-Uniformities.

Definition 3.1. A mapping $\xi : L^X \times L^X \to L$ is called a L-fuzzy semi-topogenous order on X if it satisfies the following axioms for every $f, f_1, f_i, g, g_1, g_i \in L^X$

(ST1) $\xi(f, g) \geq \bigvee_{x\in X} f(x) \to \bigwedge_{x\in X} g(x)$,

(ST2) $\xi(f, g) \leq S(f, g)$,

(ST3) if $f_1 \leq f$, $g \leq g_1$, then $\xi(f, g) \leq \xi(f_1, g_1)$.

A L-fuzzy semi-topogenous order ξ on X is called

(ST4) L-fuzzy topogenous order if for every $f_1, f_2, g_1, g_2 \in L^X$, we have

$$\xi(f_1 \odot f_2, g_1 \odot g_2) \geq \xi(f_1, g_1) \odot \xi(f_2, g_2),$$
$$\xi(f_1 \oplus f_2, g_1 \oplus g_2) \geq \xi(f_1, g_1) \odot \xi(f_2, g_2),$$

(St) stratified if for every $\alpha \in L$,

$$\xi(\alpha \odot f, g) \geq \alpha \to \xi(f, g) \text{ and } \xi(f, \alpha \to g) \geq \alpha \to \xi(f, g),$$

(P) perfect if $\xi(\bigvee_i f_i, g) \geq \bigwedge_i \xi(f_i, g)$, $\xi(f, \bigwedge_i g_i) \geq \bigwedge_i \xi(f, g_i)$,

(SE) separated if $\xi(\top_x, \top_x) = \xi(\top^*_x, \top^*_x) = \top$,

(TS) L-fuzzy topogenous space if $\xi(f, g) \leq \bigvee_{h\in L^X} \xi(f, h) \odot \xi(h, g)$.

Let ξ_X and ξ_Y be two L-fuzzy topogenous orders on X and Y, respectively. A mapping $\phi : (X, \xi_X) \to (Y, \xi_Y)$ is said to be a L-topogenous map if

$$\xi_Y(f, g) \leq \xi_X(\phi^{\leftarrow}(f), \phi^{\leftarrow}(g)) \quad \forall\, f, g \in L^Y.$$

The category of L-fuzzy topogenous orders with L-topogenous maps as morphisms is denoted by **L-FTGS**.

Remark 3.2. Let ξ be a L-fuzzy topogenous order, then by (ST1) we have

$$\xi(\top_X, \top_X) \leq \bigvee_{x\in X} \top_X(x) \to \bigwedge_{x\in X} \top_X(x) = \top \to \top = \top.$$

So, $\xi(\top_X, \top_X) = \xi(\bot_X, \bot_X) = \top$.

Theorem 3.3. Let $\mathcal{U}$ be a L-fuzzy pre-uniformity on X. Define a map $\xi_{\mathcal{U}} : L^X \times L^X \to L$ as

$$\xi_{\mathcal{U}}(f, g) = \bigwedge_{x\in X}(f(x) \to \mathcal{U}(u_{\top_x, g})) \quad \forall\, f, g \in L^X.$$

Then, the following hold

(1) $\xi_{\mathcal{U}}$ is a L-fuzzy topogenous order on X,

(2) If $\mathcal{U}$ is stratified, then $\xi_{\mathcal{U}}$ is stratified,

(3) If $\mathcal{U}$ is perfect, then $\xi_{\mathcal{U}}$ is perfect,

(4) If $\mathcal{U}$ is perfect and stratified, then

$$\xi_{\mathcal{U}}(f, g) = \bigwedge_{x,z\in X}\Big(f(x) \to (g^*(z) \to \mathcal{U}(u_{\top_x, \top^*_z}))\Big), \quad \xi_{\mathcal{U}}(f, g) = \mathcal{U}(u_{f,\, g}),$$

(5) If $\mathcal{U}$ is separated, then $\xi_{\mathcal{U}}$ is separated,

(6) If $\mathcal{U}$ is a L-fuzzy quasi-uniformity on X, then $\xi_{\mathcal{U}}$ is a L-fuzzy topogenous space on X.

Proof. (1) (ST1) By lemma 2.2 (1),(2),(5), (U1), we have

$$\begin{aligned}\xi_{\mathcal{U}}(f, g) &= \bigwedge_{x\in X}\Big(f(x) \to \mathcal{U}(u_{\top_x, g})\Big)\\ &\geq \bigwedge_{x\in X}\Big(f(x) \to (\bigvee_{x\in X} \top_x(x) \to \bigwedge_{x\in X} g(x))\Big) = \bigvee_{x\in X} f(x) \to \bigwedge_{x\in X} g(x).\end{aligned}$$

(ST2) By (U4), we have

$$\begin{aligned}\xi_{\mathcal{U}}(f,g) = \bigwedge_{x\in X}(f(x) \to \mathcal{U}(u_{\top_x,\, g})) &\leq \bigwedge_{x\in X}(f(x) \to S(\top_x, g)) \\ &= \bigwedge_{x\in X}(f(x) \to g(x)) = S(f,g).\end{aligned}$$

(ST3) If $f \leq f_1,\ g \leq g_1$, then by lemma 2.6 (2) and (U2), we have

$$\begin{aligned}\xi_{\mathcal{U}}(f_1,g_1) = \bigwedge_{x\in X}(f_1(x) \to \mathcal{U}(u_{\top_x,\, g_1})) &\geq \bigwedge_{x\in X}(f_1(x) \to \mathcal{U}(u_{\top_x,\, g})) \\ &\geq \bigwedge_{x\in X}(f(x) \to \mathcal{U}(u_{\top_x,\, g})) = \xi_{\mathcal{U}}(f,g).\end{aligned}$$

(ST4) For every $f_1, f_2, g_1, g_2 \in L^X$, by Lemma 2.6 (5), (U2),(U3), we have

$$\begin{aligned}\xi_{\mathcal{U}}(f_1,g_1) \odot \xi_{\mathcal{U}}(f_2,g_2) &= \bigwedge_{x\in X}(f_1(x) \to \mathcal{U}(u_{\top_x,\, g_1})) \odot \bigwedge_{x\in X}(f_2(x) \to \mathcal{U}(u_{\top_x,\, g_2})) \\ &\leq \bigwedge_{x\in X}\Big((f_1(x)\odot f_2(x)) \to (\mathcal{U}(u_{\top_x,\, g_1}) \odot \mathcal{U}(u_{\top_x,\, g_2}))\Big) \\ &\leq \bigwedge_{x\in X}\Big((f_1\odot f_2)(x) \to \mathcal{U}(u_{\top_x,\, g_1} \odot u_{\top_x,\, g_2})\Big) \\ &\leq \bigwedge_{x\in X}\Big((f_1\odot f_2)(x) \to \mathcal{U}(u_{\top_x,\, g_1\odot g_2})\Big) = \xi_{\mathcal{U}}(f_1\odot f_2, g_1\odot g_2).\end{aligned}$$

Similarly, by Lemma 2.5 (6) we can prove that

$$\xi_{\mathcal{U}}(f_1\oplus f_2, g_1\oplus g_2) \geq \xi_{\mathcal{U}}(f_1,g_1)\odot \xi_{\mathcal{U}}(f_2,g_2).$$

Hence, $\xi_{\mathcal{U}}$ is a L-fuzzy topogenous order on X.
(2) (St) By lemma 2.2 (5),(7), lemma 2.5 (7) and (U5), we have

$$\begin{aligned}\xi_{\mathcal{U}}(\alpha\odot f, g) &= \bigwedge_{x\in X}\Big((\alpha\odot f(x)) \to \mathcal{U}(u_{\top_x,\, g})\Big) \\ &= \bigwedge_{x\in X}\Big(\alpha \to (f(x) \to \mathcal{U}(u_{\top_x,\, g}))\Big) \\ &= \alpha \to \bigwedge_{x\in X}(f(x) \to \mathcal{U}(u_{\top_x,\, g})) = \alpha \to \xi_{\mathcal{U}}(f,g).\end{aligned}$$

$$\begin{aligned}\xi_{\mathcal{U}}(f, \alpha\to g) &= \bigwedge_{x\in X}(f(x) \to \mathcal{U}(u_{\top_x,\alpha\to g})) \\ &= \bigwedge_{x\in X}(f(x) \to \mathcal{U}(\alpha \to u_{\top_x, g})) \\ &\geq \bigwedge_{x\in X}\Big(f(x) \to (\alpha \to \mathcal{U}(u_{\top_x, g}))\Big) \\ &= \alpha \to \bigwedge_{x\in X}(f(x) \to \mathcal{U}(u_{\top_x, g})) = \alpha \to \xi_{\mathcal{U}}(f,g).\end{aligned}$$

(3) (P) By lemma 2.2 (4) and lemma 2.5 (4), we have

$$\begin{aligned}\xi_{\mathcal{U}}(\textstyle\bigvee_{i\in I} f_i, g) &= \bigwedge_{x\in X}(\bigvee_{i\in I} f_i(x) \to \mathcal{U}(u_{\top_x,\, g})) \\ &= \bigwedge_{i\in I}\bigwedge_{x\in X}(f_i(x) \to \mathcal{U}(u_{\top_x,\, g})) = \bigwedge_{i\in I}\xi_{\mathcal{U}}(f_i, g).\end{aligned}$$

$$\begin{aligned}\xi_{\mathcal{U}}(f, \textstyle\bigwedge_{i\in I} g_i) &= \bigwedge_{x\in X}(f(x) \to \mathcal{U}(u_{\top_x,\, \bigwedge_{i\in I} g_i})) \\ &= \bigwedge_{x\in X}(f(x) \to \bigwedge_{i\in I}\mathcal{U}(u_{\top_x,\, g_i})) \\ &= \bigwedge_{i\in I}\bigwedge_{x\in X}(f(x) \to \mathcal{U}(u_{\top_x,\, g_i})) = \bigwedge_{i\in I}\xi_{\mathcal{U}}(f, g_i).\end{aligned}$$

(4) By Lemma 2.5 (8), we have

$$\begin{aligned}\xi_{\mathcal{U}}(f,g) &= \bigwedge_{x\in X}(f(x) \to \mathcal{U}(u_{\top_x,\, g})) \\ &= \bigwedge_{x\in X}\Big(f(x) \to \mathcal{U}(\bigwedge_{z\in X}(g^*(z) \to u_{\top_x,\top_z^*}))\Big) \\ &\geq \bigwedge_{x\in X}\Big(f(x) \to \bigwedge_{z\in X}(g^*(z) \to \mathcal{U}(u_{\top_x,\top_z^*}))\Big) \\ &= \bigwedge_{x,z\in X}\Big(f(x) \to (g^*(z) \to \mathcal{U}(u_{\top_x,\top_z^*}))\Big).\end{aligned}$$

By Lemma 2.5 (9), we have

$$\xi_{\mathcal{U}}(f,g) = \bigwedge_{x\in X}(f(x) \to \mathcal{U}(u_{\top_x,\, g})) \leq \mathcal{U}(\bigwedge_{x\in X}(f(x) \to u_{\top_x,\, g})) = \mathcal{U}(u_{f,\, g}).$$

(5) By lemma 2.5 (1) and for each $x \in X$, we have

$$\xi_{\mathcal{U}}(\top_x, \top_x) = \bigwedge_{x\in X}\Big(\top_x(x) \to \mathcal{U}(u_{\top_x,\top_x})\Big) = \top \to \top = \top,$$
$$\xi_{\mathcal{U}}(\top_x^*, \top_x^*) = \bigwedge_{x\in X}\Big(\top_x^*(x) \to \mathcal{U}(u_{\top_x^*,\top_x^*})\Big) = \bot \to \top = \top.$$

(6) (TS) By (3), (QU) and Lemma 2.5 (3), we have

$$\begin{aligned}\bigvee_{h\in L^X} \xi_{\mathcal{U}}(f,h) \odot \xi_{\mathcal{U}}(h,g) &= \bigvee_{h\in L^X} \mathcal{U}(u_{f,h}) \odot \mathcal{U}(u_{h,g}) \\ &\geq \{\mathcal{U}(u_{f,g}) \mid u_{f,h} \circ u_{h,g} \leq u_{f,g}\} = \xi_{\mathcal{U}}(f,g).\end{aligned}$$

Hence, $\xi_{\mathcal{U}}$ is a L-fuzzy topogenous space on X.

Theorem 3.4. Let $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$ be two L-fuzzy pre-uniformities. If a map $\phi : (X, \mathcal{U}_X) \to (Y, \mathcal{U}_Y)$ is LF-uniformly continuous, then the map $\phi : (X, \xi_{\mathcal{U}_X}) \to (Y, \xi_{\mathcal{U}_Y})$ is LF-topogenous map.

Proof.

$$\begin{aligned}\xi_{\mathcal{U}_X}(\phi^{\leftarrow}(f), \phi^{\leftarrow}(g)) &= \bigwedge_{x\in X}\Big(\phi^{\leftarrow}(f)(x) \to \mathcal{U}_X(u_{\top_x,\, \phi^{\leftarrow}(g)})\Big) \\ &= \bigwedge_{x\in X}\Big(f(\phi(x)) \to \mathcal{U}_X(u_{\phi^{\leftarrow}(\top_{\phi(x)}),\, \phi^{\leftarrow}(g)})\Big) \\ &= \bigwedge_{x\in X}\Big(f(\phi(x)) \to \mathcal{U}_X((\phi\times\phi)^{\leftarrow}(u_{\top_{\phi(x)},\, g}))\Big) \\ &\geq \bigwedge_{\phi(x)=y\in Y}\Big(f(y) \to \mathcal{U}_Y(u_{\top_y,\, g})\Big) = \xi_{\mathcal{U}_Y}(f,g).\end{aligned}$$

Theorem 3.5. Let $\mathcal{V}$ be a L-fuzzy pre-uniformity on X. Define a map $\xi_{\mathcal{V}} : L^X \times L^X \to L$ as

$$\xi_{\mathcal{V}}(f,g) = \bigvee_v \Big\{\mathcal{V}(v) \mid v \leq v_{f,\,g}\Big\} \quad \forall\, f,g \in L^X.$$

Then, the following hold
(1) $\xi_{\mathcal{V}}$ is a L-fuzzy topogenous order on X,
(2) If $\mathcal{V}$ is stratified, then $\xi_{\mathcal{V}}$ is stratified,
(3) If $\mathcal{V}$ is perfect, then $\xi_{\mathcal{V}}$ is perfect,
(4) If $\mathcal{V}$ is separated, then $\xi_{\mathcal{V}}$ is separated,
(5) If $\mathcal{V}$ is a L-fuzzy quasi-uniformity on X, then $\xi_{\mathcal{V}}$ is a L-fuzzy topogenous space on X,
(6) $\xi_{\mathcal{V}^s}(f,g) = \xi_{\mathcal{V}}(g^*, f^*)$.

Proof. (1) (ST1) By (U1) and (U2), we have

$$\xi_{\mathcal{V}}(f,g) = \bigvee_v \Big\{\mathcal{V}(v) \mid v \leq v_{f,\,g}\Big\} \geq \mathcal{V}(v_{f,\,g}) \geq \bigvee_{x\in X} f(x) \to \bigwedge_{x\in X} g(x).$$

(ST2) By (U4), we have $\xi_{\mathcal{V}}(f,g) = \bigvee_v \Big\{\mathcal{V}(v) \mid v \leq v_{f,\,g}\Big\} \leq S(f,g)$.
(ST3) By lemma 2.5 (2), we have

$$\begin{aligned}\xi_{\mathcal{V}}(f_1,g_1) &= \bigvee_v \Big\{\mathcal{V}(v) \mid v \leq v_{f_1,\,g_1}\Big\} \\ &= \bigvee_v \Big\{\mathcal{V}(v) \mid v \leq v_{f_1,\,g_1} \leq v_{f,\,g}\Big\} = \xi_{\mathcal{V}}(f,g).\end{aligned}$$

(ST4) For every $f_1, f_2, g_1, g_2 \in L^X$ and by lemma 2.5 (5), (U2), we have

$$\begin{aligned}\xi_{\mathcal{V}}(f_1,g_1) \odot \xi_{\mathcal{V}}(f_2,g_2) &= \bigvee_v \Big\{\mathcal{V}(v) \mid v \leq v_{f_1,\,g_1}\Big\} \odot \bigvee_w \Big\{\mathcal{V}(w) \mid w \leq v_{f_2,\,g_2}\Big\} \\ &= \bigvee_{v\odot w} \Big\{\mathcal{V}(v) \odot \mathcal{V}(w) \mid v \odot w \leq v_{f_1,\,g_1} \odot v_{f_2,\,g_2}\Big\} \\ &\leq \bigvee_{v\odot w} \Big\{\mathcal{V}(v \odot w) \mid v \odot w \leq v_{f_1\odot f_2,\, g_1\odot g_2}\Big\} = \xi_{\mathcal{V}}(f_1 \odot f_2, g_1 \odot g_2).\end{aligned}$$

Similarly, by Lemma 2.5 (6) we can prove that

$$\xi_{\mathcal{V}}(f_1 \oplus f_2, g_1 \oplus g_2) \geq \xi_{\mathcal{V}}(f_1,g_1) \odot \xi_{\mathcal{V}}(f_2,g_2).$$

Hence, $\xi_{\mathcal{V}}$ is a L-fuzzy topogenous order on X.
(2) (St) By Lemma 2.5 (7) and (U5), we have

$$\begin{aligned}\alpha \to \xi_{\mathcal{V}}(f,g) &= \alpha \to \bigvee_v \left\{\mathcal{V}(v) \mid v \le v_{f,\,g}\right\}\\ &\le \bigvee_v \left\{\mathcal{V}(\alpha \to v) \mid \alpha \to v \le \alpha \to v_{f,\,g}\right\}\\ &= \bigvee_v \left\{\mathcal{V}(\alpha \to v) \mid \alpha \to v \le v_{\alpha \odot f,\,g}\right\} = \xi_{\mathcal{V}}(\alpha \odot f, g).\end{aligned}$$

(3) (P) By lemma 2.5 (4) and (U3), we have

$$\begin{aligned}\xi_{\mathcal{V}}(\bigvee_{i\in I} f_i, g) &= \bigvee_v \left\{\mathcal{V}(v) \mid v \le v_{\bigvee_{i\in I} f_i,\,g}\right\}\\ &= \bigvee_v \left\{\mathcal{V}(v) \mid v \le \bigwedge_{i\in I} v_{f_i,\,g}\right\}\\ &\ge \bigwedge_{i\in I} \bigvee_v \left\{\mathcal{V}(v) \mid v \le v_{f_i,\,g}\right\} = \bigwedge_{i\in I} \xi_{\mathcal{V}}(f_i, g).\end{aligned}$$

In a similar way, we can prove that $\xi_{\mathcal{V}}(f, \bigvee_{i\in I} g_i) \ge \bigvee_{i\in I} \xi_{\mathcal{V}}(f, g_i)$.
(4) (SE) By lemma 2.5 (1) and for each $x \in X$, we have

$$\xi_{\mathcal{V}}(\top_x, \top_x) = \bigwedge_v \left\{\mathcal{V}(v) \mid v \le v_{\top_x,\top_x}\right\} \ge \mathcal{V}(v_{\top_x,\top_x}) \ge \top.$$

Similarly, $\xi_{\mathcal{V}}(\top_x^*, \top_x^*) = \top$.
(5) (TS) By (U3), (QU) and lemma 2.5 (3), we have

$$\begin{aligned}\bigvee_{h\in L^X} \xi_{\mathcal{V}}(f,h) \odot \xi_{\mathcal{V}}(h,g) &= \bigvee_{h\in L^X} \bigvee_u \left\{\mathcal{V}(u) \mid u \le u_{f,\,h}\right\} \odot \bigvee_v \left\{\mathcal{V}(v) \mid v \le v_{h,\,g}\right\}\\ &= \bigvee_{h\in L^X} \bigvee_{u,v} \left\{\mathcal{V}(u) \odot \mathcal{V}(v) \mid u \le u_{f,\,h},\ v \le v_{h,\,g}\right\}\\ &\ge \bigvee_w \left\{\mathcal{V}(w) \mid u \circ v \le w,\ u_{f,\,h} \circ v_{h,\,g} \le w_{f,\,g}\right\}\\ &= \bigvee_w \left\{\mathcal{V}(w) \mid w \le w_{f,\,g}\right\} = \xi_{\mathcal{V}}(f,g).\end{aligned}$$

Hence, $\xi_{\mathcal{V}}$ is a L-fuzzy topogenous space on X.
(6) By lemma 2.5 (11) and remark 2.7 (2), we have

$$\begin{aligned}\xi_{\mathcal{V}^s}(f,g) = \bigvee_v \left\{(\mathcal{V}^s)(v) \mid v \le v_{f,\,g}\right\} &= \bigvee_v \left\{\mathcal{V}(v^{-1}) \mid v^{-1} \le v^{-1}_{f,\,g}\right\}\\ &= \bigvee_v \left\{\mathcal{V}(v^{-1}) \mid v^{-1} \le v_{g^*,\,f^*}\right\} = \xi_{\mathcal{V}}(g^*, f^*).\end{aligned}$$

Theorem 3.6. Let $(X, \mathcal{V}_X)$ and $(Y, \mathcal{V}_Y)$ be two L-fuzzy pre-uniformities. If a map $\phi : (X, \mathcal{V}_X) \to (Y, \mathcal{V}_Y)$ is LF-uniformly continuous, then the map $\phi : (X, \xi_{\mathcal{V}_X}) \to (Y, \xi_{\mathcal{V}_Y})$ is a LF-topogenous map.

Proof. For $f, g \in L^Y$, we have

$$\begin{aligned}\xi_{\mathcal{V}_Y}(f,g) &= \bigwedge_v \left\{\mathcal{V}^*_Y(v) \mid v \le v_{f,\,g^*}\right\}\\ &\ge \bigwedge_v \left\{\mathcal{V}^*_X((\phi\times\phi)^{\leftarrow}(v)) \mid (\phi\times\phi)^{\leftarrow}(v) \le (\phi\times\phi)^{\leftarrow}(v_{f,\,g^*})\right\}\\ &= \bigwedge_v \left\{\mathcal{V}^*_X((\phi\times\phi)^{\leftarrow}(v)) \mid (\phi\times\phi)^{\leftarrow}(v) \le v_{\phi^{\leftarrow}(f),(\phi^{\leftarrow}(g))^*}\right\} = \xi_{\mathcal{V}_X}(\phi^{\leftarrow}(f), \phi^{\leftarrow}(g)).\end{aligned}$$

Theorem 3.7. Let ξ be a L-fuzzy topogenous order on X. Define a map $\mathcal{U}_\xi : L^{X\times X} \to L$ as

$$\mathcal{U}_\xi(u) = \bigvee\{\xi(f,g) \mid u_{f,\,g} \le u\} \quad \forall\ f, g \in L^X.$$

Then, the following hold
(1) $\mathcal{U}_\xi$ is a L-fuzzy pre-uniformity on X,
(2) If ξ is stratified, then $\mathcal{U}_\xi$ is stratified,
(3) If ξ is separated, then $\mathcal{U}_\xi$ is separated,
(4) If ξ is perfect, then $\mathcal{U}_\xi$ is perfect,

(5) If ξ is a L-fuzzy topogenous space on X, then $\mathcal{U}_\xi$ is a L-fuzzy quasi-uniformity on X,
(6) $\xi = \xi_{\mathcal{U}_\xi}$ and $\mathcal{U}_{\xi_{\mathcal{U}}} \leq \mathcal{U}$.

Proof. (1) (U1),(U2),(U4) Easy to be proved.
(U3) For every $u, v \in L^{X\times X}$ and by (ST4) and lemma 2.5 (5), we have

$$\begin{aligned}\mathcal{U}_\xi(u) \odot \mathcal{U}_\xi(v) &= \bigvee\{\xi(f_1, g_1) \mid u_{f_1,\, g_1} \leq u\} \odot \bigvee\{\xi(f_2, g_2) \mid u_{f_2,\, g_2} \leq v\}\\ &= \bigvee\{\xi(f_1, g_1) \odot \xi(f_2, g_2) \mid u_{f_1,\, g_1} \odot u_{f_2,\, g_2} \leq u \odot v\}\\ &\leq \bigvee\{\xi(f_1 \odot f_2, g_1 \odot g_2) \mid u_{f_1,\, g_1} \odot u_{f_2,\, g_2} \leq u_{f_1\odot f_2,\, g_1\odot g_2} \leq u \odot v\} = \mathcal{U}_\xi(u \odot v).\end{aligned}$$

(2) (St) By lemma 2.2 (2), lemma 2.5 (7) and (St), we have

$$\begin{aligned}\alpha \to \mathcal{U}_\xi(u) &= \alpha \to \bigvee\{\xi(f, g) \mid u_{f,\, g} \leq u\}\\ &= \bigvee\{\alpha \to \xi(f, g) \mid \alpha \to u_{f,\, g} \leq \alpha \to u\}\\ &\leq \bigvee\{\xi(\alpha \odot f, g) \mid u_{\alpha\odot f,\, g} \leq \alpha \to u\} = \mathcal{U}_\xi(\alpha \to u).\end{aligned}$$

(3) Easily proved.
(4) (P) Suppose that there exist $u_i \in L^{X\times X}$ for all $i \in I$ such that

$$\mathcal{U}_\xi(\bigwedge_{i\in I} u_i) \not\geq \bigwedge_{i\in I} \mathcal{U}_\xi(u_i).$$

Then for each $i \in I$, there exist $u_{f_i,\, g_i} \leq u$ such that

$$\mathcal{U}_\xi(\bigwedge_{i\in I} u_i) \not\geq \bigwedge_{i\in I} \xi(f_i, g_i) \quad f_i, g_i \in L^X.$$

For each $i \in I$, there exist $j_i, k_i \in I$ with $f_{j_i} = f_i,\ u_{f_{j_i},\, g_i} \leq u_i$ and $g_{k_i} = g_i,\ u_{f_i,\, g_{k_i}} \leq u_i$, such that $J = \{j_i : i \in I\},\ K = \{k_i : i \in I\},\ J \cup K = I$, and

$$\mathcal{U}_\xi(\bigwedge_{i\in I} u_i) \not\geq \bigwedge_{i\in I} \xi(f_i, g_i) = \xi(\bigvee_{i\in I} f_i, \bigwedge_{i\in I} g_i).$$

On the other hand, since $u_{\bigvee_{j_i\in J} f_{j_i}, \bigwedge_{k_i\in K} g_{k_i}} = \bigwedge_{i\in I} u_{f_i,\, g_i} \leq \bigwedge_{i\in I} u_i$, then we have

$$\mathcal{U}_\xi(\bigwedge_{i\in I} u_i) \geq \xi(\bigvee_{j_i\in J} f_{j_i}, \bigwedge_{k_i\in K} g_{k_i}) = \xi(\bigvee_{i\in I} f_i, \bigwedge_{i\in I} g_i) \geq \bigwedge_{i\in I} \xi(f_i, g_i).$$

Which contradicts the assumption at first. Hence, $\mathcal{U}_\xi(\bigwedge_{i\in I} u_i) \geq \bigwedge_{i\in I} \mathcal{U}_\xi(u_i)$. And by (U2), we have $\mathcal{U}_\xi(\bigwedge_{i\in I} u_i) \leq \bigwedge_{i\in I} \mathcal{U}_\xi(u_i)$. Thus, $\mathcal{U}_\xi$ is perfect.

(5) (QU) Suppose that $\mathcal{U}_\xi(u) \not\leq \bigvee\{\mathcal{U}_\xi(v) \odot \mathcal{U}_\xi(w) \mid v \circ w \leq u\}$ for some $u \in L^{X\times X}$. By the definition of $\mathcal{U}_\xi(u)$ and by (U2), we have

$$\mathcal{U}_\xi(u_{f,\, g}) \leq \mathcal{U}_\xi(u) \not\leq \bigvee\{\mathcal{U}_\xi(v) \odot \mathcal{U}_\xi(w) \mid v \circ w \leq u\}.$$

By the definition of $\mathcal{U}_\xi(u)$, then we have

$$\mathcal{U}_\xi(u_{f,\, g}) \not\leq \bigvee_{h\in L^X} \mathcal{U}_\xi(u_{f,\, h}) \odot \mathcal{U}_\xi(u_{h,\, g}).$$

Since $u_{f,h} \circ u_{h,g} \leq u_{f,g} \leq u$ and by Theorem 3.6 (2), we have

$$\mathcal{U}_\xi(u_{f,\, g}) = \bigvee_{h\in L^X} \mathcal{U}_\xi(u_{f,\, h}) \odot \mathcal{U}_\xi(u_{h,\, g}) \leq \bigvee\{\mathcal{U}_\xi(v) \odot \mathcal{U}_\xi(w) \mid v \circ w \leq u\}.$$

Which is a contradiction. Hence, $\mathcal{U}_\xi$ is a L-fuzzy quasi-uniformity on X.
(6) $\xi_{\mathcal{U}_\xi}(f,g) = \bigvee_u\{\mathcal{U}_\xi(u) : u \le u_{f,\,g}\} = \xi(f,g)$,

$$\mathcal{U}_{\xi_{\mathcal{U}}}(u) = \bigvee\{\xi_{\mathcal{U}}(f,g) : u_{f,\,g} \le u\} \le \mathcal{U}(u).$$

Theorem 3.8. Let (X,ξ_X) and (Y,ξ_Y) be two L-fuzzy topogenous spaces. If a map $\phi : (X,\xi_X) \to (Y,\xi_Y)$ is a LF-topogenous map, then the map $\phi : (X,\mathcal{U}_{\xi_X}) \to (Y,\mathcal{U}_{\xi_Y})$ is LF-uniformly continuous.

Proof. For every $v \in L^{Y\times Y}$, we have

$$\begin{aligned}\mathcal{U}_{\xi_X}((\phi\times\phi)^{\leftarrow}(v)) &= \bigvee\{\xi_X(\phi^{\leftarrow}(f),(\phi^{\leftarrow}(g))) : u_{\phi^{\leftarrow}(f),\,\phi^{\leftarrow}(g)} \le (\phi\times\phi)^{\leftarrow}(v)\}\\ &= \bigvee\{\xi_X(\phi^{\leftarrow}(f),\phi^{\leftarrow}(g)) : (\phi\times\phi)^{\leftarrow}(u_{f,\,g}) \le (\phi\times\phi)^{\leftarrow}(v)\}\\ &\ge \bigvee\{\xi_Y(f,g) : u_{f,\,g} \le v\} = \mathcal{U}_{\xi_Y}(v).\end{aligned}$$

Theorem 3.9.
(1) Υ : **L-FPUNS** $\to$ **L-FTGS** defined as $\Upsilon(X,\mathcal{U}) = (X,\xi_{\mathcal{U}})$ and $\Upsilon(\varphi) = \varphi$ is a concrete functor,
(2) Φ : **L-FTGS** $\to$ **L-FPUNS** defined as $\Phi(X,\xi) = (X,\mathcal{U}_\xi)$ and $\Phi(\varphi) = \varphi$ is a concrete functor,
(3) The pair (Υ,Φ) is a Galois correspondence between **L-FPUNS** and **L-FTGS**.

Proof.
(1) Follows from theorems 3.5 and 3.6 and (2) Follows from theorems 3.7 and 3.8.
(3) By Theorem 3.7 (6), if (X,ξ) is a L-fuzzy topogenous space, then

$$(\Upsilon\circ\Phi)(X,\xi) = \Upsilon(\Phi(X,\xi)) = \Upsilon(X,\mathcal{U}_\xi) = (X,\xi_{\mathcal{U}_\xi}) = (X,\xi).$$

Hence, the identity map $\Upsilon\circ\Phi = id_\xi$ is a LF-topogenous map.
By Theorem 3.7 (6), if $(X,\mathcal{U})$ is an L-fuzzy pre-uniformity, then

$$(\Phi\circ\Upsilon)(X,\mathcal{U}) = \Phi(\Upsilon(X,\mathcal{U})) = \Phi(X,\xi_{\mathcal{U}}) = (X,\mathcal{U}_{\xi_{\mathcal{U}}}) \le (X,\mathcal{U}).$$

Hence, the identity map $\Phi\circ\Upsilon = id_{\mathcal{U}}$ is LF-uniformly continuous. Therefore, the pair (Υ,Φ) is a Galois correspondence.
Example 3.10. Let $R \in L^{X\times X}$ be a reflexive L-fuzzy relation. Define a map $\mathcal{U} : L^{X\times X} \to L$ as

$$\mathcal{U}(u) = \bigwedge_{x,y\in X}(R(x,y) \to u(x,y)).$$

Then, (U1), (U2), (U3) can be easily proved.
(U4)

$$\begin{aligned}\mathcal{U}(u_{f,\,g}) &= \textstyle\bigwedge_{x,y\in X}(R(x,y) \to u_{f,\,g}(x,y))\\ &= \textstyle\bigwedge_{x,y\in X}(R(x,y) \to (f(x) \to g(y)))\\ &\le \textstyle\bigwedge_{x\in X}(R(x,x) \to (f(x) \to g(x))) = S(f,g).\end{aligned}$$

Hence, $\mathcal{U}$ is a L-fuzzy pre-uniformity on X.
For $R_1(x,y) = \top_{X\times X}$, we obtain

$$\mathcal{U}_1(u) = \bigwedge_{x,y\in X}(R_1(x,y) \to u(x,y)) = \bigwedge_{x,y\in X}(\top_{X\times X}(x,y) \to u(x,y)) = \bigwedge_{x,y\in X}u(x,y).$$

For $R_2(x,y) = \triangle_{X\times X}$, where

$$\triangle_{X\times X}(x,y) = \begin{cases}\top, & \text{if } y = x,\\ \bot, & \text{otherwise,}\end{cases}$$

we obtain

$$\mathcal{U}_2(u) = \bigwedge_{x,y\in X}(R_2(x,y) \to u(x,y)) = \bigwedge_{x,y\in X}(\triangle_{X\times X}(x,y) \to u(x,y)) = \bigwedge_{x\in X} u(x,x).$$

(1) From theorem 3.3, we obtain a L-fuzzy topogenous order $\xi_{\mathcal{U}} : L^X \times L^X \to L$ as

$$\begin{aligned}\xi_{\mathcal{U}}(f,g) &= \textstyle\bigwedge_{x\in X}(f(x) \to \mathcal{U}(u_{\top_x,\,g}))\\ &= \textstyle\bigwedge_{x\in X}(f(x) \to \bigwedge_{x,y\in X}(R(x,y) \to u_{\top_x,\,g}(x,y)))\\ &= \textstyle\bigwedge_{x\in X}(f(x) \to \bigwedge_{x,y\in X}(R(x,y) \to (\top_x(x) \to g(y))))\\ &= \textstyle\bigwedge_{x,y\in X}(f(x) \to (R(x,y) \to g(y))) = \bigwedge_{x,y\in X}(R(x,y) \to (f(x) \to g(y))).\end{aligned}$$

For $R_1(x,y) = \top_{X\times X}$, we have

$$\begin{aligned}\xi_{\mathcal{U}_1}(f,g) &= \textstyle\bigwedge_{x,y\in X}(R_1(x,y) \to (f(x) \to g(y)))\\ &= \textstyle\bigwedge_{x,y\in X}(\top_{X\times X}(x,y) \to (f(x) \to g(y))) = \bigwedge_{x,y\in X} f(x) \to g(y).\end{aligned}$$

For $R_2(x,y) = \triangle_{X\times X}$, we have

$$\begin{aligned}\xi_{\mathcal{U}_2}(f,g) &= \textstyle\bigwedge_{x,y\in X}(R_2(x,y) \to (f(x) \to g(y)))\\ &= \textstyle\bigwedge_{x,y\in X}(\triangle_{X\times X}(x,y) \to (f(x) \to g(y))) = \bigwedge_{x\in X} f(x) \to g(x) = S(f,g).\end{aligned}$$

(2) From theorem 3.5, we obtain a L-fuzzy topogenous order $\xi_{\mathcal{V}} : L^X \times L^X \to L$ as

$$\xi_{\mathcal{V}}(f,g) = \textstyle\bigvee_v\{\mathcal{V}(v) \mid v \le v_{f,\,g}\} = \bigvee_v\{\bigwedge_{x,y\in X}(R(x,y) \to v(x,y)) \mid v \le v_{f,\,g}\}.$$

For $R_1(x,y) = \top_{X\times X}$, we have

$$\begin{aligned}\xi_{\mathcal{V}_1}(f,g) &= \textstyle\bigvee_v\{\bigwedge_{x,y\in X}(R_1(x,y) \to v(x,y)) \mid v \le v_{f,\,g}\}\\ &= \textstyle\bigvee_v\{\bigwedge_{x,y\in X}(\top_{X\times X}(x,y) \to v(x,y)) \mid v \le v_{f,\,g}\}\\ &= \textstyle\bigvee_v \bigwedge_{x,y\in X}\{v(x,y) \mid v \le v_{f,\,g}\}.\end{aligned}$$

For $R_2(x,y) = \triangle_{X\times X}$, we have

$$\begin{aligned}\xi_{\mathcal{V}_2}(f,g) &= \textstyle\bigvee_v\{\bigwedge_{x,y\in X}(R_2(x,y) \to v(x,y)) \mid v \le v_{f,\,g}\}\\ &= \textstyle\bigvee_v\{\bigwedge_{x,y\in X}(\triangle_{X\times X}(x,y) \to v(x,y)) \mid v \le v_{f,\,g}\}\\ &= \textstyle\bigvee_v \bigwedge_{x\in X}\{v(x,x) \mid v \le v_{f,\,g}\}.\end{aligned}$$

(3) From theorem 3.7, we obtain a L-fuzzy pre-uniformity $\mathcal{U}_\xi : L^{X\times X} \to L$ as

$$\begin{aligned}\mathcal{U}_{\xi_{\mathcal{U}}}(u) &= \textstyle\bigvee\{\xi_{\mathcal{U}}(f,g) : u_{f,\,g} \le u\}\\ &= \textstyle\bigvee\{\bigwedge_{x,y\in X}(R(x,y) \to (f(x) \to g(y))) : u_{f,\,g} \le u\} \le \mathcal{U}(u).\end{aligned}$$

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