Filomat 34:1 (2020), 59–66
https://doi.org/10.2298/FIL2001059E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Quaternionic Bertrand Curves in the Galilean Space

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Abstract. In this article, we introduce the notion of a spatial quaternionic Bertrand curves in G^3 and give some characterizations of such curves. Furthermore, we introduce spatial quaternionic $(1,3)-$Bertrand curves in G^4.

1. Introduction

The geometry of curves in both Euclidean and Minkowski spaces was represented for many years a popular topic in the field of classical differential geometry [4–6, 11]. Global and local properties were studied in several books and new invariants were defined for curves. Two curves which have a common principal normal vector at corresponding points are called Bertrand curves [9, 11].

In four dimensional Euclidean space E^4, Bertrand curves are generalized and characterized in [11]. Moreover, in $3-$dimensional Galilean space G^3, Bertrand curves are defined and characterized in many papers such as [1, 12].

K. Bharathi and M. Nagaraj in [3] were studied quaternionic curves in both Euclidean space E^3 and Euclidean $4-$ space E^4 and gave the Frenet formulae for quaternionic curves. For other results of quaternionic curves, we refer to the papers [7, 8, 10, 14]. Moreover, O. Kecilioglu and K. Ilarslan [9] were proved that if the bitorsion of a quaternionic curve α is no vanish, then there is no quaternionic curve in the Euclidean $4-$ space E^4 is a Bertrand curve. Therefore, they defined $(1,3)-$ type Bertrand curves for quaternionic curves in the Euclidean $4-$ space E^4 and gave some characterizations for this type in E^4 by means of the curvature functions of the curve.

In these regards, we introduce the quaternionic Bertrand curves in the Galilean space G^3 and give some characterizations of such curves. Also, we investigate generalized quaternionic Bertrand curves in the Galilean $4-$ space G^4 and deduce general characterizations of these curves.

2. Preliminaries

In this section, we give the basic elements of the theory of quaternions and quaternionic curves. Also, we introduce a brief note about Bertrand curves in the Galilean space.

2010 *Mathematics Subject Classification*. 53A35, 51A05
Keywords. Galilean space, Bertrand curves, Quaternionic Bertrand curves.
Received: 02 January 2019; Revised: 11 March 2019; Revised: 22 March 2019; Accepted: 16 June 2019
Communicated by Biljana Popović
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A real quaternion q is an expression of the form

$$q = a_1\, e_1 + a_2\, e_2 + a_3\, e_3 + a_4\, e_4$$

where a_i, $(1 \leq i \leq 4)$ are real numbers, and e_i , $(1 \leq i \leq 4)$ are quaternionic units which satisfy the non-commutative multiplication rules

$$\begin{aligned} e_i \times e_i &= -e_4, \qquad (1 \leq i \leq 3) \\ e_i \times e_j &= e_k = -e_j \times e_i, \ (1 \leq i, j, k \leq 3) \end{aligned}$$

where (ijk) is an even permutation of (123) in the Euclidean space [8].

The algebra of the quaternions is denoted by Q and its natural basis is given by (e_1, e_2, e_3, e_4). A real quaternion can be given by the form

$$q = S_q + V_q$$

where $S_q = a_4$ is scalar part and $V_q = a_1e_1 + a_2e_2 + a_3e_3$ is vector part of q.

On the other hand, the conjugate of $q = S_q + V_q$ is defined by [2]

$$\alpha\, q = S_q - V_q$$

Let p and q be any two elements of Q. Then the product of p and q is denoted by

$$p \times q = S_pS_q - \left\langle V_p, V_q \right\rangle + S_pV_q + S_qV_p + V_p \wedge V_q$$

where we have used the inner and cross products in the Galilean space G^3.

This defines the symmetric real-valued, non-degenerate, bilinear form as follows:

$$\begin{aligned} h \quad &: \quad Q \times Q \to R \\ (p, q) \quad &\to \quad h(p, q) = \frac{1}{2}\left[p \times \alpha q + q \times \alpha p\right] \end{aligned}$$

which is called the quaternion inner product. Then the norm of q is given by

$$\left\|q\right\|^2 = h(q, q) = \frac{1}{2}\left[\alpha\, q \times q \ + \ q \times \alpha\, q\right] = a_1^2 \ + \ a_2^2 \ + \ a_3^2 \ + \ a_4^2$$

If $\| q \| = 1$, then q is called unit quaternion.

q is called a spatial quaternion whenever $q + \alpha q = 0$ and called a temporal quaternion whenever $q - \alpha q = 0$. Then a general quaternion q can be given as [3]

$$q = \frac{1}{2}\left[q + \alpha q\right] + \frac{1}{2}\left[q - \alpha q\right] \tag{1}$$

Now, we consider a quaternionic curve in E^3. The Euclidean space E^3 is identified with the space of spatial quaternion $\{\beta \in Q / \beta + \alpha\beta = 0\}$ in an obvious manner. Let $I = [0, 1]$ be an interval in the real line R and $s \in I$ be the arc-length parameter along the smooth curve

$$\begin{aligned} \beta \quad &: \quad I \subset R \to Q \\ s \quad &\to \quad \beta(s) = \sum_{i=1}^{3} \beta_i(s) e_i \quad (1 \leq i \leq 3) \end{aligned}$$

The tangent vector $\beta'(s) = \mathbf{t}(\mathbf{s})$ has unit length $\|\mathbf{t}(\mathbf{s})\| = 1$ for all s. It follows

$$\mathbf{t}' \times \alpha\mathbf{t} + \mathbf{t} \times \alpha\mathbf{t}' = 0$$

which implies $\mathbf{t}'$ is orthogonal to $\mathbf{t}$ and $\mathbf{t}' \times \alpha\mathbf{t}$ is a spatial quaternion. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame of $\beta(s)$, then Frenet formulae are given by

$$\begin{aligned} \mathbf{t}' &= k\mathbf{n} \\ \mathbf{n}' &= -k\mathbf{t} + \tau\mathbf{b} \\ \mathbf{b}' &= -\tau\mathbf{n} \end{aligned}$$

where $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are the unit tangent, the unit principal normal and the unit binormal vector of a quaternionic curve β, respectively. The functions k, τ are called the principal curvature and the torsion of β, respectively [1].

Theorem 2.1. *[8, 9] The four-dimensional Euclidean space E^4 are identified with the space of unique quaternions. Let $I = [0, 1]$ be a unit interval of the real line R and*

$$\begin{aligned} \beta &: I \subset R \rightarrow Q \\ s &\rightarrow \beta(s) = \sum_{i=1}^{4} \beta_i(s) e_i \quad (1 \leq i \leq 4) \end{aligned}$$

be a smooth curve in E^4 with non zero curvatures $\{K, k, r - K\}$ and $\{T(s), N(s), B_1(s), B_2(s)\}$ denotes the Frenet frame of the curve. Then the Frenet formulae are given by

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & r-K \\ 0 & 0 & -(r-K) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{2}$$

where K is the principal curvature, k is the torsion and $(r - K)$ is the bitorsion of β.

The notion of Bertrand curves was discovered by J. Bertrand in 1850 and it plays an important role in classical differential geometry, and a lot of mathematicians have studied on the Bertrand curves in different areas [12]. For more about generalized Bertrand curves in the Galilean space and Minkowski space, we refer to [1, 4, 6, 10–13].

3. Quaternionic Bertrand curves in the Galilean 3−Space

Definition 3.1. *Let $\alpha(s)$ and $\beta(s^*)$ be two quaternionic curves in G^3. $\{T_\alpha(s), N_\alpha(s), B_\alpha(s)\}$ and $\{T_\beta(s^*), N_\beta(s^*), B_\beta(s^*)\}$ are Frenet frames of these curves, respectively. $\alpha(s)$ and $\beta(s^*)$ are called Bertrand curves if there exist a bijection $\phi : I \subset R \rightarrow I^*$, and $s \rightarrow \phi(s) = s^*$, $\frac{ds^*}{ds} \neq 0$ and the principal normal lines of $\alpha(s)$ and $\beta(s^*)$ are linearly dependent. In other words, if the principal normal lines of $\alpha(s)$ and $\beta(s^*)$ at $s \in I$ are parallel.*

Theorem 3.2. *Let $\alpha(s)$ and $\beta(s^*)$ be a spatial quaternionic Bertrand curves in G^3. Then $d(\alpha(s), \beta(s^*)) =$ constant for all $s \in I$.*

Proof. From definition (3.1), we can write

$$\beta(s^*) = \alpha(s) + \lambda(s) N_\alpha(s) \tag{3}$$

differentiating with respect to s and using Frenet equations, we get

$$T_\beta(s^*) \frac{ds^*}{ds} = T_\alpha(s) + \lambda'(s) N_\alpha(s) + \lambda(s)\tau_\alpha(s) B_\alpha(s)$$

Let us put $\frac{ds^*}{ds} = \varphi'(s) \neq 0$, then we have

$$T_\beta(s^*) = \frac{1}{\varphi'(s)} [T_\alpha(s) + \lambda'(s) N_\alpha(s) + \lambda(s)\tau_\alpha(s) B_\alpha(s)] \tag{4}$$

Since $\{N_\alpha(s), N_\beta(s^*)\}$ is a linearly dependent set, we get $h(N_\alpha(s), N_\beta(s^*)) \neq 0$ hence $\lambda'(s) = 0$ and λ is a constant function on I. □

Theorem 3.3. *Let $\alpha(s)$ and $\beta(s^*)$ be a spatial quaternionic Bertrand curves in G^3. Then the measure of the angle between the tangent vector field of $\alpha(s)$ and $\beta(s^*)$ is constant.*

Proof. Let $\alpha(s)$ and $\beta(s^*)$ be a spatial quaternionic Bertrand curves in G^3 with arc length s and s^*, respectively. Define $T_\beta(s^*)$ as follows

$$T_\beta(s^*) = \cos\theta(s) T_\alpha(s) + \sin\theta(s) B_\alpha(s) \tag{5}$$

where θ is the angle between $T_\alpha(s)$ and $T_\beta(s^*)$.

Differentiating with respect to s, we obtain

$$T'_\beta(s^*)\varphi'(s) = \frac{d\cos\theta(s)}{ds} T_\alpha(s) + \cos\theta(s) T'_\alpha(s) + \frac{d\sin\theta(s)}{ds} B_\alpha(s) + \sin\theta(s) B'_\alpha(s)$$

Since $\{T_\alpha(s), B_\alpha(s), N_\alpha(s)\}$ is the Frenet frame on G^3 along $\alpha(s)$ and $\beta(s^*)$ is a Bertrand mate of $\alpha(s)$, so

$$\begin{aligned} h(N_\alpha(s), T_\alpha(s)) &= h(N_\alpha(s), B_\alpha(s)) = h(T_\alpha(s), B_\alpha(s)) = 0 \\ h(T_\alpha(s), T_\alpha(s)) &= h(B_\alpha(s), B_\alpha(s)) = h(N_\alpha(s), N_\alpha(s)) = 1 \end{aligned}$$

and hence, we obtain

$$\frac{d\sin\theta(s)}{ds} = 0 \Rightarrow \sin\theta(s) = constant$$
$$\frac{d\cos\theta(s)}{ds} = 0 \Rightarrow \cos\theta(s) = constant$$

Therefore, we can deduce that the angle θ is constant. □

Theorem 3.4. *Let $\alpha(s)$ be a spatial quaternionic curve in G^3 with arc length s. $\alpha(s)$ is a spatial quaternionic Bertrand curve if and only if $\alpha(s)$ has a constant torsion.*

Proof. Let $\alpha(s)$ and $\beta(s^*)$ be a spatial quaternionic Bertrand curves in G^3. From theorem (3.2), we obtain

$$\varphi'(s) T_\beta(s^*) = T_\alpha(s) + \lambda\tau_\alpha(s) B_\alpha(s) \tag{6}$$

Also, from theorem (3.3), we get

$\varphi'(s) = \frac{1}{\cos\theta(s)}$ and $\varphi'(s)\sin\theta(s) = \lambda\tau_\alpha(s)$. Therefore, $\tan\theta(s) = \lambda\tau_\alpha(s)$ which implies that $\tau_\alpha(s) = \frac{\tan\theta(s)}{\lambda}$. Since θ is constant we obtain that τ_α is constant. Conversely, assume that $\tau_\alpha(s)$ is constant and define $\beta(s^*) = \alpha(s) + \lambda(s) N_\alpha(s)$. So, we have

$$\varphi'(s) T_\beta(s^*) = T_\alpha(s) + \lambda\tau_\alpha(s) B_\alpha(s)$$

which implies that

$$(\varphi'(s))^2 = 1 + \lambda^2\tau_\alpha^2 \tag{7}$$

Differentiating with respect to s , we get

$$(\varphi'(s))^2 K_\beta(s^*) N_\beta(s^*) = \left(K_\alpha(s) - \lambda\tau_\alpha^2(s)\right) N_\alpha(s)$$

and therefore,

$$K_\beta(s^*) N_\beta(s^*) = \frac{K_\alpha(s) - \lambda\tau_\alpha^2}{1 + \lambda^2\tau_\alpha^2} N_\alpha(s) \tag{8}$$

which means that $N_\beta(s^*)$ and $N_\alpha(s)$ are linearly dependent, and according to definition (3.1), it determines that $\alpha(s)$ and $\beta(s^*)$ are spatial quaternionic Bertrand pair. □

Corollary 3.5. *Let $\alpha(s)$ be a spatial quaternionic curve in G^3 with arc- length parameter s. $\alpha(s)$ is a spatial quaternionic Bertrand curve if $K_\beta(s^*) = LK_\alpha(s) + M$ where L and M are constants.*

Theorem 3.6. *Let $(\alpha(s), \beta(s^*))$ be a spatial quaternionic Bertrand pair in G^3. Then $\frac{\tau_\alpha(s)}{\tau_\beta(s^*)}$ = constant provided that $\tau_\beta(s^*) \neq 0$.*

Proof. If we take $\alpha(s)$ instead of $\beta(s^*)$, then we can write equation (3) in the form:

$$\alpha(s) = \beta(s^*) - \lambda N_\beta(s^*) \tag{9}$$

differentiating with respect to s and using Frenet equations, we obtain

$$T_\alpha(s) = \varphi'(s)T_\beta(s^*) - \lambda\varphi'(s)\, \tau_\beta(s^*)B_\beta(s^*) \tag{10}$$

which implies that

$$\begin{aligned} 1 &= (\varphi'(s))^2 - \lambda^2\varphi'(s)^2\, \tau_\beta^2(s^*) \\ &= (\varphi'(s))^2 \left[1 - \lambda^2\tau_\beta^2(s^*)\right] \end{aligned}$$

Now, from equation (7), we can get

$$\tau_\alpha^2(s) = \tau_\beta^2(s^*)\left(1 + \lambda^2\tau_\alpha^2(s)\right)$$

which gives

$$\frac{\tau_\alpha(s)}{\tau_\beta(s^*)} = \sqrt{1 + \lambda^2\tau_\alpha^2(s)}$$

Therefore, $\frac{\tau_\alpha(s)}{\tau_\beta(s^*)}$ is constant. □

4. Quaternionic Bertrand curves in the Galilean 4−Space

Special Bertrand curves in 4D Galilean space was introduced by Oztekin [13]. In this section we introduce the spatial quaternionic Bertrand curves in G^4. Let $\alpha(s)$ be a quaternionic curve in the Galilean space G^4, if $K_{3\alpha}(s) \neq 0$, then there is no quaternionic curve in G^4 is a Bertrand curve. Therefore, we can introduce the following definition.

Definition 4.1. *Let $\alpha(s)$ and $\beta(s^*)$ be two spatial quaternionic curves in G^4, $\{T_\alpha(s), N_\alpha(s), B_{1\alpha}(s), B_{2\alpha}(s)\}$ and $\left\{T_\beta(s^*), N_\beta(s^*), B_{1\beta}(s^*), B_{2\beta}(s^*)\right\}$ are Frenet frames of these curves, respectively. $\alpha(s)$ and $\beta(s^*)$ are called spatial quaternionic $(1,3)$−Bertrand curves if there exist a bijection*

$$\begin{aligned} \phi &: I \to I^* \\ s &\to \phi(s) = s^*, \frac{ds^*}{ds} \neq 0 \end{aligned}$$

and the plane spanned by $N_\alpha(s)$, $B_{2\alpha}(s)$ at each point $\alpha(s)$ of the curve α coincides with the plane spanned by $N_\beta(s^*)$, $B_{2\beta}(s^*)$ at corresponding point $\beta(s^*) = \beta(\phi(s))$ of the curve β.

Theorem 4.2. *Let $\alpha(s)$ be a spatial quaternionic curve in G^4 with curvature functions $K_{1\alpha}(s)$, $K_{2\alpha}(s)$, $K_{3\alpha}(s)$ and $K_{3\alpha}(s) \neq 0$. Then $\alpha(s)$ is a $(1,3)$−Bertrand curve if there exist constant real numbers ρ, σ, ζ and δ satisfying*

1. $\rho\, K_{2\alpha}(s) - \sigma\, K_{3\alpha}(s) \neq 0$,
2. $\zeta\, (\rho\, K_{2\alpha}(s) - \sigma\, K_{3\alpha}(s)) = 1$,
3. $\delta\, K_{3\alpha}(s) = \zeta\, K_{1\alpha}(s) - K_{2\alpha}(s)$

Proof. We assume that $\alpha(s)$ is a $(1,3)-$ Bertrand curve parametrized by arc length s. The $(1,3)-$ Bertrand mate $\beta(s)$ is given by

$$\beta(s^*) = \beta(\phi(s)) = \alpha(s) + \rho(s)N_\alpha(s) + \sigma(s)B_{2\alpha}(s) \tag{11}$$

for all $s \in I$, where $\rho(s)$ and $\sigma(s)$ are $C^\infty-$ functions on I and s^* is the arc length parameter of β.

Differentiating equation (11) with respect to s and using the Frenet equations, we obtain

$$\phi'(s)T_\beta(s^*) = T_\alpha(s) + \rho'(s)N_\alpha(s) + [\rho(s)K_{2\alpha}(s) - \sigma(s)K_{3\alpha}(s)]B_{1\alpha}(s) + \sigma'(s)B_{2\alpha}(s)$$

for all $s \in I$.

Since the plane spanned by $N_\alpha(s)$ and $B_{2\alpha}(s)$ coincides with the plane spanned by $N_\beta(s^*)$ and $B_{2\beta}(s^*)$, we can put

$$N_\beta(s^*) = \cos\theta(s)N_\alpha(s) + \sin\theta(s)B_{2\alpha}(s) \tag{12}$$

and

$$B_{2\beta}(s^*) = (-\sin\theta(s))N_\alpha(s) + \cos\theta(s)B_{2\alpha}(s) \tag{13}$$

and we notice that $\sin\theta(s) \neq 0$ for all $s \in I$, also we obtain

$$\rho'(s)\cos\theta(s) + \sigma'(s)\sin\theta(s) = 0$$

$$-\rho'(s)\sin\theta(s) + \sigma'(s)\cos\theta(s) = 0$$

from these equations, we get $\sigma'(s) = 0$ and $\rho'(s) = 0$, hence $\rho(s)$ and $\sigma(s)$ are constant functions on I with values ρ and σ, respectively.

Thus, for all $s \in I$, eqation (11) can be rewritten in the form

$$\beta(s^*) = \beta(\phi(s)) = \alpha(s) + \rho N_\alpha(s) + \sigma B_{2\alpha}(s) \tag{14}$$

differentiating with respect to s and using the Frenet equations, we obtain

$$\phi'(s)T_\beta(s^*) = T_\alpha(s) + [\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s)]B_{1\alpha}(s)$$

and

$$\left(\phi'(s)\right)^2 = 1 + (\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s))^2 \neq 0 \tag{15}$$

for all $s \in I$.

Therefore, we can put

$$T_\beta(s^*) = (\cos\psi(s))\,T_\alpha(s) + (\sin\psi(s))\,B_{1\alpha}(s) \tag{16}$$

and we obtain $\cos\psi(s) = \frac{1}{\phi'(s)}$ and $\sin\psi(s) = \frac{\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s)}{\phi'(s)}$ where $\psi(s)$ is a $C^\infty-$ functions on I.

Differentiating equation (16) and using Frenet equations, we get

$$\begin{aligned}\phi'(s)K_{1\beta}(s^*)N_\beta(s^*) &= [\cos\psi(s)K_{1\alpha}(s) - \sin\psi(s)K_{2\alpha}(s)]\,N_\alpha(s) + \frac{d\cos\psi(s)}{ds}T_\alpha(s)\\ &+ \frac{d\sin\psi(s)}{ds}B_{1\alpha}(s) + \sin\psi(s)K_{3\alpha}(s)B_{2\alpha}(s)\end{aligned}$$

Since $N_\beta(s^*) = N_\beta(\phi(s))$ is expressed by a linear combination of $N_\alpha(s)$ and $B_{2\alpha}(s)$, it holds that

$$\begin{aligned}\frac{d\cos\psi(s)}{ds}T_\alpha(s) &= 0 \Longrightarrow \frac{d\cos\psi(s)}{ds} = 0\\ \frac{d\sin\psi(s)}{ds}B_{1\alpha}(s) &= 0 \Longrightarrow \frac{d\sin\psi(s)}{ds} = 0\end{aligned}$$

i.e., $\psi(s)$ is a constant function on I with value ψ, and hence

$$T_\beta(s^*) = T_\alpha(s)\cos\psi + B_{1\alpha}(s)\sin\psi \tag{17}$$

$$\sin\psi = \frac{\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s)}{\phi'(s)}$$

$$\cos\psi = \frac{1}{\phi'(s)}$$

which implies that

$$\sin\psi = (\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s))\cos\psi \tag{18}$$

for all $s \in I$.

If $\sin\psi = 0$, then it holds $\cos\psi = \mp 1$ and hence equation (17) becomes

$$T_\beta(s^*) = \mp T_\alpha(s) \tag{19}$$

Differentiating with respect to s and using the Frenet equations, we obtain

$$\phi'(s)K_{1\beta}(s^*)N_\beta(s^*) = \mp K_{1\alpha}(s)N_\alpha(s)$$

i.e.,

$$N_\beta(s^*) = \mp N_\alpha(s) \tag{20}$$

for all $s \in I$, and this is a contradiction. Hence, we must consider only the case of $\sin\psi \neq 0$, then equation (18) implies that

$$\rho\, K_{2\alpha}(s) - \sigma\, K_{3\alpha}(s) \neq 0 \tag{21}$$

and therefore, we obtain the first relation.

The fact $\sin\psi \neq 0$ and equation (18) imply that

$$\zeta\,(\rho\, K_{2\alpha}(s) - \sigma\, K_{3\alpha}(s)) = 1 \tag{22}$$

where $\zeta = \frac{\cos\psi}{\sin\psi}$ is a constant number and hence we obtain the second relation.

Differentiating equation (17) with respect to s and using the Frenet equations, we obtain

$$\left(\phi'(s)K_{1\beta}(s^*)\right)^2 = [\cos\psi K_{1\alpha}(s) - \sin\psi K_{2\alpha}(s)]^2 + (\sin\psi K_{3\alpha}(s))^2 \tag{23}$$

and from equations (22) and (23), we deduce

$$\begin{aligned}\left(\phi'(s)K_{1\beta}(s^*)\right)^2 &= \left[(\zeta K_{1\alpha}(s) - K_{2\alpha}(s))^2 + (K_{3\alpha}(s))^2\right].\\ &\quad (\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s))^2\left(\phi'(s)\right)^{-2}\end{aligned}$$

for all $s \in I$.

From equation (15) and the second relation, we get

$$\left(\phi'(s)\right)^2 = \left(\varsigma^2 + 1\right)(\lambda K_{2\alpha}(s) - \mu K_{3\alpha}(s))^2$$

Thus, we obtain

$$\left(\phi'(s)K_{1\beta}(s^*)\right)^2 = \frac{1}{\zeta^2+1}\left[(\zeta K_{1\alpha}(s) - K_{2\alpha}(s))^2 + (K_{3\alpha}(s))^2\right] \tag{24}$$

From equations (22),(23) and the second relation, we can put

$$N_\beta(s^*) = N_\beta(\phi(s)) = \cos\eta(s)N_\alpha(s) + \sin\eta(s)B_{2\alpha}(s) \tag{25}$$

where

$$\cos\eta(s) = \frac{(\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s))\,(\zeta K_{1\alpha}(s) - K_{2\alpha}(s))}{K_{1\beta}(\phi(s))\left(\phi'(s)\right)^2} \tag{26}$$

and

$$\sin\eta(s) = \frac{K_{3\alpha}(s)\,(\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s))}{K_{1\beta}(\phi(s))\left(\phi'(s)\right)^2} \tag{27}$$

for all $s \in I$. Here, η is a $C^\infty-$ function on I.

Differentiating equation (25) with respect to s and using Frenet equations, we obtain

$$\frac{d\cos\eta(s)}{ds} = 0,\ \frac{d\sin\eta(s)}{ds} = 0$$

that is, η is a constant function on I with value η_0. From equations (26) and (27), we obtain

$$K_{3\alpha}(s)\frac{\cos\eta_0}{\sin\eta_0} = (\zeta K_{1\alpha}(s) - K_{2\alpha}(s))$$

Let $\delta = \frac{\cos\eta_0}{\sin\eta_0}$ be a constant number, we then get

$$\delta K_{3\alpha}(s) = \zeta\, K_{1\alpha}(s) - K_{2\alpha}(s)$$

which is the third relation. □

Acknowledgement: The authors wish to express their sincere thanks to referee for making several useful comments.

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