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Constraction of a Core Regular Double *MS*-Algebra

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Abstract. In this paper, we introduce and characterize a core regular double *MS*-algebra. A construction of a core regular double *MS*-algebra $M^{[2]}$ via a de Morgan algebra M is given. A one to one correspondence between the class of de Morgan algebras and the class of core regular double *MS*-algebras is obtained. According to such construction we investigate many properties of a core regular double *MS*-algebra deal with subalgebras, homomorphisms, atoms and dual atoms. A description of an atomic core regular double *MS*-algebra is established. Also, we discuss some properties of a complete core regular double *MS*-algebra.

1. Introduction

De Morgan Stone algebra (briefly *MS*-algebra) is introduced by T.S. Blyth and J.C. Varlet [8] as a common properties of a de Morgan algebra and a Stone algebra. T.S. Blyth and J.C. Varlet [9] described the lattice of all subclasses of the class **MS** of all *MS*-algebras which contains twenty subclasses, for examples, the class **S** of all Stone algebras and the class **M** of all de Morgan algebras. Also, T.S. Blyth and J.C. Varlet [10] presented the class **DMS** of all double *MS*-algebras which containing the class **DS** of all double Stone algebras. J.C. Varlet [18] studied a regular variety of type (2,2,1,1,0,0). T. Katriňāk [16] presented a construction of a regular double Stone algebra from a suitable Boolean algebra B and a filter F of B. S.D. Comer [14] proved the existence and uniqueness of perfect extensions of a regular double stone algebra using Katriňāk′s construction [16]. Recently, A. Badawy [2] introduced and characterized the class of double *MS*-algebras satisfying the generalized complement property (briefly DMS^{gc}-algebras) which includes the class of double *MS*-algebras satisfying the complement property presented by L. Congwen [13]. Also, A. Badawy [2] gave a construction of DMS^{gc}-algebras generalizing the construction due to T. Katriňāk [11] for regular double Stone algebras. Many important properties of *MS*-algebras and double *MS*-algebras deal with homomorphisms, subalgebras, filters and congruences are studied in [3-7].

In this paper, we introduce and characterize a subclass of the class of double MS^{gc}-algebras which is called core regular double *MS*-algebras. In fact the class **CRDMS** of all core regular double *MS*-algebras includes the class **CRDS** of all core regular double Stone algebras due to R. Kumar et al. [17]. A construction of a core regular double *MS*-algebra from a suitable de Morgan algebra is obtained. Also, we construct

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a core regular double Stone algebra from a suitable Boolean algebra. We observe that there is a one to one correspondence between the class **M** of all de Morgan algebras and the class **CRDMS**. We give many applications of such construction. Characterizations of homomorphisms and subalgebras of core regular double *MS*-algebras are obtained. We describe atoms and dual atoms of a core regular double *MS*-algebra by using this construction. A description of atomic core regular double *MS*-algebras is given. We observe that the completeness of a core regular double *MS*-algebra L depends on only the completeness of its skeleton $L^{\circ\circ}$, in particular the last two applications of our construction are to discuss complete homomorphisms and complete subalgebras of core regular double *MS*-algebras.

2. Preliminaries

In this section, we recall certain definitions and important results. We refer the reader to the references [5], [7], [8], [9], [10],[11],[12] and [15] as a guide references.

Definition 2.1. [15] *An algebra $(L; \wedge, \vee)$ of type (2,2) is said to be a lattice if for every $a, b, c \in L$, it satisfies the following properties:*

(1) $a \wedge a = a, a \vee a = a$ (Idempotency),

(2) $a \wedge b = b \wedge a, a \vee b = b \vee a$ (Commutativity),

(3) $(a \wedge b) \wedge c = a \wedge (b \wedge c), (a \vee b) \vee c = a \vee (b \vee c)$ (Associativity),

(4) $(a \wedge b) \vee a = a, (a \vee b) \wedge a = a$ (Absorption).

If a lattice L has a greatest element (denoted by 1) and a smallest element (denoted by 0), then L is said to be a bounded lattice.

Definition 2.2. [15] *A lattice L is called distributive if it satisfies either of the following equivalent distributive laws:*

(1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, for all $a, b, c \in L$.

Definition 2.3. [11] *A de Morgan algebra is an algebra $(L; \vee, \wedge, ^{-}, 0, 1)$ of type (2,2,1,0,0) where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $^{-}$ the unary operation of involution satisfies:*

$$\overline{\overline{x}} = x, \overline{(x \vee y)} = \overline{x} \wedge \overline{y}, \overline{(x \wedge y)} = \overline{x} \vee \overline{y}.$$

Definition 2.4. [12] *A Stone algebra is a universal algebra $(L; \vee, \wedge, ^{*}, 0, 1)$ of type $(2, 2, 1, 0, 0)$, where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation * of pseudocomplementation has the properties that $x \wedge a = 0 \Leftrightarrow x \leq a^{*}$ and $x^{**} \vee x^{*} = 1$.*

Definition 2.5. [16] *A dual Stone algebra is a universal algebra $(L; \vee, \wedge, ^{+}, 0, 1)$ of type $(2, 2, 1, 0, 0)$, where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation $^{+}$ of dual pseudocomplementation has the properties that $x \vee a = 1 \Leftrightarrow x \geq a^{+}$ and $x^{++} \wedge x^{+} = 0$.*

Definition 2.6. [16] *A double Stone algebra is an algebra $(L; ^{*}, ^{+})$ such that $(L; ^{*})$ is a Stone algebra, $(L; ^{+})$ is a dual Stone algebra and for every $x \in L, x^{*+} = x^{**}, x^{+*} = x^{++}$.*

Definition 2.7. [8] *An MS-algebra is an algebra $(L; \vee, \wedge, ^{\circ}, 0, 1)$ of type (2,2,1,0,0) where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation $^{\circ}$ satisfies:*

$$x \leq x^{\circ\circ}, (x \wedge y)^{\circ} = x^{\circ} \vee y^{\circ}, 1^{\circ} = 0.$$

Definition 2.8. [10] *A dual MS-algebra is an algebra $(L; \vee, \wedge, ^{+}, 0, 1)$ of type (2,2,1,0,0) where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation $^{+}$ satisfies:*

$$x \geq x^{++}, (x \wedge y)^{+} = x^{+} \vee y^{+}, 0^{+} = 1.$$

Definition 2.9. [10] *A double MS-algebra is an algebra* $(L;^{\circ},^{+})$ *such that* $(L;^{\circ})$ *is an MS-algebra,* $(L;^{+})$ *is a dual MS-algebra, and the unary operations* $^{\circ},^{+}$ *are linked by the identities* $x^{+\circ} = x^{++}$ *and* $x^{\circ +} = x^{\circ\circ}$*, for all* $x \in L$.

The class **DS** of all double Stone algebras is a subclass of the class **DMS** of all double *MS*-algebras and is characterized by the identities $x \wedge x^{\circ} = 0$ and $x \vee x^{+} = 1$.

Throughout this paper, we adopt the following rules of computation in a double *MS*-algebra $(L; \vee, \wedge,^{\circ},^{+}, 0, 1)$ (see [8] and [10]).

Theorem 2.10. *For any two elements* a, b *of a double MS-algebra L, we have*

(1) $0^{\circ\circ} = 0$ *and* $1^{\circ\circ} = 1$	(1_d) $0^{++} = 0$ *and* $1^{++} = 1$,
(2) $a \leq b \Rightarrow b^{\circ} \leq a^{\circ}$	(2_d) $a \leq b \Rightarrow b^{+} \leq a^{+}$,
(3) $a^{\circ\circ\circ} = a^{\circ}$	(3_d) $a^{+++} = a^{+}$,
(4) $a^{\circ\circ\circ\circ} = a^{\circ\circ}$	(4_d) $a^{++++} = a^{++}$,
(5) $(a \vee b)^{\circ} = a^{\circ} \wedge b^{\circ}$	(5_d) $(a \vee b)^{+} = a^{+} \wedge b^{+}$,
(6) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$	(6_d) $(a \vee b)^{++} = a^{++} \vee b^{++}$,
(7) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$	(7_d) $(a \wedge b)^{++} = a^{++} \wedge b^{++}$.

Theorem 2.11. [9] *Let* $(L; \vee, \wedge,^{\circ},^{+}, 0, 1)$ *be a double MS-algebra. Then*

(1) $L^{\circ\circ} = \{x \in L : x = x^{\circ\circ}\} = \{x \in L : x = x^{++}\} = \{x \in L : x^{\circ} = x^{+}\}$ *is a de Morgan subalgebra of L,*

(2) $L^{\vee} = \{x \vee x^{\circ} : x \in L\} = \{x \in L : x \geq x^{\circ}\}$ *is an increasing subset (dual order ideal) of L,*

(3) $L^{\circ\circ\vee} = \{a \vee a^{\circ} : a \in L^{\circ\circ}\} = L^{\circ\circ} \cap L^{\vee}$.

Definition 2.12. [15] *Let* $L = (L; \vee, \wedge, 0, 1)$ *and* $L_1 = (L_1; \vee, \wedge, 0, 1)$ *be bounded lattices. A mapping* $f : L \rightarrow L_1$ *is called a* $\{0, 1\}$*-lattice homomorphism if* $f(0) = 0$, $f(1) = 1$, $f(x \vee y) = f(x) \vee f(y)$ *and* $f(x \wedge y) = f(x) \wedge f(y)$ *for all* $x, y \in L$. *A* $\{0, 1\}$*-lattice homomorphism is called an isomorphism if f is a bijective mapping, in this case, we call L and* L_1 *are isomorphic lattices and write* $L \cong L_1$.

3. Core regular double *MS*-algebras

In this section, we introduce the concept of core regular double *MS*-algebras that includes the class of core regular double Stone algebras.

Definition 3.1. [2] *A double MS-algebra* $(L;^{\circ},^{+})$ *is said to be a regular double MS-algebra (or simply RDMS-algebra) if for any* $x, y \in L$, $x^{\circ} = y^{\circ}$ *and* $x^{+} = y^{+}$ *imply* $x = y$.

A relation $\Phi_{\circ}^{+}$ defined by $(x, y) \in \Phi_{\circ}^{+} \Leftrightarrow x^{\circ} = y^{\circ}$ and $x^{+} = y^{+}$ is a congruence relation on a double *MS*-algebra L.

A characterization of regular double *MS*-algebra in terms of the congruence $\Phi_{\circ}^{+}$ is given in the following.

Theorem 3.2. *Let L be a double MS-algebra. Then L is regular if and only if* $\Phi_{\circ}^{+} = \omega$*, where* $\omega = \{(x, x) : x \in L\}$.

Proof. Let L be a regular double *MS*-algebra. Let $(x, y) \in \Phi_{\circ}^{+}$. Then $x^{\circ} = y^{\circ}$ and $x^{+} = y^{+}$ and hence by regularity of L, we get $x = y$. Therefore $\Phi_{\circ}^{+} = \omega$. Conversely, let $\Phi_{\circ}^{+} = \omega$. Let $x^{\circ} = y^{\circ}$ and $x^{+} = y^{+}$. Then $(x, y) \in \omega$. So, $x = y$ and L is regular. □

Definition 3.3. [1] *Let L be an MS-algebra. An element* $d \in L$ *is called a dense element of L if* $d^{\circ} = 0$*, the set of all dense elements of L is denoted by* $D(L)$.

Definition 3.4. *Let L be a dual MS-algebra. An element $d \in L$ is called a dual dense element of L if $d^+ = 1$, the set of all dual dense elements of L is denoted by $\overline{D(L)}$.*

Lemma 3.5. *Let L be a double MS-algebra. Then $D(L)$ is a filter of L and $\overline{D(L)}$ is an ideal of L.*

Proof. It is observed that $D(L)$ is a filter of L (see [1]). Let $x, y \in \overline{D(L)}$. Then $x^+ = y^+ = 1$. So by Theorem 2.10(5_d), $(x \vee y)^+ = x^+ \wedge y^+ = 1$. Hence $x \vee y \in \overline{D(L)}$. Now, let $z \leq x \in \overline{D(L)}$ and $z \in L$. Then by Theorem 2.10(2_d), $z^+ \geq x^+ = 1$. This means that $z \in \overline{D(L)}$. Therefore $\overline{D(L)}$ is an ideal of L. □

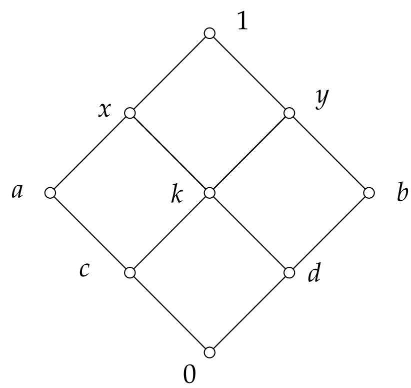
Definition 3.6. *Let L be a double MS-algebra. The set $K(L) = D(L) \cap \overline{D(L)}$ is called the core of L.*

Definition 3.7. *A core regular double MS-algebra (briefly CRDMS-algebras) is a regular double MS-algebra with non empty core, that is, $K(L) \neq \phi$.*

Lemma 3.8. *Let L be a CRDMS-algebra. Then $|K(L)| = 1$.*

Proof. Let $k_1, k_2 \in K(L)$. Then $k_1^\circ = k_2^\circ = 0$ and $k_1^+ = k_2^+ = 1$. Hence by regularity of L, $k_1 = k_2$. Therefore $K(L)$ has a unique element and hence $|K(L)| = 1$. □

We will denote the core element by k. The core element k will play an important role throughout the rest of this paper.



L

Figure 1: L is a bounded distributive lattice.

Example 3.9.

(1) Every core regular double Stone algebra is a core regular double MS-algebra.

(2) Consider the bounded distributive lattice L in Figure 1. Define unary operations $^\circ, ^+$ on L by

$$k^\circ = x^\circ = y^\circ = 1^\circ = 0, d^\circ = b^\circ = b, c^\circ = a^\circ = a, 1^\circ = 0$$

and

$$k^+ = c^+ = d^+ = 0^+ = 1, y^+ = b^+ = b, x^+ = a^+ = a, 1^+ = 0.$$

It is observed that $(L; ^\circ, ^+)$ is a regular double MS-algebra. We have $D(L) = \{k, x, y, 1\}$, $\overline{D(L)} = \{0, c, d, k\}$ and $K(L) = \{k\}$. Then L represents a CRDMS-algebra. Since $c^\circ \wedge c \neq 0$ and $c^+ \vee c \neq 1$ then L is not a double Stone algebra. This example deduce that **CRDS** $\subsetneq$ **CRDMS**.

(3) Consider the bounded distributive lattice L in Figure 1. Define unary operations $^\circ, ^+$ on L by

$$x^\circ = 1^\circ = 0, k^\circ = y^\circ = c, d^\circ = b^\circ = a, a^\circ = b, c^\circ = y, 1^\circ = 0$$

and

$$d^+ = 0^+ = 1, k^+ = c^+ = y, x^+ = a^+ = b, b^+ = a, y^+ = c, 1^+ = 0.$$

Clearly $(L;^\circ,^+)$ *is a regular double MS-algebra. We have* $D(L) = \{x, 1\}$, $\overline{D(L)} = \{0, d\}$ *and* $K(L) = \phi$*. Then L is not a core regular double MS-algebra.*

Definition 3.10. [2] *A double MS-algebra L is called a double MS-algebra satisfying the generalized complement property (or briefly* DMS^{gc}*-algebra) if*

(1) L is a regular double MS-algebra,

(2) Given $a, b \in L^{\circ\circ}$ *and a filter F of* $L^{\circ\circ}$ *containing* $L^{\circ\circ\vee}$ *such that* $a \leq b$ *and* $a \vee b^\circ \in F$*, then there exists an element* $x \in L$ *such that* $x^{++} = a$ *and* $x^{\circ\circ} = b$.

Lemma 3.11. *Every CRDMS-algebra with core element k is a* DMS^{gc}*-algebra.*

Proof. We can choose $F = L^{\circ\circ}$. Let $a, b \in L^{\circ\circ}$ be such that $a \leq b$. Clearly $a \vee b^\circ \in F$ as $F = L^{\circ\circ}$. Set $x = (a \vee k) \wedge b$. Then $x^{++} = a$ and $x^{\circ\circ} = b$. Then condition (ii) of Definition 3.9 holds. Then L is a DMS^{gc}-algebra. □

Now we illustrate an example to show that the converse of the above Lemma is not true, that is, the class **CRDMS** of all core regular double *MS*-algebras is a proper subclass of the class of $\mathbf{DMS^{gc}}$ of all DMS^{gc}-algebras.

Example 3.12. *Consider* $L = \{0 < c < a < d < 1\}$ *be a five element chain and* $a = a^\circ = c^\circ = a^+ = d^+, d^\circ = 1^\circ = 0, 0^+ = c^+ = 1$*. It is clear that* $(L;^\circ,^+)$ *is a regular double MS-algebra,* $L^{\circ\circ} = \{0, a, 1\}$ *and* $L^{\circ\circ\vee} = \{a, 1\}$*. A filter* $F = \{a, 1\}$ *of* $L^{\circ\circ}$ *contains* $L^{\circ\circ\vee}$*. It is observed that* $(L,^\circ,^+)$ *is a* DMS^{gc}*-algebra. Since* $D(L) = \{1, d\}$ *and* $\overline{D(L)} = \{0, c\}$ *then* $K(L) = D(L) \cap \overline{D(L)} = \phi$*. Then L is not a CRDMS-algebra.*

4. The construction

The construction of a core regular double *MS*-algebra from a suitable de Morgan algebra is given in the following.

Theorem 4.1. (**Construction Theorem**)

Let $(M; \vee, \wedge, ^-, 0, 1)$ *be a de Morgan algebra. Then*

$$M^{[2]} = \{(a, b) \in M \times M : a \leq b\}$$

is a core regular double CRDMS-algebra with core element $(0, 1)$*, whenever*

$$\begin{aligned}
(a, b) \vee (c, d) &= (a \vee c, b \vee d), \\
(a, b) \wedge (c, d) &= (a \wedge c, b \wedge d), \\
(a, b)^+ &= (\bar{a}, \bar{a}), \\
(a, b)^\circ &= (\bar{b}, \bar{b}) \\
0_{M^{[2]}} &= (0, 0) \\
1_{M^{[2]}} &= (1, 1).
\end{aligned}$$

Moreover, M is isomorphic to $D(M^{[2]})$ *as well as* $\overline{D(M^{[2]})}$ *as lattices.*

Proof. T.S. Blyth and J.c. Varlet [10] observed that $M^{[2]} = (M^{[2]}; \vee, \wedge, ^\circ, ^+, (0,0), (1,1))$ is a double MS-algebra. Let $(a,b)^\circ = (c,d)^\circ$ and $(a,b)^+ = (c,d)^+$. Then $(\bar{b},\bar{b}) = (\bar{d},\bar{d})$ and $(\bar{a},\bar{a}) = (\bar{c},\bar{c})$ imply $a = c$ and $b = d$. Thus $(a,b) = (c,d)$. Therefore $M^{[2]}$ is a regular double MS-algebra. By Theorem 3.5, $D(M^{[2]})$ is a filter of $M^{[2]}$ and $\overline{D(M^{[2]})}$ is an ideal of $M^{[2]}$. We observe that

$$\begin{aligned} D(M^{[2]}) &= \{(x,y) \in M^{[2]} : (x,y)^\circ = (0,0)\} \\ &= \{(x,y) \in M^{[2]} : (\bar{y},\bar{y}) = (0,0)\} \\ &= \{(x,y) \in M^{[2]} : \bar{y} = 0\} \\ &= \{(x,y) \in M^{[2]} : y = 1\} \\ &= \{(x,1) \in M^{[2]} : x \in M\}, \end{aligned}$$

and

$$\begin{aligned} \overline{D(M^{[2]})} &= \{(x,y) \in M^{[2]} : (x,y)^+ = (1,1)\} \\ &= \{(x,y) \in M^{[2]} : (\bar{x},\bar{x}) = (1,1)\} \\ &= \{(x,y) \in M^{[2]} : \bar{x} = 1\} \\ &= \{(x,y) \in M^{[2]} : x = 0\} \\ &= \{(0,y) \in M^{[2]} : y \in M\}. \end{aligned}$$

Now, we prove that the element $(0,1)$ is the core element of $M^{[2]}$. Since $(0,1)^\circ = (0,0)$, then $(0,1) \in D(L)$. We claim that $D(L)$ is a principal filter of $M^{[2]}$ generated by $(0,1)$. Let $(x,1)$ be any element of $D(L)$. Then $x \geq 0$ implies $(x,1) \geq (0,1)$. Therefore $(0,1)$ is the smallest element of $D(L)$ and $D(L) = [(0,1))$. Similarly, we can prove that $\overline{D(L)}$ is a principal ideal of $M^{[2]}$ generated by $(0,1)$. Thus $\overline{D(L)} = ((0,1)]$. Consequently, the core of $M^{[2]}$ is $K(M^{[2]}) = D(M^{[2]}) \cap \overline{D(M^{[2]})} = [(0,1)) \cap ((0,1)] = \{(0,1)\}$. To prove that the lattices M and $D(M^{[2]})$ are isomorphic, define a map $f : M \to D(M^{[2]})$ by $f(a) = (a,1)$. Clearly $f(0) = (0,1)$ and $f(1) = (1,1)$. For every $a, b \in M$, we have

$$f(a \wedge b) = (a \wedge b, 1) = (a,1) \wedge (b,1) = f(a) \wedge f(b).$$

Also, $f(a \vee b) = f(a) \vee f(b)$. Therefore f is a $\{0,1\}$-lattice homomorphism. Obviously f is a bijective map. Therefore f is an isomorphism and $M \cong D(M^{[2]})$. Similarly, we can deduce that $M \cong \overline{D(M^{[2]})}$ under the lattice isomorphism $a \mapsto (0,a)$. Therefore $D(M^{[2]})$ and $\overline{D(M^{[2]})}$ are also isomorphic lattices. □

We illustrate the above construction on the following example.

Example 4.2. Let M be the four-element de Morgan algebra (see Fig. 2).

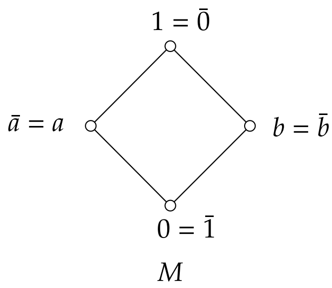


Figure 2: M is a de Morgan algebra.

Using the construction Theorem (theorem 4.1), we obtain a core regular double MS-algebra $M^{[2]}$ in figure 3.
Where $^\circ$ and $^+$ are given as follows:

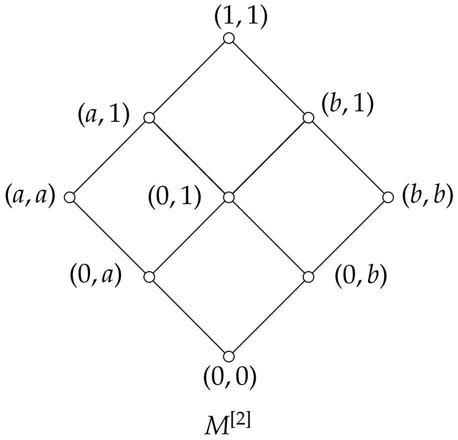


Figure 3: $M^{[2]}$ is a *CRDMS*-algebra with core $(0, 1)$.

$(0,b)^\circ = (b,b)^\circ = (b,b), (0,1)^\circ = (b,1)^\circ = (a,1)^\circ = (1,1)^\circ = (0,0), (0,a)^\circ = (a,a)^\circ = (a,a), (0,0)^\circ = (1,1)$ and $(0,k)^+ = (0,a)^+ = (0,b)^+ = (0,0)^+ = (1,1), (a,a)^+ = (a,1)^+ = (a,a), (b,b)^+ = (b,1)^+ = (a,a), (1,1)^+ = (0,0)$. Clearly, $(M^{[2]})^{\circ\circ} = \{(0,0),(a,a),(b,b),(1,1)\}$ is isomorphic to M under a map $(a,a) \mapsto a$ and $D(M^{[2]}) = \{(0,1),(a,1),(b,1),(1,1)\}$ is isomorphic to M under a map $(x,1) \mapsto x$.

Definition 4.3. *A mapping $f : M \to M_1$ of a de Morgan algebra M into a de Morgan algebra M_1 is said to be a homomorphism if f is a $\{0,1\}$-lattice homomorphism satisfying $f(\overline{x}) = \overline{(f(x))}$. A bijective homomorphism of de Morgan algebras is called isomorphism.*

Corollary 4.4. *M is isomorphic to $(M^{[2]})^{\circ\circ}$ as de Morgan algebras.*

Proof. It is known that $((M^{[2]})^{\circ\circ}, \vee, \wedge, ^\circ, (0,0), (1,1))$ is a de Morgan subalgebra of $M^{[2]}$ (by Theorem 2.11(1)). Let $(a,b) \in (M^{[2]})^{\circ\circ}$. Then $(a,b)^{\circ\circ} = (a,b)$ implies $(b,b) = (a,b)$. Hence $a = b$. Therefore

$$(M^{[2]})^{\circ\circ} = \{(a,a) : a \in M\}.$$

Then clearly a map $a \mapsto (a,a)$ is an isomorphism of M onto $(M^{[2]})^{\circ\circ}$. Consequently, $M \cong (M^{[2]})^{\circ\circ}$. □

For a core regular double Stone algebra, we have.

Corollary 4.5. *If $B = (B; \vee, \wedge, ', 0, 1)$ is a Boolean algebra, then $B^{[2]}$ is a core regular double Stone algebra and $(B^{[2]})^{\circ\circ}$ is a Boolean subalgebra of $B^{[2]}$, where $'$ is a unary operation of complementation on B.*

Proof. For any element x of a Boolean algebra B, we have the facts $x \vee x' = 1$ and $x \wedge x' = 0$. Since each Boolean algebra is a de Morgan algebra, then according to the above Theorem 4.1, $B^{[2]} = \{(a,b) : a \le b\}$ is a core regular double *MS*-algebra with core element $(0,1)$. We prove that $(a,b) \wedge (a,b)^\circ = (0,0)$ and $(a,b) \vee (a,b)^+ = (1,1)$ for all $(a,b) \in B^{[2]}$.

$$\begin{aligned}
(a,b) \wedge (a,b)^\circ &= (a,b) \vee (b',b') \\
&= (a \wedge b', b \wedge b') \\
&= (a \wedge b', 0) \in B^{[2]} \text{ as } b \wedge b' = 0 \\
&= (0,0) \text{ as } a \wedge b' \le 0 \Rightarrow a \wedge b' = 0 \\
(a,b) \vee (a,b)^+ &= (a,b) \vee (a',a') \\
&= (a \vee a', b \vee a') \\
&= (1, b \vee a') \in B^{[2]} \text{ as } a \vee a' = 1 \\
&= (1,1) \text{ as } 1 \le b \vee a' \Rightarrow b \vee a' = 1.
\end{aligned}$$

Therefor $B^{[2]}$ is a core double Stone algebra. By Theorem 2.11(1), $(B^{[2]})^{\circ\circ}$ is a de Morgan subalgebra of $B^{[2]}$. From corollary 4.2, $(B^{[2]})^{\circ\circ} = \{(a,a) : a \in B\}$. Since $(a,a) \vee (a,a)^{\circ} = (1,1)$ and $(a,a) \wedge (a,a)^{\circ} = (0,0)$ for all $(a,a) \in (B^{[2]})^{\circ\circ}$, then $(B^{[2]})^{\circ\circ}$ is a Boolean subalgebra of $B^{[2]}$. □

Definition 4.6. *A mapping $f : L \to L_1$ of a CRDMS-algebra L with core element k into a CRDMS-algebra L_1 with core element k_1 is called a homomorphism if*

(1) f is a $\{0,1\}$-lattice homomorphism,

(2) $f(k) = k_1$, $f(x^{\circ}) = (f(x))^{\circ}$ and $f(x^{+}) = (f(x))^{+}$.
A bijective homomorphism of CRDMS-algebras is called isomorphism.

Theorem 4.7. *A CRDMS-algebra L with core element k is isomorphic to $L^{\circ\circ[2]}$.*

Proof. Since $L^{\circ\circ}$ is a de Morgan algebra, then by Theorem 4.1, $L^{\circ\circ[2]} = \{(a,b) \in L^{\circ\circ} \times L^{\circ\circ} : a \leq b\}$ is a *CRDMS*-algebra with core element $(0,1)$. Define $\varphi : L \to L^{\circ\circ[2]}$ by $\varphi(x) = (x^{++}, x^{\circ\circ})$. Since $x^{++} \leq x^{\circ\circ}$, then $\varphi(x) \in L^{\circ\circ[2]}$. To prove that φ is an injective map, let $\varphi(x) = \varphi(y)$. Then $(x^{++}, x^{\circ\circ}) = (y^{++}, y^{\circ\circ})$. Hence $x^{++} = x^{++}$ and $x^{\circ\circ} = x^{\circ\circ}$. Then by Theorem 2.10($3_d$),(3), we have $x^{+} = x^{+++} = y^{+++} = y^{+}$ and $x^{\circ} = x^{\circ\circ\circ} = y^{\circ\circ\circ} = y^{\circ}$. By regularity of L, $x = y$. Now, we prove that φ is surjective. For all $(a,b) \in L^{\circ\circ[2]}$, we have $a \leq b$ and $a, b \in L^{\circ\circ}$. Set $d = (a \vee k) \wedge b$. Using (6),(6_d),(7)and (7_d) of Theorem 2.10, and k^{+}=1, $k^{\circ} = 0$, we have

$$d^{++} = ((a \vee k) \wedge b)^{++} = (a^{++} \vee k^{++}) \wedge b^{++} = (a \vee 0) \wedge b = a \wedge b = a,$$

and

$$d^{\circ\circ} = ((a \vee k) \wedge b)^{\circ\circ} = (a^{\circ\circ} \vee k^{\circ\circ}) \wedge b^{\circ\circ} = (a \vee 1) \wedge b = 1 \wedge b = b.$$

Thus $\varphi(d) = (d^{++}, d^{\circ\circ}) = (a,b)$. Therefore φ is a bijective mapping. Clearly, $\varphi(0) = (0,0)$, $\varphi(1) = (1,1)$ and $\varphi(k) = (0,1)$. For all $x, y \in L$, we get

$$\begin{aligned}
\varphi(x \wedge y) &= ((x \wedge y)^{++}, (x \wedge y)^{\circ\circ}) \\
&= (x^{++} \wedge y^{++}, x^{\circ\circ} \wedge y^{\circ\circ}) \text{ by Theorem 2.10(7),(}7_d\text{)} \\
&= (x^{++}, x^{\circ\circ}) \wedge (y^{++}, y^{\circ\circ}) \\
&= \varphi(x) \wedge \varphi(y,) \\
\varphi(x \vee y) &= ((x \vee y)^{++}, (x \vee y)^{\circ\circ}) \\
&= (x^{++} \vee x^{++}, x^{\circ\circ} \vee y^{\circ\circ}) \text{ by Theorem 2.10(6),(}6_d\text{)} \\
&= (x^{++}, x^{\circ\circ}) \vee (y^{++}, y^{\circ\circ}) \\
&= \varphi(x) \vee \varphi(y).
\end{aligned}$$

Therefore φ is a $\{0,1\}$-lattice homomorphism. Now, for all $x \in L$ we have

$$\begin{aligned}
\varphi(x^{+}) &= (x^{+++}, x^{+\circ\circ}) \\
&= (x^{+++}, x^{+++}) \text{ as } x^{+\circ} = x^{++} \\
&= (x^{++}, x^{\circ\circ})^{+} \\
&= (\varphi(x))^{+}, \\
\varphi(x^{\circ}) &= (x^{\circ++}, x^{\circ\circ\circ}) \\
&= (x^{\circ\circ\circ}, x^{\circ\circ\circ}) \text{ as } x^{\circ+} = x^{\circ\circ} \\
&= (x^{++}, x^{\circ\circ})^{\circ} \\
&= (\varphi(x))^{\circ}.
\end{aligned}$$

Then φ preserves $^{+}$ and $^{\circ}$. Consequently, φ is an isomorphism of a *CRDMS*-algebra L onto a *CRDMS*-algebra $L^{\circ\circ[2]}$. So $L \cong L^{\circ\circ[2]}$. □

From the above discussion, we immediately get the following important result.

Theorem 4.8. *There is a one to one correspondence between the class of core regular double MS-algebras and the class of de Morgan algebras.*

Now, we give another useful characterization of a core regular double *MS*-algebra.

Theorem 4.9. *Let L be a RDMS-algebra. Then the following statements are equivalent.*

(i) L has core element,

(ii) For $a, b \in L^{\circ\circ}$ and $a \leq b$, there exists an element $x \in L$ such that $x^{++} = a$ and $x^{\circ\circ} = b$.

Proof. $(i) \Rightarrow (ii)$: Let L has core element k. Let $a \leq b$, $a, b \in L^{\circ\circ}$. Set $x = (a \vee k) \wedge b$. It is clear that $x^{++} = a$ and $x^{\circ\circ} = b$. Then condition (ii) holds.
$(ii) \Rightarrow (i)$: Let L be a regular double *MS*-algebra satisfying the condition (ii). Then by Theorem 4.1, $L^{\circ\circ[2]} = \{(a, b) \in L^{\circ\circ} \times L^{\circ\circ} : a \leq b\}$ is a core regular double *MS*-algebra with core element $(0, 1)$. Define a map $\varphi : L \to L^{\circ\circ[2]}$ by $\varphi(x) = (x^{++}, x^{\circ\circ})$. In the proof of Theorem 4.7, we show that φ is an injective mapping of L into $L^{\circ\circ[2]}$. Now we show that φ is a surjective mapping using (ii). Let $(a, b) \in L^{\circ\circ[2]}$. Then $a \leq b$ and $a, b \in L^{\circ\circ}$. By (ii) there exists $x \in L$ such that $x^{++} = a$ and $x^{\circ\circ} = b$. Then $\varphi(x) = (x^{++}, x^{\circ\circ}) = (a, b)$. Therefore φ is a bijective mapping of L onto $L^{\circ\circ[2]}$. We claim that the inverse image of the core element $(0, 1)$ of $L^{\circ\circ[2]}$ is the core element of L. Suppose that $d = \varphi^{-1}(0, 1)$. Then $\varphi(d) = (0, 1)$ implies $(d^{++}, d^{\circ\circ}) = (0, 1)$. Thus $d^{++} = 0$ and $d^{\circ\circ} = 1$. It follows that $d^{+} = 1$ and $d^{\circ} = 0$. This deduce that d is the core element of L. □

Now, for any de Morgan algebra $M = (M; \vee, \wedge, ^{-}, 0, 1)$ and any filter F of M containing $M^{\vee}$, the author proved in [2] that $(L; \vee, \wedge, ^{\circ}, ^{+}, (0, 0), (1, 1))$ forms a DMS^{gc}-algebra, where

$$L = (M, F) = \{(a, b) : a \leq b, a \vee \bar{b} \in F\}$$

and the operations $\vee, \wedge, ^{\circ}$ and $^{+}$ are given as in Theorem 4.1.

The following result gives the necessary and sufficient condition for a DMS^{gc}-algebra $L = (M, F)$ to become a core regular double *MS*-algebra.

Theorem 4.10. *A DMS^{gc}-algebra $L = (M, F)$ is a CRDMS-algebra iff $F = M$.*

Proof. Let $F = M$. Then $L = (M, M) = M^{[2]}$. Thus by Theorem 4.1, $L = M^{[2]}$ is a core regular double *MS*-algebra with core element (0,1). Conversely, Let $L = (M, F)$ is a core regular double *MS*-algebra with core element (a, b). Then $(a, b) \in D(L) \cap \overline{D(L)}$ and $a \vee \bar{b} \in F$. Hence $(a, b)^{\circ} = (0, 0)$ and $(a, b)^{+} = (1, 1)$. It follows that $(\bar{b}, \bar{b}) = (0, 0)$ and $(\bar{a}, \bar{a}) = (1, 1)$, respectively. Then $\bar{b} = 0$ and $\bar{a} = 1$ implies $b = 1$ and $a = 0$, respectively. Then $(a, b) = (0, 1)$ and hence $0 = 0 \vee \bar{1} = a \vee \bar{b} \in F$. Therefore $F = M$. □

5. Applications of the construction Theorem

We start this section with subalgebras of a *CRDMS*-algebra.

Definition 5.1. *A bounded sublattice H of a CRDMS-algebra L with core element k is said to be a subalgebra of L if*

(1) $x^{\circ}, x^{+} \in H$ for all $x \in H$,

(2) $k \in H$.

It is observed that $\{0, k, 1\}$ is the smallest subalgebra of any *CRDMS*-algebra L.

The subalgebras of a *CRDMS*-algebra L in example 3.9(2) are $\{0, k, 1\}$, $\{0, c, a, k, x, 1\}$, $\{0, d, b, k, y, 1\}$ and L.

Theorem 5.2. *There is one to one correspondence between the set of all subalgebras of a de Morgan algebra M and the set of all subalgebras of a CRDMS-algebra $M^{[2]}$.*

Proof. Let M_1 be a subalgebra of M. We prove that a set $M_1^{[2]} = \{(a,b) \in M_1 \times M_1 : a \leq b\}$ is a subalgebra of $M^{[2]}$. Since $0, 1 \in M_1$, then $(0,0)$, $(1,1)$ and $(0,1)$ are belong to $M_1^{[2]}$. For every $(a,b), (c,d) \in M_1^{[2]}$. Then $a, b, c, d \in M_1$ and hence $a \vee c, b \vee d, a \wedge c, b \wedge d \in M_1$. Thus

$$\begin{aligned}(a,b) \vee (c,d) &= (a \vee c, b \vee d) \in M_1^{[2]} \text{ as } a \vee c \leq b \vee d,\\ (a,b) \wedge (c,d) &= (a \wedge c, b \wedge d) \in M_1^{[2]} \text{ as } a \wedge c \leq b \wedge d.\end{aligned}$$

Therefore $M_1^{[2]}$ is a bounded sublattice of $M^{[2]}$. Let $(a,b) \in M_1^{[2]}$. Then $a, b \in M_1$ and hence $\bar{a}, \bar{b} \in M_1$ (as M_1 is a subalgebra of M). Thus

$$\begin{aligned}(a,b)^+ &= (\bar{a}, \bar{a}) \in M_1^{[2]},\\ (a,b)^\circ &= (\bar{b}, \bar{b}) \in M_1^{[2]}.\end{aligned}$$

The core element $(0,1)$ of $M^{[2]}$ belongs to $M_1^{[2]}$. Therefore $M_1^{[2]}$ is a subalgebra of $M^{[2]}$.
Conversely, let L_1 be a subalgebra of $M^{[2]}$. Consider a subset M_1 of M as follows:

$$M_1 = \{a \in M : (a,a) \in L_1\}.$$

We claim that M_1 is a subalgebra of M. Since $(0,0)$, $(1,1) \in L_1$, then $0, 1 \in M_1$. Let $x, y \in M_1$. Hence $(x,x), (y,y) \in L_1$. Now

$$\begin{aligned}(x,x) \wedge (y,y) &= (x \wedge y, x \wedge y) \in L_1 \Rightarrow x \wedge y \in M_1,\\ (x,x) \vee (y,y) &= (x \vee y, x \vee y) \in L_1 \Rightarrow x \vee y \in M_1,\\ (x,x)^\circ &= (\bar{x}, \bar{x}) \in L_1 \Rightarrow \bar{x} \in M_1.\end{aligned}$$

Therefore M_1 is a subalgebra of a de Morgan algebra M. □

A clarification of the correspondence between subalgebras of a de Morgan algebra M and a *CRDMS*-algebra $M^{[2]}$ is provided in the following example.

Example 5.3. *Consider a de Morgan algebra M and a CRDMD-algebra $M^{[2]}$ in example 4.2. We observe that the subalgebras $M_1 = \{0, 1\}, M_2 = \{0, a, 1\}, M_3 = \{0, b, 1\}, M_4 = M$ of a de Morgan algebra M are corresponding to the subalgebras $M_1^{[2]} = \{(0,0), (0,1), (1,1)\}$,*
$M_2^{[2]} = \{(0,0), (0,a), (0,1), (a,a), (a,1), (1,1)\}$, $M_3^{[2]} = \{(0,0), (0,b), (b,b)(0,1), (b,1), (1,1)\}$,
$M_4^{[2]} = M^{[2]}$ of a CRDMS-algebra $M^{[2]}$, respectively.

Definition 5.4. *A subalgebra L_1 of a CRDMS-algebra L is said to be a Stone subalgebra if $x^\circ \vee x^{\circ\circ} = 1$ and $x^+ \wedge x^{++} = 0$ for all $x \in L_1$.*

Corollary 5.5. *There is one to one correspondence between the set of all Boolean subalgebras of a de Morgan algebra M and the set of all Stone subalgebras of the CRDMS-algebra $M^{[2]}$.*

Proof. Let M_1 is a Boolean subalgebra of a de Morgan algebra M. Then $x \wedge \bar{x} = 0$ and $x \vee \bar{x} = 1$ for all $x \in M_1$. Theorem 5.2 shows that $M_1^{[2]}$ is a subalgebra of $M^{[2]}$. We need to prove that the Stone identities hold in $M_1^{[2]}$. For all $(x,y) \in M_1^{[2]}$, we get

$$\begin{aligned}(x,y)^+ \wedge (x,y)^{++} &= (\bar{x}, \bar{x}) \wedge (x,x) = (\bar{x} \wedge x, \bar{x} \wedge x) = (0,0)\\ (x,y)^\circ \vee (x,y)^{\circ\circ} &= (\bar{y}, \bar{y}) \vee (y,y) = (\bar{y} \vee y, \bar{y} \vee y) = (1,1).\end{aligned}$$

Conversely, let L_1 is a Stone subalgebra of $M^{[2]}$. Then by Theorem 5.2, $M_1 = \{a \in M : (a,a) \in L_1\}$ is a subalgebra of a de Morgan algebra M. To prove M_1 is a Boolean subalgebra of M, we have to show that $a \vee \bar{a} = 1$ and $a \wedge \bar{a} = 0$ for $a \in M_1$. Let $a \in M_1$. Then $(a,a) \in L_1$. Since L_1 is a Stone subalgebra of $M^{[2]}$ then $(1,1) = (a,a)^\circ \vee (a,a)^{\circ\circ} = (\bar{a} \vee a, \bar{a} \vee a)$. Therefore $a \vee \bar{a} = 1$. Also, $(0,0) = (a,a)^\circ \wedge (a,a)^{\circ\circ} = (\bar{a} \wedge a, \bar{a} \wedge a)$ implies that $\bar{a} \wedge a = 0$. □

It is known that the center $Z(M) = \{x \in M : x \vee \bar{x} = 1\}$ of a de Morgan algebra M forms a Boolean subalgebra of M.

Corollary 5.6. $(Z(M))^{[2]}$ *is the greatest Stone subalgebra of* $M^{[2]}$.

Example 5.7. *Consider a de Morgan algebra M and a CRDMD-algebra $M^{[2]}$ in example 4.2. The center $Z(M) = \{0, 1\}$ of M correspond to the greatest Stone subalgebra $M_1^{[2]} = \{(0, 0), (0, 1), (1, 1)\}$ of a CRDMS-algebr $M^{[2]}$.*

Let $h : L \to L_1$ be a homomorphism of a *CRDMS*-algebra L into a *CRDMS*-algebra L_1. We will denote by $h_{L^{\circ\circ}}$, $h_{D(L)}$ and $h_{\overline{D(L)}}$ to the restrictions of h on $L^{\circ\circ}$, $D(L)$ and $\overline{D(L)}$, respectively. It is easy to show the following.

Lemma 5.8. *Let $h : L \to L_1$ be a homomorphism of a CRDMS-algebra L into a CRDMS-algebra L_1. Then*

(1) $h_{L^{\circ\circ}}$ is a homomorphism of a de Morgan algebras $L^{\circ\circ}$ into a de Morgan algebra $L_1^{\circ\circ}$,

(2) $h_{D(L)}$ is a $\{0, 1\}$-lattice homomorphism of a lattice $D(L)$ into a lattice $D(L_1)$,

(3) $h_{\overline{D(L)}}$ is a $\{0, 1\}$-lattice homomorphism of a lattice $\overline{D(L)}$ into a lattice $\overline{D(L_1)}$.

Theorem 5.9. *Let M and M_1 be de Morgan algebras. If $f : M \to M_1$ is a homomorphism, then a map $h : M^{[2]} \to M_1^{[2]}$ defined by $h(a, b) = (f(a), f(b))$ is a homomorphism of a CRDMS-algebra $M^{[2]}$ into a CRDMS-algebra $M_1^{[2]}$. Conversely, if $h : M^{[2]} \to M_1^{[2]}$ is a homomorphism of CRDMS-algebras, then $f : M \to M_1$ defined by $f(a) = b \Leftrightarrow h_{(M^{[2]})^{\circ\circ}}(a, a) = (b, b)$ for all $a \in M$ is homomorphism of de Morgan algebras.*

Proof. Let $f : M \to M_1$ be a homomorphism between de Morgan algebras M and M_1. It is ready seen that a map $h : M^{[2]} \to M_1^{[2]}$ defined by $h(a, b) = (f(a), f(b))$ is a homomorphism of a *DMS*-algebra $M^{[2]}$ into a *DMS*-algebra $M_1^{[2]}$. Since $h(0, 1) = (f(0), f(1)) = (0, 1)$, then h is a homomorphism of *CRDMS*-algebra $M^{[2]}$ into a *CRDMS*-algebra $M_2^{[2]}$.
Conversely, let $h : M^{[2]} \to M_1^{[2]}$ be a homomorphism of $M^{[2]}$ into $M_1^{[2]}$. Define a map $f : M \to M_1$ as follows:

$$f(a) = b \Leftrightarrow h_{(M^{[2]})^{\circ\circ}}(a, a) = h(a, a) = (b, b) \text{ for all } a \in M.$$

Using Lemma 5.8(1), $h(a, a) = (b, b) \in M_1^{[2]}$. Then $f(a) = b \in M_1$ for all $a \in M$. Since $h(0, 0) = (0, 0)$ and $h(1, 1) = (1, 1)$, then $f(0) = 0$ and $f(1) = 1$, respectively. For all $x, y \in M$, by Lemma 5.8(1), we have $h(x, x) = (x_1, x_1)$ and $h(y, y) = (y_1, y_1)$. Then $f(x) = x_1$ and $f(y) = y_1$. Now,

$$\begin{aligned} h(x \wedge y, x \wedge y) &= h((x, x) \wedge (y, y)) \\ &= h(x, x) \wedge h(y, y) \\ &= (x_1, x_1) \wedge (y_1, y_1) \\ &= (x_1 \wedge y_1, x_1 \wedge y_1). \end{aligned}$$

Then $f(x \wedge y) = x_1 \wedge y_1 = f(x) \wedge f(y)$. Using similar way, we get $f(x \vee y) = f(x) \vee f(y)$. Since $h((x, x)^{\circ}) = (h(x, x))^{\circ}$, then $h(\bar{x}, \bar{x}) = (x_1, x_1)^{\circ} = (\bar{x_1}, \bar{x_1})$. Hence $f(\bar{x}) = \bar{x_1} = \overline{f(x)}$. Therefore f is a homomorphism of de Morgan algebra M into a de Morgan algebra M_1. □

Definition 5.10. [10] *An element a of a lattice L with 0 is said to be an atom of L if $a \neq 0$ and for any $x \in L$, $x \leq a$, then either $x = 0$ or $x = a$. Dually, an element d of a lattice L with 1 is said to be a coatom (dual atom) of L if $d \neq 1$ and for any $x \in L$, $d \leq x$, then either $x = 1$ or $x = d$. Let $At(L)$ be the set of all atoms of L. A lattice L with zero element is said to be atomic if for every nonzero element x of L, there exists an atom a of L such that $a \leq x$.*

Now, we obtain many properties of atoms and coatoms of *CRDMS*-algebras that should be useful for further discussion.

Lemma 5.11. *For a CRDMS-algebra $M^{[2]}$, we have*

(1) $x = (a, b) \in M^{[2]}$ *is an atom of* $M^{[2]}$ *if and only if* $b \in At(M)$ *and* $a = 0$*,*

(2) $x = (a, b) \in M^{[2]}$ *is a coatom of* $M^{[2]}$ *if and only if a is a coatom of M and* $b = 1$*.*

Proof. (1). Suppose that $x = (a, b) \in M^{[2]}$ is an atom of $M^{[2]}$. If b is not an atom of M, there exists $0 < b_1 < b$ and $y = (b_1 \wedge a, b_1) \in M^{[2]}$. Thus $y < x$, which contradicts with the fact that x is an atom of $M^{[2]}$. Hence b is an atom of M. Now, since $a \leq b$ and b is an atom of M, we have $a = 0$ or $a = b$. If $a = b$ then $(0, 0) < (0, b) < (a, b)$, which contradicts with that (a, b) is an atom of $M^{[2]}$. Then $a = 0$. Conversely, let b is an atom of M and $b = 0$. Then we have to show that $x = (0, b)$ is an atom of $M^{[2]}$. Let $y = (c, d)$ is an element of $M^{[2]}$ such that $y \leq x$. Then $c = 0$ and $d \leq b$. Since b is an atom of M, then $d = 0$ and $y = (0, 0)$. Therefore $x = (0, b)$ is an atom of $M^{[2]}$ as claimed.

(2) By duality of (1). □

Corollary 5.12.

(1) b is an atom of M if and only if $(0, b)$ *is an atom of* $M^{[2]}$*,*

(2) b is a coatom of M if and only if $(b, 1)$ *is a coatom of* $M^{[2]}$*,*

(3) there is a one to one correspondence between the set of all atoms (coatoms) of M and the st of all atoms (coatoms) of $M^{[2]}$*.*

Theorem 5.13. *Let M be a de Morgan algebra and* $a \in M$*. Then*

(1) $(0, a)$ *is an atom of* $M^{[2]}$ *implies* $(\bar{a}, 1)$ *is a coatom of* $M^{[2]}$*,*

(2) $(a, 1)$ *is a coatom of* $M^{[2]}$ *implies* $(0, \bar{a})$ *is an atom of* $M^{[2]}$*,*

(3) there is a one to one correspondence between the set of all atoms of $M^{[2]}$ *and the set of all coatoms of* $M^{[2]}$*.*

Proof. (1). Let $(0, a)$ is an atom of $M^{[2]}$. Then by Corollary 5.12(1), a is an atom of M. Clearly $(\bar{a}, 1) \in M^{[2]}$. Let $(x, y) \geq (\bar{a}, 1)$ for some $(x, y) \in M^{[2]}$. Then $x \geq \bar{a}$ and $y = 1$ implies $\bar{x} \leq a$ and $y = 1$. Since a is an atom of M, then $\overline{x} = 0$ or $\overline{x} = a$. It follows that $x = 1, y = 1$ or $x = \bar{a}, y = 1$. Hence $(x, y) = (1, 1)$ or $(x, y) = (\bar{a}, 1)$. Therefore $(\bar{a}, 1)$ is a coatom of $M^{[2]}$.
The proof of (2) is similar to that of (1)and the proof of (3) follows (1) and (2). □

Theorem 5.14. *A de Morgan algebra M is atomic if and only if* $M^{[2]}$ *is atomic.*

Proof. Let M be an atomic de Morgan algebra. Let (a, b) is a nonzero element of $M^{[2]}$. Then $a \leq b$. Hence $a = 0$ or $a \neq 0$ but $b \neq 0$. If $a = 0$, then there exist atom of M say c such that $c \leq b$. Then by Corollary 5.10(1), $(0, c)$ is an atom of $M^{[2]}$ and $(0, c) \leq (0, b) = (a, b)$. If $a \neq 0$ then there exists an atom of M say x such that $x \leq a$. Hence $(0, x)$ is an atom of $M^{[2]}$ with $(0, x) \leq (a, a) \leq (a, b)$. Therefore $M^{[2]}$ is an atomic core regular double *MS*-algebra. Conversely, let $M^{[2]}$ is atomic. Let $0 \neq a \in M$. Then (a, a) is a nonzero element of $M^{[2]}$. Thus there exists an atom of $M^{[2]}$ say $(0, y)$ with $(0, y) \leq (a, a)$. Consequently y is an atom of M with $y \leq a$. Therefore M is atomic. □

In the following example, we clarify the properties of atoms and coatoms of M and $M^{[2]}$.

Example 5.15. *Consider a de Morgan algebra M and a CRDMD-algebra* $M^{[2]}$ *in example 4.2. We observe the following.*

(1) $At(M) = \{a, b\}$ *and* $At(M^{[2]}) = \{(0, a), (0, b)\}$*, where* a, b *are corresponding to* $(0, a), (0, b)$*, respectively.*

(2) $\{a, b\}$ *and* $\{(a, 1), (b, 1)\}$ *are the sets of coatoms of M and* $At(M^{[2]})$*, respectively. Also,* a, b *are corresponding to* $(a, 1), (b, 1)$*, respectively.*

(3) The atoms $(0, a), (0, b)$ *of* $M^{[2]}$ *are corresponding to the coatoms* $(a, 1), (b, 1)$ *of* $M^{[2]}$*, respectively.*

(4) It is ready seen that M is an atomic de Morgan algebra and $M^{[2]}$ *is an atomic CRDMS-algebra.*

Definition 5.16. [7] *A lattice L is called complete if* $\inf_L H$ *and* $\sup_L H$ *exist for each* $\phi \neq H \subseteq L$*.*

A *CRDMS*-algebra L is called complete if considered as a lattice it is complete.

Let $H = \{x_i = (a_i, b_i) : i \in I\} \subseteq M^{[2]}$. We can write $\sup_L H = \bigvee_{i\in I} x_i$ and $\inf_L H = \bigwedge_{i\in I} x_i$.

Theorem 5.17. *If M is a complete de Morgan algebra, then $M^{[2]}$ is complete CRDMS-algebra.*

Proof. Let $\phi \neq H \subseteq M^{[2]}$. Consider $H = \{(a_i, b_i) \in M^{[2]}, i \in I\}$. Since M is complete, then $\bigvee_{i\in I} a_i$ and $\bigvee_{i\in I} a_i$ exist. Hence $a_i \leq \bigvee_{i\in I} a_i$ and $b_i \leq \bigvee_{i\in I} b_i$. So, $(a_i, b_i) \leq (\bigvee_i a_i, \bigvee_i b_i)$ and hence $(\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i)$ is an upper bound of H. Let (x, y) be an upper bound of H. Then $(a_i, b_i) \leq (x, y)$ implies $a_i \leq x$ and $b_i \leq y$. Therefore $\bigvee a_{i\in I} \leq x$ and $\bigvee b_{i\in I} \leq y$ and $(\bigvee a_{i\in I}, \bigvee b_{i\in I}) \leq (x, y)$. Then $(\bigvee a_{i\in I}, \bigvee b_{i\in I}) = \sup H$. Similarly, we can show that $(\bigwedge a_{i\in I}, \bigwedge b_{i\in I}) = \inf H$. Then $M^{[2]}$ is complete. □

Theorem 5.18. *Let $M^{[2]}$ be a complete CRDMS-algebra. Then*

(1) $(M^{[2]})^{\circ\circ}$ is complete,

(2) M is complete.

Proof. (1). Let $\phi \neq H \subseteq (M^{[2]})^{\circ\circ}$. Since $M^{[2]}$ is complete and $H \subseteq M^{[2]}$, then $\sup H$ and $\inf H$ exist in $M^{[2]}$. Assume that $(a, b) = \sup_{M^{[2]}} H$ and $(c, d) = \inf_{M^{[2]}} H$. We prove that $(b, b) = \sup_{(M^{[2]})^{\circ\circ}} H$. Since $(a, b) = \sup_{M^{[2]}} H$, then $(h, h) \leq (a, b)$ for all $h \in H$. Thus $(h, h) = (h, h)^{++} \leq (a, b)^{++} = (a, a)$ and hence (a, a) is an upper bound of H. Since $(a, b) = \sup_{M^{[2]}} H$, then $(a, b) \leq (a, a)$ implies $b \leq a$. But $a \leq b$ as $(a, b) \in M^{[2]}$. Therefore $a = b$ and $(a, b) = (b, b) \in (M^{[2]})^{\circ\circ}$ and $(b, b) = \sup H_{(M^{[2]})^{\circ\circ}}$. Similarly, we can show that $\inf H \in (M^{[2]})^{\circ\circ} = (d, d)$. Therefor $(M^{[2]})^{\circ\circ}$ is complete de Morgan algebra.

(2) Let $\phi \neq C \subseteq M$. Since M isomorphic to $(M^{[2]})^{\circ\circ}$ (see Corollary 4.4) then $\acute{C} = \{(c, c) : c \in C\} \subseteq (M^{[2]})^{\circ\circ}$ corresponds to C. Since by (1), $(M^{[2]})^{\circ\circ}$ is complete and $\acute{C} \subseteq (M^{[2]})^{\circ\circ}$ then $\sup_{(M^{[2]})^{\circ\circ}} \acute{C}$ and $\sup_{(M^{[2]})^{\circ\circ}} \acute{C}$ exist. Assume $(x, x) = \sup_{(M^{[2]})^{\circ\circ}} \acute{C}$ and $(y, y) = \inf_{(M^{[2]})^{\circ\circ}} \acute{C}$. Then $(c, c) \leq (x, x)$ for all $(c, c) \in \acute{C}$ implies $c \leq x$ for all $c \in C$. Thus x is an upper bound of C. Let y be an upper bound of C. Then $c \leq y$ for all $c \in C$ implies $(c, c) \leq (y, y)$ for all $(c, c) \in \acute{C}$. Hence (y, y) is an upper bound of $\acute{C}$. Then $(x, x) \leq (y, y)$ as $(x, x) = \sup_{(M^{[2]})^{\circ\circ}} \acute{C}$. Therefore $x \leq y$ and $x = \sup_M C$. Using a similar way, we get $y = \inf_M C$. Then M is complete. □

Combining Theorem 5.17 and Theorem 5.18(2), we have

Theorem 5.19. *A de Morgan algebra M is complete if and only if $M^{[2]}$ is a complete CRDMS-algebra.*

Now, we give two examples of complete atomic *CRDMS*-algebras, the first one is finite and the second one is infinite.

Example 5.20.

(1) Consider a CRDMS-algebra L in example 3.9(2). We have $At(L) = \{c, d\}$. It is clear that L is a finite complete atomic CRDMS-algebra.

(2) Let $M = \{\overline{0}\} \oplus [0, 1] \oplus \overline{1}$ be an infinite chain, where $[0, 1]$ is a real closed interval and $\oplus$ stands for the ordinal sum. Then $(M; \vee, \wedge, \overline{0}, \overline{1})$ forms a bounded distributive lattice, where $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x, y \in [0, 1]$ and $\overline{0}, \overline{1}$ are the smallest and the greatest elements of M, respectively. Define a negation $\sim$ on M by $\sim x = 1 - x$ for all $x \in [0, 1]$, $\sim \overline{0} = \overline{1}$ and $\sim \overline{1} = \overline{0}$. Since $At(M) = \{0\}$, then M is atomic. As $\sup_M H$ and $\inf_M H$ exist, for $\phi \neq H \subseteq M$, then M is complete. Therefore M is a complete atomic de Morgan algebra. Using the construction Theorem, we obtain

the core regular double MS-algebra $M^{[2]}$, where

$$
\begin{aligned}
M^{[2]} \;=\;& \{(\overline{0},\overline{0}),(\overline{0},0),...,(\overline{0},1/2),...,(\overline{0},\overline{1}),\\
& (0,0),...,(0,1/2),...(0,1),(0,\overline{1}),\\
&.\\
&.\\
&.\\
& (1/2,1/2),...,(1/2,3/4),...,(1/2,1),(1/2,\overline{1}),\\
&.\\
&.\\
&.\\
& (1,1),(1,\overline{1}),\\
& (\overline{1},\overline{1})\}.
\end{aligned}
$$

and $(x,y)^{\circ} = (\sim y, \sim y)$, $(x,y)^{+} = (\sim x, \sim x)$ for all $(x,y) \in M^{[2]}$. Also, $K(M^{[2]}) = D(L) \cap \overline{D(L)} = \{(x,\overline{1}) : x \in M\} \cap \{(\overline{0},y) : y \in M\} = \{(\overline{0},\overline{1})\}$. We have $At(M^{[2]}) = \{(\overline{0},0)\}$. By Theorem 5.14, $M^{[2]}$ is atomic. Also, $M^{[2]}$ is complete (see Theorem 5.17). Therefore $M^{[2]}$ is an infinite complete atomic CRDMS-algebra.

Lemma 5.21. *Let $M^{[2]}$ be a complete CRDMS-algebra and $x_i = (a_i, b_i) \in M^{[2]}$ for all $i \in I$. Then*

(1) $\bigvee_{i\in I} x_i = (\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i)$,

(2) $\bigwedge_{i\in I} x_i = (\bigwedge_{i\in I} a_i, \bigwedge_{i\in I} b_i)$,

(3) $(\bigvee_{i\in I} x_i)^{\circ} = \bigwedge_{i\in I} x_i^{\circ}$,

(4) $(\bigwedge_{i\in I} x_i)^{+} = \bigvee_{i\in I} x_i^{+}$.

Proof. (1). Since $M^{[2]}$ is complete, then by Theorem 5.6, M is also complete. Hence $\bigvee_{i\in I} a_i$ and $\bigvee_{i\in I} b_i$ exist in M. Then $a_i \le \bigvee_{i\in I} a_i$ and $b_i \le \bigvee_{i\in I} b_i$ imply $(a_i, b_i) \le (\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i)$. Hence $(\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i)$ is an upper bound of x_i for all $i \in I$. Let (a, b) be an upper bound of x_i. Therefore $a_i \le a$ and $b_i \le b$ for all $i \in I$. Hence a is an upper bound of a_i and b is an upper bound of b_i for all $i \in I$. So, $(\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i) \le (a, b)$ and $(\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i)$ is the least upper bound of x_i for all $i \in I$.

(2) The proof is similar to that of (1).

(3) Since $\bigvee_{i\in I} x_i \ge x_i$, then $(\bigvee_{i\in I} X_i)^{\circ} \le x_i^{\circ}$. Hence $(\bigvee_{i\in I} X_i)^{\circ}$ is a lower bound of x_i°. Let y be a lower bound of x_i°. Then $y \le x_i^{\circ}$ implies $y^{\circ} \ge x_i^{\circ\circ} \ge x_i$. Then y° is an upper bound of x_i and this gives $\bigvee_{i\in I} x_i \le y^{\circ}$. Therefore $(\bigvee_{i\in I} x_i)^{\circ} \ge y^{\circ\circ} \ge y$. Then we deduce that $(\bigvee_{i\in I} x_i)^{\circ}$ is the greatest lower bound of x_i° and $(\bigvee_{i\in I} x_i)^{\circ} = \bigwedge_{i\in I} x_i^{\circ}$.

(4) The proof is similar to that of (3). □

Definition 5.22. *A subalgebra L_1 of a complete CRDMS-algebra L is called a complete subalgebra of L if $\inf_L H \in L_1$ and $\sup_L H \in L_1$ for every subset H of L_1.*

Theorem 5.23. *Let $M_1^{[2]}$ be a subalgebra of a complete CRDMS-algebra $M^{[2]}$. Then $M_1^{[2]}$ is complete subalgebra of $M^{[2]}$ if and only if M_1 is a complete subalgebra of M.*

Proof. Let $M_1^{[2]}$ is a complete subalgebra of $M^{[2]}$. Let $\phi \neq H \subseteq M_1$. Consider the subset $\acute{H} = \{x_i = (a_i, a_i) : a_i \in H, i \in I\}$ of $M_1^{[2]}$ corresponding to H. Since $M^{[2]}$ is complete, then by Lemma 5.21(1), (2), we have

$$
\begin{aligned}
\sup_M \acute{H} &= \bigvee_{i\in I} x_i = (\bigvee_{i\in I} a_i, \bigvee_{i\in I} a_i) \in M_1^{[2]},\\
\inf_M \acute{H} &= \bigwedge_{i\in I} x_i = (\bigwedge_{i\in I} a_i, \bigvee_{i\in I} a_i) \in M_1^{[2]}.
\end{aligned}
$$

Hence $\bigvee_{i\in I} a_i \in M_1$ and $\bigwedge_{i\in I} a_i \in M_1$. Then M_1 is complete complete subalgebra of M. Conversely, let M_1 is a complete subalgebra of a complete de Morgan algebra M. Let $\phi \neq H \subseteq M_1^{[2]}$. Then

$$H = \{x_i = (a_i, b_i) \in M_1^{[2]}, i \in I\} \subseteq M_1^{[2]}.$$

Since M_1 is complete subalgebra of M, then we have

$$\bigvee_{i\in I} a_i \in M_1 \text{ and } \bigvee_{i\in I} b_i \in M_1.$$

Also, $\bigwedge_{i\in I} a_i \in M_1$ and $\bigwedge_{i\in I} b_i \in M_1$. Then by Lemma 5.21,(1),(2), respectively, we get

$$\begin{aligned} \bigvee_{M^{[2]}} H &= \bigvee_{i\in I} x_i = (\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i) \in M_1^{[2]}, \\ \bigwedge_{M^{[2]}} H &= \bigwedge_{i\in I} x_i = (\bigwedge_{i\in I} a_i, \bigwedge_{i\in I} b_i) \in M_1^{[2]}. \end{aligned}$$

Then $M_1^{[2]}$ is a complete subalgebra of a *CRDMS*-algebra $M^{[2]}$.
□

Definition 5.24. [6] *A lattice homomorphism $h : L \to L_1$ of a complete lattice L into a complete lattice L_1 is called complete if*

$$h(\inf_L H) = \inf_{L_1} h(H) \text{ and } h(\sup_L H) = \sup_{L_1} h(H) \text{ for each } \phi \neq H \subseteq L.$$

A homomorphism $h : L \to L_1$ of a complete *CRDMS*-algebra L into a complete *CRDMS*-algebra L_1 is called complete if it is complete as a lattice homomorphism.

Theorem 5.25. *Let M and M_1 are complete de Morgan algebras. If $f : M \to M_1$ is a complete homomorphism, then $h : M^{[2]} \to M_1^{[2]}$ defined by $h(a, b) = (f(a), f(b))$ is a complete homomorphism of $M^{[2]}$ into $M_1^{[2]}$. Conversely, if $g : M^{[2]} \to M_1^{[2]}$ is a complete homomorphism, then $f : M \to M_1$ defined by $f(a) = b \Leftrightarrow g(a, a) = (b, b)$ is a complete homomorphism of de Morgan algebras.*

Proof. Let $f : M \to M_1$ is a complete homomorphism. Then by Theorem 5.9, $h : M^{[2]} \to M_1^{[2]}$ defined by $h(a, b) = (f(a), f(b))$ is a homomorphism of *CRDMS*-algebras $M^{[2]}$ and $M_1^{[2]}$. We prove that $\sup_{M_1^{[2]}} h(H) = h(\sup_{M^{[2]}} H)$ for $\phi \neq H \subseteq M^{[2]}$. Consider $H = \{x_i = (a_i, b_i) \in M^{[2]} : i \in I\}$ for $\phi \neq H \subseteq M^{[2]}$. Using Lemma 5.21(1), we get $\sup_{M^{[2]}} H = \bigvee_{i\in I} x_i = \bigvee_{i\in I}(a_i, b_i) = (\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i)$. Thus

$$\begin{aligned} h(\sup_{M^{[2]}} H) &= h(\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i) \\ &= (f(\bigvee_{i\in I} a_i), f(\bigvee_{i\in I} b_i)) \\ &= (\bigvee_{i\in I} f(a_i), \bigvee_{i\in I} f(b_i)) \\ &= \bigvee_{i\in I} (f(a_i), f(b_i)) \\ &= \bigvee_{i\in I} h(a_i, b_i) \\ &= \sup_{M_1^{[2]}} h(H), \end{aligned}$$

Using Lemma 5.21(2), we can get $\inf^{[2]}_{M_1} = h(\inf_{M^{[2]}} H)$. Therefore h is complete. Conversely, let $g : M^{[2]} \to M_1^{[2]}$ is a complete homomorphism of a *CRDMS*-algebra $M^{[2]}$ into $M_1^{[2]}$. Then by Theorem 5.9, a mapping $f : M \to M_1$ defined by $f(a) = b \Leftrightarrow g(a, a) = (b, b)$ is a homomorphism of M into M_1. We have to show that f is complete. Let $\phi \neq H = \{a_i : i \in I\} \subseteq M$, we prove that $f(\inf_M H) = \inf_{M_1} f(H)$. Consider a subset $\acute{H} = \{x_i = (a_i, a_i) : a_i \in H, i \in I\}$ of $M^{[2]}$ corresponding to H. Since M and $M^{[2]}$ are complete, then by Lemma 5.21(2), we get

$$\inf_{M^{[2]}} \acute{H} = \bigwedge_{i\in I} x_i = (\bigwedge_M a_i, \bigwedge_{i\in I} a_i)$$

Let $g(a_i, a_i) = (b_i, b_i)$. Then by definition of f, we have $f(a_i) = b_i$. Since g is complete, then $g(\inf_{M^{[2]}} \acute{H}) = \inf_{M_1^{[2]}} g(\acute{H})$. Now

$$\begin{aligned} g(\inf_{M^{[2]}} \acute{H}) &= g(\bigwedge_{i\in I}(a_i, a_i)) = g(\bigwedge_{i\in I} a_i, \bigwedge_{i\in I} a_i), \\ \inf_{M_1^{[2]}} g(\acute{H}) &= \bigwedge_{i\in I} g(a_i, a_i)) = \bigwedge_{i\in I}(b_i, b_i)) = (\bigwedge_{i\in I} b_i, \bigwedge_{i\in I} b_i). \end{aligned}$$

Then $g(\bigwedge_{i\in I} a_i, \bigwedge_{i\in I} a_i) = (\bigwedge_{i\in I} b_i, \bigwedge_{i\in I} b_i)$ implies $f(\inf_M H) = f(\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} b_i = \inf_{M_1} f(H)$. Similarly, we can show that $f(\sup_M H) = \sup_{M_1}(f(H))$. Then f is complete. $\square$

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