Filomat 34:1 (2020), 249–255
https://doi.org/10.2298/FIL2001249S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Automorphism Group of Homogeneous Structures

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Abstract. A relational structure $\mathcal{A}$ with a countable universe is defined to be homogeneous iff every finite partial isomorphism of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{A}$. Endow the universe of $\mathcal{A}$ with the discrete topology. Then the automorphism group $Aut(\mathcal{A})$ of $\mathcal{A}$ becomes a topological group (with the subspace topology inherited from the suitable topological power of the discrete topology on $\mathcal{A}$). Recall, that a tuple $\langle g_0, ..., g_{n-1}\rangle$ of elements of $Aut(\mathcal{A})$ is defined to be weakly generic iff its diagonal conjugacy class (in the group theoretic sense) is dense in the topological sense, and further, the $\langle g_0, ..., g_{n-1}\rangle$-orbit of each $a \in A$ is finite. Investigations about weakly generic automorphisms have model theoretic origins (and reasons); however, the existence of weakly generic automorphisms is closely related to interesting results in finite combinatorics, as well.

In this work we survey some connections between the existence of weakly generic automorphisms and finite combinatorics, group theory and topology. We will recall some classical results as well as some more recently obtained ones.

1. Introduction

A relational structure $\mathcal{A}$ with a countable universe is defined to be *homogeneous* iff every finite partial isomorphism of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{A}$ (structures with this property are sometimes called ultrahomogeneous, as well). Studying homogeneous structures is a classical area of model theory, for further details and for historical remarks we refer to Section 7.1 of [5] and the more recent excellent survey [10].

In this survey paper we are summing up some classical as well as some more recently obtained results on a particular property of the group of automorphisms of a homogeneous structure. This particular property is *having weakly generic automorphisms* and is defined as follows.

Endow the universe of $\mathcal{A}$ with the discrete topology. Then the automorphism group $Aut(\mathcal{A})$ of $\mathcal{A}$ becomes a topological group (with the subspace topology inherited from the suitable topological power of the discrete topology on $\mathcal{A}$). Recall, that a tuple $\langle g_0, ..., g_{n-1}\rangle$ of elements of $Aut(\mathcal{A})$ is defined to be *weakly generic* iff it satisfies conditions (1) and (2) below:

(1) the diagonal conjugacy class of $\langle g_0, ..., g_{n-1}\rangle$ (in the group theoretic sense) is dense in the topological sense, that is,

2010 *Mathematics Subject Classification*. Primary 03C15, 20B27, 08A35, 20E18.
Keywords. Weakly generic automorphism, Homogeneous structures, Extension Property for finite partial isomorphisms.
Received: 31 March 2019; Revised: 18 July 2019; Revised: 23 October 2019; Accepted: 30 October 2019
Communicated by Biljana Popović
Supported by Hungarian National Foundation for Scientific Research grant K129211.
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$$\{\langle f^{-1}g_0f, ..., f^{-1}g_{n-1}f\rangle : f \in Aut(\mathcal{A})\} \quad \text{is a dense subset of } Aut(\mathcal{A})^n;$$

(2) if G is the subgroup of $Aut(\mathcal{A})$ generated by $\{g_0, ..., g_{n-1}\}$ then the G-orbit of each $a \in A$ is finite.

Originally, the problem of existence of tuples of automorphisms satisfying (1) was motivated by the model theoretic Small Index Property; in this respect we refer to [6], [3], [9] and the references therein. Existence of weakly generic automorphisms is closely related to interesting results in finite combinatorics; in this respect we refer e.g. to [16], [8], [2], [4], [1], [7], [14] and [12].

In the present work we discuss some connections between the existence of weakly generic automorphisms and finite combinatorics, group theory and topology. The structure of the paper is rather simple. We close this section by summing up our system of notation. In Section 2 we present the connection with finite combinatorics, in Section 3 we discuss the connections with group theory, and in Section 4 we present some topological aspects of the existence of weakly generic automorphisms. As we mentioned, the present work is a survey.

Notation

Our notation is mostly standard, but the following list may be helpful.

Throughout ω denotes the set of natural numbers and for every $n \in \omega$ we have $n = \{0, 1, ..., n-1\}$. Let A and B be sets. Then ${}^AB$ denotes the set of functions whose domain is A and whose range is a subset of B. For a topological space X and $n \in \omega$, the n^{th} power of X (in the topological sense) will be denoted by X^n.

In addition, $|A|$ denotes the cardinality of A and $\mathcal{P}(A)$ denotes the power set of A, that is, $\mathcal{P}(A)$ consists of all subsets of A.

Throughout we use function composition in such a way that the rightmost factor acts first. That is, for functions f, g we define $f \circ g(x) = f(g(x))$.

If $\mathcal{G}$ is a group and $f_0, ..., f_{n-1} \in G$ then $\langle f_0, ..., f_{n-1}\rangle$ denotes the subgroup of G generated by $\{f_0, ..., f_{n-1}\}$. If G is a group acting on a set X and $a \in X$ then $O_G(a)$ denotes the orbit of a with respect to the action of G.

We extend the notion of orbits to certain monoids of functions as follows. If X is a set and $F = \{f_0, ..., f_{n-1}\}$ is a set of injective partial functions on X then we say that $a, b \in X$ are in the same F-orbit, iff there exists a finite sequence $g_0, ..., g_{m-1} \in F \cup \{f^{-1} : f \in F\}$ such that $g_0 \circ ... \circ g_{m-1}(a) = b$ (that is, every partial function is defined in all relevant points and a is mapped onto b). The relation "being in the same orbit" is an equivalence relation (even in the case when the elements of F are partial functions), and the equivalence class of a will also be denoted by $O_{\langle f_0,...,f_{n-1}\rangle}(a)$.

If $\mathcal{A}$ and $\mathcal{B}$ are structures then $\mathcal{A} \leq \mathcal{B}$ denotes the fact that $\mathcal{A}$ is a substructure of $\mathcal{B}$. In addition, $Aut(\mathcal{A})$ denotes the automorphism group of $\mathcal{A}$.

2. Connections with finite combinatorics

Let $\mathcal{L}$ be a finite, first order relational language and let $\mathcal{A} = \langle A, R_i\rangle_{i\in\mathcal{L}}$ be an $\mathcal{L}$-structure. A partial function f on A is defined to be a *partial isomorphism* of $\mathcal{A}$ iff f is an isomorphism between the substructures of $\mathcal{A}$ generated by the domain and range of f. Let K be a class of finite $\mathcal{L}$-structures. K is defined to have the *Extension Property for Partial Isomorphisms* (*EP* for short) iff for all $\mathcal{A} \in K$ and all sequences $\langle f_0, ..., f_{n-1}\rangle$ of partial isomorphisms of $\mathcal{A}$ there exist $\mathcal{B} \in K$ such that $\mathcal{A}$ can be embedded into $\mathcal{B}$ and each f_i can be extended to an automorphism of $\mathcal{B}$.

Connections with the existence of weakly generic automorphisms will be recalled in Section 4 (see Lemma 4.2 below). During the last decades, the extension problem for partial isomorphisms has been thoroughly investigated. More concretely,

• Truss in [16] proved that each finite graph $\mathcal{G}$ with one partial isomorphism f can be embedded into another finite graph $\mathcal{G}'$ such that f extends to an automorphism of $\mathcal{G}'$;

• Hrushovski in [8] proved that the class of finite graphs has the *EP*;
• Herwig and Lascar in [2] and [4] obtained similar results for the class of finite structures of arbitrary (finite) relational languages. They obtained more general results e.g. for the class of finite triangle free graphs, etc.
• Solecki in [15] proved the analogous result for the class of finite metric spaces.

For more recent related investigations we refer to [1] and [9].

Hrushovski's approach (later generalized by Herwig) was as follows. Keeping the notation introduced so far, let $\mathcal{A} \in K$ and let G be any group. Then one can define an equivalence relation $\sim$ on $A \times G$ as follows. If $\langle a, f\rangle, \langle b, g\rangle \in A \times G$ then $\langle a, f\rangle \sim \langle b, g\rangle$ iff $g \circ f^{-1}(a) = b$. Further, G acts on $A \times G$ in a natural way: if $\langle a, f\rangle \in A \times G$ and $g \in G$ then $g^*(\langle a, f\rangle)$ is defined to be $\langle a, gf\rangle$. Now the crucial point is to find a suitable group G such that

(a) the monoid of partial isomorphisms of $\mathcal{A}$ can be "embedded" into G and
(b) $(A \times G)/\sim$ can be extended to an $\mathcal{L}$-structure in which $\mathcal{A}$ can be embedded.

In more detail, according to point (a), we have to find a finite group G and a function φ from the partial isomorphisms of $\mathcal{A}$ into G such that φ preserves composition and forming inverses. Finiteness of G is essential, otherwise the so obtained extension of $\mathcal{A}$ may not remain finite.

As we mentioned, if G is an arbitrary group, then G acts on $A \times G$ by a natural way: for $\langle a, f\rangle \in A \times G$ and $g \in G$ define $g^*(\langle a, f\rangle)$ to be $\langle a, gf\rangle$. According to point (b) we have to endow $(A \times G)/\sim$ by basic relations in order to make it to an $\mathcal{L}$-structure $\mathcal{A}^*$ in which $\mathcal{A}$ can be embedded by the function $A \ni a \mapsto \langle a, 1^G\rangle/\sim$. We would like to achieve $g^* \in Aut(\mathcal{A}^*)$ for all $g \in G$. Let $R \in \mathcal{L}$ be any relation symbol in our language and let $\bar{a}, \bar{b} \in A$ be tuples whose length is the same as the arity of R. As $R^{\mathcal{A}^*}$ (in fact, any definable relation of $\mathcal{A}^*$) is a union of certain orbits of $Aut(\mathcal{A}^*)$, we have to define $R^{\mathcal{A}^*}$ "orbit-wise", that is, if $\bar{a}, \bar{b}$ are in the same G^*-orbit, then either $\bar{a}, \bar{b} \in R^{\mathcal{A}^*}$ or $\bar{a}, \bar{b} \notin R^{\mathcal{A}^*}$ should hold. If the orbit of $\bar{a}$ intersects A, then we do not have any choice, if the orbit of $\bar{a}$ does not intersect A, then we may choose between the two options above.

This approach can be implemented only if the following condition is satisfied: if $\bar{a}, \bar{b} \in A$ and they lie in the same G^*-orbit, then the function mapping $\bar{a}$ onto $\bar{b}$ coordinatewise is a partial isomorphism of $\mathcal{A}$. Thus, denoting the monoid of partial isomorphisms of $\mathcal{A}$ by P, we require

$$(*) \qquad O_P(\bar{a}) = O_P(\bar{b}) \text{ whenever } O_{G^*}(\bar{a}) = O_{G^*}(\bar{b});$$

if this does not hold, then the "orbitwise" definition of $R^{\mathcal{A}^*}$ described in the previous paragraph may be meaningless (as it should satisfy contradictory requirements). So $(*)$ is another requirement which should (and may) be satisfied by a careful choice of G. This is the main difficulty which would be handled in item (b).

This approach was examined in [8] and it was further generalized in [3]. Let $\mathcal{G}$ be a graph on n vertices. In [8] Hrushovski investigated the size (number of vertices) of the smallest graph G^* in which $\mathcal{G}$ can be embedded such that each partial isomorphism of $\mathcal{G}$ extends to an automorphism of $\mathcal{G}^*$. He obtained an upper bound for the size of $\mathcal{G}^*$ which is doubly exponential in n.

In [4] Lascar and Herwig established a rather smaller upper bound by providing another proof for Hrushovski's extension theorem. Their construction is very clever, short, completely elementary and can be generalized to structures with arbitrary first order relational languages. The main idea is to find an infinite family $\langle \mathcal{A}_n : n \in \omega\rangle$ of finite $\mathcal{L}$-structures such that

• each $\mathcal{A}_n$ is symmetric enough: if p is a partial isomorphism of $\mathcal{A}_n$, then p can be extended to an automorphism of $\mathcal{A}_n$ and
• if $\mathcal{B}$ is any finite $\mathcal{L}$-structure, then there is an n such that $\mathcal{B}$ can be embedded into $\mathcal{A}_n$.

3. Connections with group theory

In [4] there are rather involved group theoretic methods and results that provide connections between the existence of generic automorphisms and the profinite topology on the free groups. The main results in [4] were motivated to prove *EP* not only for the class of finite graphs, but many other classes of structures. The strategy of the proofs is essentially the same as described in (a) and (b) of Section 2, but to find a suitable group G satisfying (b), deep group theoretical methods have been utilized (instead of Hrushovski's original construction, which has a combinatorial character). To survey the group-theoretical method we need some further preparations.

Throughout, $\mathcal{F}^g(P)$ and $\mathcal{F}^s(P)$ will denote the free group and respectively, the free semigroup generated by the set of free generators P.

Again, let $\mathcal{A}$ be any finite relational structure with universe (underlying set) A. Let r be the maximum of the arities of the basic relations of $\mathcal{A}$. Let $n \in \omega$, $P = \{p_0, ..., p_{n-1}\}$ and let $h_0, ..., h_{n-1}$ be partial isomorphisms of $\mathcal{A}$. Let $\langle A_0, ..., A_{m-1}\rangle$ be an enumeration of the orbits of the monoid generated by $\{h_0, ..., h_{n-1}\}$. For each $i < m$ choose an (arbitrary) element $x_i \in A_i$. Then, x_i can be mapped onto each element of A_i by a suitable element of the monoid generated by $\{h_0, ..., h_{n-1}\}$. Thus, intuitively, x_i can be considered as the "origin" of A_i, and we can fix a "name" for each element x of A_i; by a name w_x of x we mean a word, that is, an element of $\mathcal{F}^s(P)$, such that the function naturally associated to w_x maps x_i onto x.

More precisely, let $*$ be the semigroup homomorphism on $\mathcal{F}^s(P)$ that maps each p_i onto h_i. Then, for each $i < m$ and $x \in A_i$ there exists $w_x \in \mathcal{F}^s(P)$ with $w_x^*(x_i) = x$. In addition, for any $x \in A$, let $j(x) \in m$ be the unique number for which $x \in A_{j(x)}$.

As explained in [4], there is a "correspondence" between

- certain m-tuples $\langle H_0, ..., H_{m-1}\rangle$ of subgroups of $\mathcal{F}^g(P)$ and
- certain (not necessarily finite) structures $\mathcal{B}$ in which $\mathcal{A}$ can be embedded so that each h_i extends to an automorphism of $\mathcal{B}$.

The underlying set of $\mathcal{B}$ will be the disjoint union of the sets of left cosets of the H_i in which $\mathcal{F}^g(P)$ acts naturally (the details will be explained a few paragraphs below). Intuitively, each member H_i of the sequence $\langle H_0, ..., H_{m-1}\rangle$ represents a unique orbit of $Aut(\mathcal{B})$ (acting on the universe B of $\mathcal{B}$ in the usual way). As it is well known, if G is a group acting on a set B, and $f, g \in G$ and $a \in B$ then $f^*(a) = g^*(a)$ if and only if f and g lie in the same left coset of the subgroup $G_a = \{h \in G : h^*(a) = a\}$ of G. We have to choose $\langle H_0, ..., H_{m-1}\rangle$ carefully in order not to destroy the orbit structure determined by the partial isomorphisms of the small structure $\mathcal{A}$. This careful choice can be done by referring to involved group theoretical results. We give a more precise description what we sketched so far in the present paragraph and then return to how to choose a suitable sequence $\langle H_0, ..., H_{m-1}\rangle$.

So, more precisely, and in more detail, an m-tuple $\langle H_0, ..., H_{m-1}\rangle$ of subgroups of $\mathcal{F}^g(P)$ is called a *shadow* of $\langle \mathcal{A}, h_0, ..., h_{n-1}\rangle$ iff the following stipulations are satisfied for all $i < n$ and $j < m$:

(1) if $x, y \in A_j$ and $h_i(x) = y$ then $w_y^{-1} \cdot h_i \cdot w_x \in H_j$;

(2) if $x, y \in A_j$ and $x \neq y$ then $w_y^{-1} \cdot w_x \notin H_j$;

(3) if $\overline{u}, \overline{v} \in {}^{\leq r}A$ are tuples with same length lying in different orbits of $\langle h_0, ..., h_{n-1}\rangle$, then there is no $h \in \mathcal{F}^g(P)$ such that for all $k < |\overline{u}|$ we have

$$h \cdot w_{u_k} \cdot H_{j(u_k)} = w_{v_k} \cdot H_{j(v_k)}$$

(recall, for example, that $j(u_k)$ is the unique number $i \in m$ for which $u_k \in H_i$).

Let $\mathcal{H}$ be the set of all shadows of $\langle \mathcal{A}, h_0, ..., h_{n-1}\rangle$.

Keeping the notation introduced so far, here (1) corresponds to the requirement, that if h_i maps x onto y, then the names of x and y should be interrelated: $h_i w_x$ should be in the coset $w_y H_j$. This guarantees, that x_j (the origin of the orbit represented by H_j) will be mapped onto the same point by $(h_i w_x)^*$ and w_y^*. Further, (2) guarantees, that the function $A \ni x \mapsto w_x H_{j(x)}$ is injective, while (3) is the adaptation (reformulation) of (b) from Section 2.

Now we turn to study (not necessarily finite) structures $\mathcal{B}$ in which $\mathcal{A}$ can be embedded such that each partial isomorphism of $\mathcal{A}$ extends to an automorphism of $\mathcal{B}$. We describe such structures by specifying the universes of them, and specifying a permutation group on it which we would like to make a subgroup of $Aut(\mathcal{B})$ using the above (3), or equivalently (b) from Section 2.

Let $\mathcal{E}$ be the set of (the isomorphism types of) structures of the form $\langle Y, h'_0, ..., h'_{n-1}\rangle$ where Y is a set with $|Y| \leq \aleph_0$, $A \subseteq Y$ and for all $i < n$, h'_i is a permutation of Y extending h_i.

For any $h = \langle H_0, ..., H_{m-1}\rangle \in \mathcal{H}$ we associate a structure $\nabla(h) = \langle Y, h'_0, ..., h'_{n-1}\rangle$ as follows: the universe of $\nabla(h)$ is Y and Y is defined to be the disjoint union of the sets $\mathcal{F}^g(P)/H_j$; further, for each $i < n$, $j < m$ and $w \in \mathcal{F}^g(P)$ let

$$h'_i(wH_j) = p_i w H_j,$$

thus, each h'_i is a function from Y into Y. In addition, define $\phi : A \to Y$ to be $\phi(x) = w_x H_{j(x)}$ for all $x \in A$.

Conversely, for any $y = \langle Y, h'_0, ..., h'_{n-1}\rangle \in \mathcal{E}$ we associate a sequence $\Delta(y) = \langle H_0, ..., H_{m-1}\rangle$ of subgroups of $\mathcal{F}^g(P)$ as follows: let $\psi : \mathcal{F}^g(P) \to Sym(Y)$ be the group homomorphism mapping each p_i onto h'_i and for all $j < m$ let

$$H_j = \{w \in \mathcal{F}^g(P) : \ (w)^{\psi}(x_j) = x_j\}.$$

Finally, let $\Delta(y) = \langle H_0, ..., H_{m-1}\rangle$.

Now, as explained on page 1990 of [4], the following are true:

(4) The function ϕ is injective, and hence each $x \in A$ may be identified by $\phi(x) = w_x H_{j(x)}$. In this way, A can be treated as a subset of the universe of $\nabla(h)$;

(5) if $h \in \mathcal{H}$ then $\nabla(h) \in \mathcal{E}$;

(6) if $y \in Y$ then $\Delta(y) \in \mathcal{H}$;

(7) using the identification described in (4), if $\overline{u}, \overline{v} \in {}^{\leq r}A$ are tuples of the same length, then $\overline{u}$ and $\overline{v}$ are lying in the same orbit under $\{h_0, ..., h_{n-1}\}$ iff they are lying in the same orbit under $\{h'_0, ..., h'_{n-1}\}$.

Suppose $h = \langle H_0, ..., H_{m-1}\rangle \in \mathcal{H}$. By construction, $\nabla(h)$ is finite iff each H_i has finite index. Thus, *EP* can be established for a variety of classes of finite relational structures by showing that there are shadows $\langle H_0, ..., H_{m-1}\rangle$ such that each H_i is of finite index. Several deep results in group theory can be used to find suitable shadows. A particularly useful tool for this is the Ribes-Zaleskii theorem: if $H_0, ..., H_{n-1}$ are finitely generated subgroups of $\mathcal{F}^g(P)$ then $H_0 \cdot ... \cdot H_{n-1}$ is a closed set in the profinite topology of $\mathcal{F}^g(P)$ (for further details and for the original result we refer to [11]; for how to use this theorem to find suitable shadows we refer to [4]).

This group theoretical approach has been examined in [4], particularly, it was used to prove *EP* for the class of all finite triangle-free graphs. We also note, that according to subsection 3.1 of [4], the Ribes-Zaleskii theorem can be derived relatively quickly from the fact, that a certain class of finite relational structures has the *EP*.

4. Connections with topology

First we survey the connection between the existence of weakly generic automorphisms and the Extension Property *EP*. We start by recalling a notation and a definition.

If K is a class of relational structures and $n \in \omega$ then

$$K_p^n = \{\langle \mathcal{A}, f_0, ..., f_{n-1}\rangle : \mathcal{A} \in K, f_0, ..., f_{n-1} \text{ are partial isomorphisms of } \mathcal{A}\}.$$

In addition, $K_p = \bigcup_{n \in \omega} K_p^n$.

Definition 4.1. *A class K of relational structures is defined to have the uniform joint embedding property (UJEP for short) iff for all $\mathcal{A}, \mathcal{B} \in K$ with disjoint universe there exist $C \in K$ and embeddings $f : A \to C, g : B \to C$*

with the following property. If R is a k-ary relation symbol in the language of K and $\bar{a}, \bar{a}' \in A \cup B$ are k-tuples such that they intersect both A and B and, in addition, for all $i < k$ we have $a_i \in A$ iff $a'_i \in A$ then $C \models R((f \cup g)(\bar{a}))$ iff $C \models R((f \cup g)(\bar{a}'))$, where $f \cup g(x) = f(x)$ if $x \in A$ and $f \cup g(x) = g(x)$ if $x \in B$. (Note, that if the functions f and g are treated as sets of pairs, then $f \cup g$ is again a set of pairs, and since A and B are assumed to be disjoint, $f \cup g$ is exactly the function described at the end of the previous sentence).

It is easy to see, that if K satisfies *UJEP* then K_p satisfies the Joint Embedding property (but the converse is not true in general). Further, *UJEP* can be easily verified for many classes of structures, for example, the class of finite tournaments satisfies *UJEP*.

Throughout this section K is a Fraïssé class, that is, K is a class of (isomorphism types of) finite structures (with the same finite relational language) such that K is closed under taking substructures, and K has the Joint Embedding and Amalgamation properties. In addition, $\mathcal{M}_K$ denotes the Fraïssé limit of K.

The following lemma describes the connection between *EP* and the existence of weakly generic automorphisms. Its proof can be found e.g. in [12] (see Lemma 2.4 therein).

Lemma 4.2. *Assume K satisfies UJEP (or K_p satisfies the Joint Embedding property) and let $n \in \omega$. Then the following are equivalent.*

(1) For any $\langle \mathcal{A}, f_0, ..., f_{n-1} \rangle \in K_p^n$ there exists $\langle \mathcal{B}, h_0, ..., h_{n-1} \rangle \in K_p^n$ such that $\mathcal{A} \leq \mathcal{B}, f_0 \subseteq h_0, ..., f_{n-1} \subseteq h_{n-1}$ and $h_0, ..., h_{n-1} \in Aut(\mathcal{B})$;

(2) There exists a weakly generic n-tuple $\langle g_0, ..., g_{n-1} \rangle \in Aut(\mathcal{M}_K)^n$.

Recall, that according to Theorem 1.1 of [9], $Aut(\mathcal{M}_K)^n$ contains an element g whose conjugacy class is dense iff K_p^n satisfies the Joint Embedding property. Hence, in Lemma 4.2, (2) implies (1) without assuming that K satisfies *UJEP* (or without assuming, that K_p satisfies the Joint Embedding Property).

In the converse direction however, if K_p^n has the Joint Embedding Property, then Theorem 1.1 of [9] does not imply (2): g is not necessarily weakly generic, because the orbits of g may not remain finite.

As we mentioned, originally, the problem of existence of weakly generic automorphisms was motivated by model theoretic problems and it was treated as follows: first the extension property *EP* had been established for a certain class (by the method surveyed in the previous sections), and secondly, as a consequence of (a version of) Lemma 4.2 one concluded, that the appropriate Fraïssé limit has generic automorphisms.

In [12] the opposite approach has been examined: in that paper the authors first establish topological properties of the automorphism group (particularly, they show that certain subsets of $Aut(\mathcal{A})$ are nowhere dense) and as a corollary of these topological investigations they obtain the Extension Property, among others, for the class of all finite tournaments (see Theorem 3.13 of [12]).

Often it is much more easy to establish the existence of a single weakly generic automorphism than the existence of tuples of weakly generic automorphisms of arbitrary finite length. In fact, the combinatorial and group theoretic methods surveyed in the earlier sections support the feeling, that if there exists a pair $\langle g_0, g_1 \rangle$ of weakly generic automorphisms, then there exist tuples of weakly generic automorphisms of arbitrary finite length. The aim of [13] is to make this impression more precise. Recall, that a topological group G has *topological rank* at most r iff there exists a dense subgroup G_0 of G such that G_0 can be generated by r elements. By Theorem 2.11 of [13] if $Aut(\mathcal{A})$ has finite topological rank r (and satisfies a further, mild technical condition inspired by *UJEP*) then the existence of a weakly generic tuple in $Aut(\mathcal{A})^r$ implies the existence of weakly generic tuples in $Aut(\mathcal{A})^n$ for all natural number $n \geq 1$. As a corollary of this result, in Theorem 3.2 of [13] it was shown that if $\mathcal{A}$ is a countable model of an $\aleph_0$-categorical, simple theory in which all types over the empty set are stationary and $\mathcal{A}$ has a pair of weakly generic automorphisms then it has tuples of weakly generic automorphisms of arbitrary finite length. In order to obtain this result, at the technical level group theoretic and topological methods had been combined.

Acknowledgements. We are very grateful to the anonymous referee for his/her valuable remarks.

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