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Fibrewise Partial Groups

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Abstract. In this paper, we introduce the concept of fibrewise partial group and discuss some of its basic properties. Also, we introduce the product and the quotient in fibrewise partial groups.

1. Introduction

The fibrewise viewpoint is standard in the theory of fibre bundles. However, it has been recognized only recently that the same viewpoint is also of great value in other theories, such as general topology. Let B be a set. Then, a fibrewise set over B consists of a set X together with a map $p : X \to B$, called the projection, and B is called the base set. For each b of B the fibre over b is the subset $X_b = p^{-1}(b)$ of X. In [6] A. E. Radwan and et al introduced the notion of fibrewise group and discussed the properties of this concept. Also, they introduced the quotients in fibrewise group and the direct product of fibrewise groups. In the present paper, we introduced the notion of fibrewise partial group and discussed the quotients in fibrewise partial group and the direct product of fibrewise partial groups.

2. Preliminaries

We collect for sake of reference the needed definitions and results appeared in the given references.

Definition 2.1. *[4] Let S be a semigroup. Then, $x \in S$ is called an idempotent element if $x \cdot x = x$. The set of all idempotent elements in S is denoted by $E(S)$.*

Definition 2.2. *[2, 3] Let S be a semigroup and $x \in S$. Then, an element $e \in S$ is called a partial identity of x if:*

(i) $ex = xe = x$,

(ii) *If $e'x = xe' = x$, for some $e' \in S$, then $ee' = e'e = e$.*

Theorem 2.1. *[2, 3] Let S be a semigroup. Then,*

(i) *If $x \in S$ has a partial identity, then it is unique*

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(ii) $E(S)$ is the set of all partial identities of the elements of S.

We will denote by e_x the partial identity of the element $x \in S$.

Definition 2.3. *[2, 3] Let S be a semigroup and $x \in S$ has a partial identity element e_x. Then, $y \in S$ is called a partial inverse of x if:*

(i) $xy = yx = e_x$,

(ii) $e_x y = y e_x = y$.

We will denote by x^{-1} the partial inverse of $x \in S$.

Definition 2.4. *[2] A semigroup S is called a partial group if:*

(i) Every $x \in S$ has a partial identity e_x

(ii) Every $x \in S$ has a partial inverse x^{-1}

(iii) The map $e_S : S \to S, x \mapsto e_x$ is a semigroup homomorphism

(iv) The map $\gamma : S \to S, x \mapsto x^{-1}$ is a semigroup antihomomorphism.

So, every group is a partial group.

Definition 2.5. *[1] Let S be a partial group. Then,a subsemigroup N of S is called a subpartial group, if $\forall\ x \in N$, we have that $x^{-1}, e_x \in N$.*

Definition 2.6. *[4] Let S be a partial group and $x \in S$. Then, we define $S_x = \{y \in S : e_x = e_y\}$.*

Theorem 2.2. *[2] Let S be a partial group and $x \in S$. Then,*

(i) S_x is a maximal subgroup of S which has identity e_x

(ii) $S = \bigcup\{S_x : x \in S\}$.

Definition 2.7. *[1] Let N be a subpartial group of S. Then, N is called normal (denoted by $N \trianglelefteq S$) if it is wide ($E(S) \subseteq N$) and $xyx^{-1} \in N$, $\forall\ x \in S,\ y \in N$.*

Corollary 2.3. *[2] Every partial group is a disjoint union of a family of groups.*

Definition 2.8. *[2] Let S and T be partial groups. Then, the map $\varphi : S \to T$ is called a partial group homomorphism if $\varphi(xy) = \varphi(x)\varphi(y), \forall x, y \in S$.*

Definition 2.9. *[2] Let $\varphi : S \to T$ be a partial group homomorphism. Then, we define $\ker\varphi = \{x \in S :\ \varphi(x) = e_{\varphi(x)}\}$ and $Im\varphi = \{\varphi(x) :\ x \in S\}$.*

Theorem 2.4. *[2] Let $\varphi : S \to T$ be a partial group homomorphism and $x \in S$. Then,*

(i) $\varphi(e_x) = e_{\varphi(x)}$.

(ii) $\varphi(x^{-1}) = \varphi(x)^{-1}$.

(iii) $\ker\varphi$ is a subpartial group of S.

(iv) $Im\varphi$ is a subpartial group of T.

(v) $\varphi(S_x)$ is a subpartial group of $T_{\varphi(x)}$.

(vi) $\varphi^{-1}(T_{e_x})$ *is a subpartial group of S.*

Definition 2.10. *[5] Let B be a set. Then, a fibrewise set over B consists of a set X together with a map* $p : X \to B$*, called the projection, and B is called the base set.*

For each $b \in B$, the fiber over b is the subset $X_b = p^{-1}(b)$ of X. Also, for each subset W of B, we regard $X_W = p^{-1}(W)$ is a fibrewise set over W with the projection determined by p.

Definition 2.11. *[6] Let X and Y be fibrewise sets over B, with projections p and q, respectively. Then, a map* $\varphi : X \to Y$ *is called fibrewise map if* $q\varphi = p$*. In other words,* $\varphi(X_b) \subseteq Y_b$*, for eachb* $\in B$.

Definition 2.12. *[6] Let B be a group. Then, a fibrewise group over B is a fibrewise set G with any binary operation makes G a group such that the projection* $p : G \to B$ *is a group homomorphism.*

For Example: The product $G \times B$ can be a regarded as a fibrewise group over B, using the second projection.

3. Fibrewise Partial Groups

In this section, we introduce the notion of fibrewise partial groups and discuss some of their basic properties.

Definition 3.1. *Let K be a partial group. Then, a fibrewise partial group over K is a fibrewise set S with any binary operation makes S a partial group such that the projection* $p : S \to K$ *is a partial group homomorphism.*

So, every fibrewise group is a fibrewise partial group.

Definition 3.2. *Let S be a fibrewise partial group over K. Then, any subpartial group N of S is a fibrewise partial group over K with projection* $\acute{p} = p|N : N \to K$*, we call this partial group a fibrewise subpartial group of S over K.*

Theorem 3.1. *Let S be a fibrewise partial group over K. Then,*

(i) *The fibre* S_e *over the partial identity* $e \in E(T)$*, is a fibrewise subpartial group of S.*

(ii) *If N is a subpartial group of the partial group K, then the set* S_N *is a fibrewise subpartial group of the fibrewise partial group S.*

(iii) *If* $x \in S_y, y \in K$*, then* $x^{-1} \in S_{y^{-1}}$.

(iv) *If* $x, y \in S_z; x \neq y$*, then* $xy^{-1} \in S_{e_z}$*, where* $e_z \in E(T)$.

Proof. (i) Since S is a fibrewise partial group over K , then the map $p : S \to K$ is a partial group homomorphism. So, $p^{-1}(e) = S_e$ is a subpartial group of S, and then S_e is a fibrewise subpartial group of S with the projection $\acute{p} = p|S_e : S_e \to K$

(ii) Similar above.

(iii) Since $x \in S_y$, then $e_x = e_y$. Since S is a fibrewise partial group over K, then the map $p : S \to K$ is a partial group homomorphism. So, $e_{x^{-1}} = e_x = e_y = e_{y^{-1}}$. Hence, $x^{-1} \in S_{y^{-1}}$.

(iv) Since $x, y \in S_z; x \neq y$, then $e_x = e_y = e_z$. Since S is a fibrewise partial group over K, then the map $p : S \to K$ is a partial group homomorphism. So, $p(xy^{-1}) = p(x)p(y^{-1}) = e_z e_z = e_z$. Hence, $xy^{-1} \in S_z$. □

Definition 3.3. *Let S and T be fibrewise partial groups over K. Then, the map* $\varphi : S \to T$ *is called a fibrewise partial group homomorphism if* φ *is a partial group homomorphism and a fibrewise map.*

Definition 3.4. *A bijective fibrewise partial group homomorphism is called a fibrewise isomorphism.*

Theorem 3.2. *Let S be a fibrewise partial group over K and N be a normal subpatial fibrewise of S. Then, the partial group S/N is a fibrewise partial group over K, with the projection $q : S/N \to B$ such that $q\rho_N = p$, where $\rho_N : S \to S/N$ is the quotient map.*

Proof. Let $xN, yN) \in S/N$. Now, $q(xNyN) = q(\rho_N(x)\rho_N(y)) = q(\rho_N(xy)) = p(xy) = p(x)p(y) = q\rho_N(x)q\rho_N(y) = q(xN)q(yN)$. That means that q is a partial group homomorphism. Hence, S/N is a fibrewise partial group over K, with the projection $q : S/N \to K$. □

Lemma 3.3. *Let S and T be fibrewise partial groups over K, with projections p and q, respectively and let $\varphi : S \to T$ be a fibrewise map. Then, $\varphi(kerp) \subseteq kerq$.*

Proof. Let $x \in kerp$. Then, $p(x) = e_{p(x)}$. Then, $(p\varphi)(x) = e_{p(x)}$, and so $\varphi(x) \in kerq$. Hence, $\varphi(kerp) \subseteq kerq$. □

Proposition 3.4. *Let S and T be fibrewise partial groups over K, with projections p and q, respectively and let $\varphi : S \to T$ be a fibrewise map. Then:*

(i) If q is injective, then φ is a fibrewise partial group homomorphism, and consequently:

(ii) $\varphi(e_x) = e_{\varphi(x)}$, where e_x is the partial identity of the element $x \in S$.

(iii) $\varphi(kerp) = E(T)$.

Proof. (i) Since S is a fibrewise partial group over K, then $p(xy) = p(x)p(y)$. Since $\varphi : S \to T$ is a fibrewise map, then $q\varphi = p$. Now,$(q\varphi)(xy) = (q\varphi)(x)(q\varphi)(y)$. So, $q(\varphi(xy)) = q(\varphi(x))q(\varphi(y))$. Since q is a partial group homomorphism and injective, then $\varphi(xy) = \varphi(x)\varphi(y)$. Hence, φ is a fibrewise partial group homomorphism.

(ii) Since S is a fibrewise partial group over K, with the projection $p : S \to K$, then p is a partial group homomorphism, and so $p(e_x) = e_{p(x)}$. Since φ is a fibrewise map, then $q\varphi = p$, and so $q(\varphi(e_x)) = e_{q(\varphi(x))}$. Since T is a fibrewise partial group over K, with the projection $q : T \to K$ and φ is injective, then $\varphi(e_x) = e_{\varphi(x)}$.

(iii) $\varphi(kerp) = \{\varphi(x) : x \in kerp\} = \{\varphi(x) : p(x) = e_{p(x)}\} = E(T)$.
□

Proposition 3.5. *Let S and T be fibrewise partial groups over K, with projections p and q, respectively and let $\varphi : S \to T$ be a fibrewise map. If q is injective, then:*

(i) If p is bijective and q is injective, then T is abelian if S is abelian.

(ii) If N is a fibrewise subpartial group of S, then $\varphi(N)$ is a fibrewise subpartial group of T.

(iii) If $\acute{N}$ is a fibrewise subpartial group of T, then $\varphi^{-1}(\acute{N})$ is a fibrewise subpartial group of S.

(iv) If N is a fibrewise normal subpartial group of S, then $\varphi(N)$ is a fibrewise normal subpartial group of T.

(v) If p and q are bijective, then φ is a fibrewise isomorphism.

Proof. (i) Let $x, y \in T$. Then, $q(x), q(y) \in K$. So, $p^{-1}(q(x)), p^{-1}(q(y)) \in S$. Since S is an abelian partial group, then $p^{-1}(q(x))p^{-1}(q(y)) = p^{-1}(q(y))p^{-1}(q(x))$. That means that $p^{-1}(q(x)q(y)) = p^{-1}(q(y)(q(x))$. So, $q(x)q(y) = q(y)q(x)$. Since q is a partial group homomorphism and injective, then $xy = yx$. Hence, T is abelian.

(ii) Since $\acute{q} = q|\varphi(N) : \varphi(N) \to K$ is a partial group homomorphism, then $\varphi(N)$ is a fibrewise subpartial group of T.

(iii) Similar (ii)

(iv) Let $z \in T$ and $y \in \varphi(N)$. Then, $zyz^{-1} \in \varphi(N)$.

(v) It is obvious.
□

4. Fibrewise Direct Product of Fibrewise Partial Groups.

In this section, we introduce a fibrewise direct product of fibrewise partial groups.

Definition 4.1. *Let S and T be fibrewise partial groups over K, with projections p and q, respectively. Then, the fibrewise direct product of S and T, is the fibrewise product $S \times_K T = \{(x, y) : p(x) = q(y), x \in S, y \in T\}$.*

We define the operation on $S \times_K T$ by $(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$.

Theorem 4.1. *Let S and T be fibrewise partial groups over K, with projections p and q, respectively. Then, the fibrewise direct product $S \times_K T$ over K, is a partial group. Moreover, If K is an abelian partial group, then $S \times_K T$ is fibrewise partial group over K with the projection $\psi : S \times_K T \to K, (x, y) \mapsto p(x)q(y)$.*

Proof. Firstly, we want to show that $S \times_K T$ is a partial group. Let $(x_1, y_1), (x_2, y_2) \in S \times_K T$. Then, $p(x_1) = q(y_1)$ and $p(x_2) = q(y_2)$. Since $p(x_1x_2) = p(x_1)p(x_2) = q(y_1)q(y_2) = q(y_1y_2)$, then $(x_1, y_1)(x_2, y_2) \in S \times_K T$. It clear that $S \times_K T$ is associative.

The element $e_x = (e_{x_1}, e_{y_1})$ is the partial identity of the element $x = (x_1, y_1)$. The partial inverse of $x = (x_1, y_1) \in S \times_K T$ is $x^{-1} = (x_1^{-1}, y_1^{-1}) \in S \times_K T$.

The map $e_{S\times_K T} : S \times_K T \to S \times_K T, (x, y) \mapsto (e_x, e_y)$ is a semigroup homomorphism and also the map $\gamma_{S\times_K T} : S \times_B T \to S \times_B T, (x, y) \mapsto (x^{-1}, y^{-1})$ is a semigroup antihomomorphism.

So, $S \times_K T$ is a partial group. The map $\psi : S \times_K T \to K$ is a partial group homomorphism since $\psi((x_1, y_1)(x_2, y_2)) = \psi(x_1, y_1)\psi(x_2, y_2)$. □

5. Conclusion

This paper has covered about the same material as in [6], but with different approach. We introduced a new concept, called fibrewise partial group and introduced the quotient and product in this concept.

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