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Quality Control for Feedback M/M/1/N Queue with Balking and Retention of Reneged Customers

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Abstract. In this paper, we provide a simple and direct approach to determine quality control policy and optimization of the Markovian feedback customers that has single-server and limited system capacity under steady-state situation adding the concepts of balking and retention of reneged customers. We use the iterative method to obtain the probability that there are n customers in the system, the probability of empty system and some performance measures. Some important queueing systems are derived as special cases and the optimization of the model is performed.

1. Introduction

Quality control for the feedback queuing models with balking and retention of reneged customers has been well studied in the last decades and widely used in a variety of academic disciplines such as communication, network models, production and manufacturing systems, and numerous other fields of science. Tapiero and Hsu [18] studied the infinite single-channel Markovian queue under quality control technique and feedback process. Also, Hus and Tapiero [7] provided a conceptual frame work for quality control in queuing facilities and they derived the specific results relating to the M/G/1 queue. Kotb et. al. [13] addressed and concerned statistical sampling design in the quality control process for the queuing system of units, adding the concept of balking via steady–state situation. Ancker and Gafarian [3] and [4] studied some queuing problems with balking, reneging and performed its steady-state analysis. A single-server, finite capacity Markovian feedback queuing system with retention of reneged customers and balking is performed by Kumar and Sharma [15] in which the inter-arrival, service and reneging times are assumed to be exponentially distributed. Other related studies are treated by Hsu and Tapiero [8][9][10], Fan-Orzechowski and Feinberg [5], Mcgrath and Gross [17], Klimenok and Dudina [12] and Jau-Chuan Ke et. al. [11].

The goal of this research is to study the optimization of various parameters in the quality control single-server truncated Markovian feedback queueing system of units, adding the balking and retention of reneged customers concepts via steady-state conditions. The probability that there are n units in the system and the probability of empty system are obtained. Some performance measures have been computed. Some important particular cases of the system have been deduced and discussed. Finally, the economic optimization analysis is performed to illustrate the numerical applications of the control process and conclusion is presented.

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2. Basic Notations and Assumptions

To derive the mathematical model of this problem, we use the following notations:
$p_n(t) \equiv$ Transient-state probability that there are n units in the system, both waiting and in service.
$p_0(t) \equiv$ Probability of empty system at time t.
$p_n \equiv$ Steady-state probability that there are n units in the system.
$p_0 \equiv$ Steady-state probability that there are no units in the system.
$n \equiv$ Number of units in the system, $0 \leq n \leq N$.
$N \equiv$ System capacity.
$\acute{N} \equiv$ The number of customers in steady-state.
$L = E(\acute{N}) \equiv$ Expected number of units in the system.
$L_q = E(\acute{N} - 1) \equiv$ Expected number of waiting units to be served where $\acute{N} > 0$ (i.e., the system will be able to present a service if there exists at least one customer in the system).
$L_s = L - L_q \equiv$ Average number of occupied (units in) service.
$W \equiv$ Expected waiting time in the system.
$W_q \equiv$ Expected waiting time in the queue.
$W_s = L - W_q \equiv$ Expected service time.
$R_r \equiv$ Average rate of reneging.
$R_R \equiv$ Average rate of retention.
$C_s \equiv$ Service cost per unit time.
$C_h \equiv$ Holding cost per unit time.
$C_l \equiv$ Cost associated with each lost unit.
$C_r \equiv$ Cost associated with each reneged unit.
$C_R \equiv$ Retaining a reneged customer cost.
$C_{s1} \equiv$ Serving a feedback customer cost.
$R \equiv$ Revenue earned by providing service to a customer.
$TEC \equiv$ Total expected cost per unit time of the system.
$TER \equiv$ Total expected revenue per unit time of the system.
$TEP \equiv$ Total expected profit per unit time of the system.

The following assumptions describe the mathematical model:

1. Customers arrive at the service facility one by one according to a Poisson process with rate $\lambda(> 0)$ and mean inter-arrival time is $1/\lambda$.
2. Service times of the customers are independent and identically distributed (iid) exponential random variables with rate $\mu(> 0)$ and mean service time $1/\mu$, $0 < \lambda < \mu$.
3. Customers are served according to first-come, first-served (FCFS) discipline.
4. After completion of each service the customer either joins at the end of the original queue as a feedback customer with probability $(1 - q)$, the probability that a processing job is defective or departure the system with probability q (reliability parameter),$0 \leq q < 1$.
5. For the feedback situation, let ω_n be the random event of inspecting a unit, such that $\omega_n = 1$ reflects the event that a unit inspected when there are n jobs in the system and $\omega_n = 0$, otherwise, $0 \leq n \leq N$.
6. After joining the queue each customer will wait a certain length of time for the service to begin with probability $(1 - p)$. If the service have not began with them, the customer will get impatient and leave the queue without getting service with probability $(n - 1)p\alpha$, $2 \leq n \leq N$.
7. On arrival a unit may join the queue with probability β and may balk with probability $1 - \beta$ when n units are ahead of him, $0 \leq n < 1$ for $1 \leq n \leq N - 1$ and $\beta = 1$ otherwise.

3. Model Formulation

Applying Markov conditions and using the above assumptions, we obtain the following system of probability differential–difference equations:

$$\acute{p}_0(t) = -\lambda p_0(t) + \mu q \omega_1 p_1(t), \qquad n = 0 \tag{1}$$

$$\acute{p}_1(t) = -(\beta\lambda + \mu q\omega_1)p_1(t) + \lambda p_0(t) + (\mu q\omega_2 + \alpha p)p_2(t), \qquad n = 1 \tag{2}$$

$$\acute{p}_n(t) = -(\beta\lambda + \mu q\omega_n + (n-1)\alpha p)p_n(t) + \beta\lambda p_{n-1}(t) + (\mu q\omega_{n+1} + n\alpha p)p_{n+1}(t), \qquad 1 < n < N \tag{3}$$

$$\acute{p}_N(t) = -(\mu q\omega_N + (N-1)\alpha p)p_N(t) + \beta\lambda p_{N-1}(t), \qquad n = N \tag{4}$$

4. Steady-State Solution

As in the usual technique, the steady-state probability difference equations are given by:

$$-\lambda p_0 + \mu q\omega_1 p_1 = 0, \qquad n = 0 \tag{5}$$

$$-(\beta\lambda + \mu q\omega_1)p_1 + \lambda p_0 + (\mu q\omega_2 + \alpha p)p_2 = 0, \qquad n = 1 \tag{6}$$

$$-(\beta\lambda + \mu q\omega_n + (n-1)\alpha p)p_n + \beta\lambda p_{n-1} + (\mu q\omega_{n+1} + n\alpha p)p_{n+1} = 0, \qquad 1 < n < N \tag{7}$$

$$-(\mu q\omega_N + (N-1)\alpha p)p_N + \beta\lambda p_{N-1} = 0, \qquad n = N \tag{8}$$

Solving these probability difference equations iteratively, we get:
$(\mu q\omega_{n+1} + n\alpha p)p_{n+1} - \beta\lambda p_n = (\mu q\omega_n + (n-1)\alpha p)p_n - \beta\lambda p_{n-1} = ... = (\mu q\omega_2 + \alpha p)p_2 - \beta\lambda p_1 = \mu q\omega_1 p_1 - \lambda p_0 = 0$, then:

$$p_n = \frac{\beta\lambda}{\mu q\omega_n + (n-1)\alpha p} p_{n-1}$$

generally, we obtain:

$$p_n = \frac{\lambda(\beta\lambda)^{n-1}}{[\mu q\omega_n + (n-1)\alpha p] \times [\mu q\omega_{n-1} + (n-2)\alpha p] \times [\mu q\omega_2 + \alpha p] \times [\mu q\omega_1]} p_0$$

thus the probability that there are n units in the system is given by:

$$p_n = \begin{cases} p_0, & n = 0, \\ \frac{\delta^n}{\beta \prod_{i=0}^{n-1}(\gamma\omega_{i+1} + i)}, & 1 \le n \le N, \end{cases} \tag{9}$$

where: $\delta = \frac{\beta\lambda}{\alpha p}$ and $\gamma = \frac{\mu q}{\alpha p}$.
To find the probability that no units are in the service department p_0, we use the boundary condition $\sum_{n=0}^{N} p_n = 1$. Then, we get:
$1 = \sum_{n=0}^{N} p_n = p_0 + \sum_{n=1}^{N} p_n = p_0 + \sum_{n=1}^{N} \frac{\delta^n}{\beta \prod_{i=0}^{n-1}(\gamma\omega_{i+1} + i)}$,
therefore:

$$p_0^{-1} = 1 + \frac{1}{\beta} \sum_{n=1}^{N} \frac{\delta^n}{\prod_{i=0}^{n-1}(\gamma\omega_{i+1} + i)} \tag{10}$$

5. Performance Measures

Using Abou El-Ata's theorems of the moments [1] and [2], which stated that:
$L = \sum_{n=1}^{\infty} np_n = -\lambda \frac{\partial \ln p_0}{\partial\lambda}$
then, the expected number of units in the system is given by:

$$L = E(\acute{N}) = -\lambda \frac{\partial \ln p_0}{\partial\lambda} = \lambda p_0 \frac{\partial p_0^{-1}}{\partial\lambda} = \frac{p_0}{\beta} \sum_{n=1}^{N} \frac{n\delta^n}{\prod_{i=0}^{n-1}(\gamma\omega_{i+1} + i)} \tag{11}$$

where $\acute{N}$ is the number of customers in the steady-state.
Also, the expected number of units in the queue is:

$$L_q = L - (1 - p_0) \tag{12}$$

The average number of occupied service is:

$$L_s = L - L_q = 1 - p_0 \tag{13}$$

The expected waiting time in the system is:

$$W = \frac{L}{\lambda} \tag{14}$$

and, the expected waiting time in the queue is:

$$W_q = \frac{L_q}{\lambda} \tag{15}$$

Relations (14) and (15) are called Littel's formulae [16].
Therefore, the expected service time is:

$$W_s = W - W_q = \frac{L - L_q}{\lambda} = \frac{1 - p_0}{\lambda} \tag{16}$$

where Little's formulae are valid for an $M/M/1/N$ except for the particular case for an $M/M/1/1$. The average reneging rate is given by:

$$R_r = \sum_{n=1}^{N} (n-1)\alpha p p_n \tag{17}$$

and the average retention rate is:

$$R_R = \sum_{n=1}^{N} (n-1)\alpha q p_n \tag{18}$$

where p_n and p_0 are given in relations (9) and (10) respectively.

6. Special Cases

Some important queuing systems are derived as special cases of this model as follows:

Case 1 Let $\beta = 1$, this is the quality control queue: $M/M/1/N$ with feedback and retention of reneged customers.

Case 2 Put $\alpha = 0$, the queuing system reduces to quality control for $M/M/1/N$ queue with feedback and balking.

Case 3 When $p = 1$, this is the quality control queue: $M/M/1/N$ with feedback, balking and reneging.

Case 4 Assume $q = 1$, the queuing system reduces to quality control for $M/M/1/N$ queue with balking and retention of reneged customers.

Case 5 When $\omega_n = 1$ the queuing system reduces to quality control for feedback $M/M/1/N$ queue via balking and retention of reneged customers concepts.

Case 6 If $N \to \infty$, this is the quality control queue: M/M/1 with feedback, balking and retention of reneged customers.

Case 7 Let $\alpha = 0$ and $N \to \infty$, the queuing system reduces to quality control for $M/M/1$ queue, which is the same Tapierot and Hsu work [18].

Case 8 If $q = \omega_n = 1$ and $\alpha = 0$, we get the simple queue: $M/M/1/N$ without any concepts, which is the same Gross and Harris work [6].

7. Economic Model Analysis

This section is devoted to study the economic analysis of an $M/M/1/N$ model under feedback, balking and retention of reneged customers concepts. The total expected cost per unit time, the total expected revenue per unit time and the total expected profit per unit time functions in terms of λ, μ and N parameters are obtained.
The total expected cost is composed of five components (the expected service cost, the expected holding cost, the expected associated cost with each lost unit, the expected associated cost with each reneged unit and the expected retaining a reneged customer cost) according to the basic notations and assumptions of the model which provided by Kumar et. al. [14]:
$TEC = \mu(C_s + P_1 C_{s1}) + C_h l + C_l \lambda p N + C_r R_r + C_R R_R$

$$= \mu(C_s + \frac{\lambda C_{s1}}{\mu q \omega_1} p_0) + \frac{C_h}{\beta} \sum_{n=1}^{N} \frac{n\delta^n}{\prod_{i=0}^{n-1}(\gamma\omega_{i+1} + i)} p_0 + \frac{C_l \lambda \delta^N}{\beta \prod_{i=0}^{N-1}(\gamma\omega_{i+1} + i)} p_0 + (C_r p + C_R q) \frac{\alpha}{\beta} \sum_{n=1}^{N} \frac{(n-1)\delta^n}{\prod_{i=0}^{n-1}(\gamma\omega_{i+1} + i)} p_0 \quad (19)$$

thus, the total expected profit of the system is equal to the total expected revenue minus the total expected cost, i.e.,:

$$TEP = TER - TEC \quad (20)$$

where:

$$TER = R\mu(1 - p_0) \quad (21)$$

and TEC is given in relation (19).

8. Model Optimization

The economic model analysis is performed numerically by using the total expected profit, the total expected revenue and the total expected cost functions and the results discussed via optimization technique.

8.1. Relation Between L and α

Assume the parameters of the queueing model are given by: $\lambda = 4, \mu = 3, N = 4, p = 0.1, q = 0.9, \beta = 0.25, 0.50, 1$ and $\omega_i = 0$ or 1, $i = 2, 4$ or $i = 1, 3$ respectively in relation (11), we get Table 1 for some values of β as: Solution of the system may be determined more readily by plotting L against α for some values of β as given in Figure 1.

Table 1: The values of L For different values of $\beta = 0.25, 0.5, 1$

α	L at $\beta = 0.25$	L at $\beta = 0.50$	L at $\beta = 1$
0.05	3.908215707	3.972583594	3.991214503
0.06	3.890814894	3.967179081	3.989463849
0.07	3.873712476	3.961800335	3.987715335
0.08	3.856900382	3.956447143	3.985968953
0.09	3.840370834	3.951119303	3.984224700
0.10	3.824116337	3.945816613	3.982482564
0.11	3.808129647	3.940538871	3.980742550
0.12	3.792403780	3.935285879	3.979004644
0.13	3.776931992	3.930057444	3.977268846
0.14	3.761707769	3.924853369	3.975535147
0.15	3.746724820	3.919673460	3.973803547

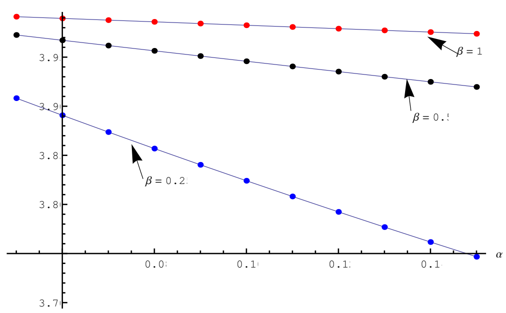

Figure 1: The relationship between L and α when $\beta = 0.25$ and $\beta = 1$

Table 2: Optimal system capacity

N	1	2	3*	4	5	6	7	8	9	10
TEP	396.357	401.128	454.681	432.120	454.184	431.489	451.666	430.465	448.977	429.343

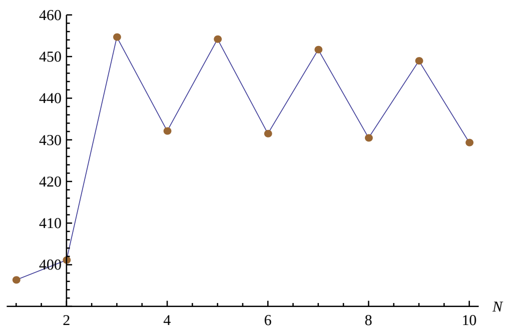

Figure 2: TEP as a function of N

Table 3: Optimal values TER^*, TEC^* and TEP^*

		$\mu = 3$				$\mu = \mu^*$			
q	C_R	TER	TEC	TEP	$N = N^*$	μ^*	TER^*	TEC^*	TEP^*
0.2	8	300	63.058	236.942	6	4.0336	403.360	67.184	336.175
0.3	12	300	64.243	235.756	6	4.3571	435.710	69.656	366.053
0.4	14	299.99	60.993	239.006	5	4.8562	485.619	67.870	418.750
0.5	20	299.99	63.754	236.245	4	4.3156	431.559	69.002	362.557
0.6	25	299.99	64.644	235.355	4	4.7304	473.039	71.543	401.496
0.7*	32	299.99	59.728	240.270	3	5.4472	544.716	66.409	478.307
0.8	36	299.99	59.602	240.395	3	5.3135	531.346	65.587	465.759
0.9	40	299.98	64.753	235.235	2	5.2453	524.511	73.735	450.775
1	45	299.98	65.053	234.936	2	5.1345	513.431	73.591	439.840

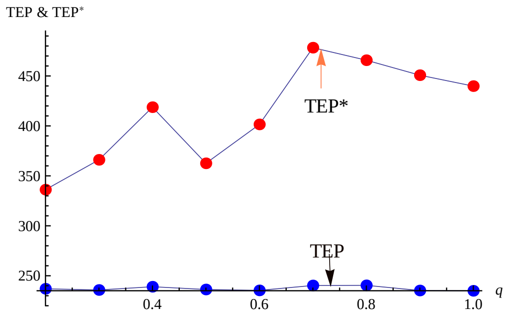

Figure 3: TEP andTEP^* as a function of q

8.2. *Optimal System Capacity and Total Expected Profit*

The decision variables N^* and μ^* should be computed to confirm that the the total expected profit of the system is maximize. Assuming the parameters of the queueing model are: $\lambda = 4, \alpha = 0.1, \beta = 0.25, p = 0.6, q = 0.9, C_h = 3, C_l = 12, C_R = 25, C_r = 8, C_s = 4, C_{s1} = 2$ and $\omega_i = 0$ or 1, $i = 2, 4, 6, 8, 12$ or $i = 1, 3, 5, 7, 9, 11$ respectively.

For different values of N, the optimal results of N^* and TEP^* are shown in Table 2.

Solution of the model may be determined more readily by plotting TEP against N for some values of system parameters as given in Figure 2.

An optimization technique was devised to determine the optimal solution for the model, system capacity ($N^* = 3$), service rate ($\mu^* = 5.0195473$) and maximum total expected profit ($TEP = 454.681$) instead of some previous models, using MAPLE and MATHEMATICA programs.

8.3. *Optimization of the Total Costs*

We shall compute the three decision variables N^*, q^* and μ^* whose values are to be determined to maximize the total costs TER^*, TEC^* and TEP^*, for some different values of q and C_R.

It follows that, the optimal values of $N^* = 3$, $q^* = 0.7$, $\mu^* = 5.4472$ and maximum total costs $TER^* = 544.716$, $TEC^* = 66.409$ and $TEP^* = 478.307$ are shown in Table 3:

Results of the comparative analysis of the total costs (TER^*, TEC^* and TEP^*) with respect to the reliability parameter (q) are given in Figure 3.

9. Conclusion and Future Works

This paper investigated how feedback, balking, retention of reneged customers concepts and quality control approach affect the queueing model. An analytical solution of the quality control for feedback

$M/M/1/N$ model is derived using steady-state situation and iterative approach. In addition, eight queueing models are obtained as particular cases. An economic queueing model analysis was devised to determine the total expected profit of the system and was performed to compute the optimal solution for the queueing model, system capacity, service rate, reliability parameter and total expected costs.
Possible future extension of this work was include state-dependent service rate and transient behavior of the model and introducing another concepts as a decision variables.

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