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Some Topological Structures of Fractals and their Related Graphs

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Abstract. The aim of this paper is to introduce a topological model of fractals. Self similar fractals will be approached as inverse limit of finite one dimensional topological spaces with alpha continuous bonding functions. The second approach is to investigate topological graphs in terms nano topological spaces for Lellis Thivagar. From these approximations, the dynamics of Julia sets as a special type of self similar fractals will be studied and some physical properties of fractals through their nano topological graphs will be applied.

1. Introduction and Preliminaries

The Nobel prize 2016 in physics was gifted to three scientists in phase transitions and topological phases of matter, this event has directed the attention to the need of more knowledge about the topology. Topology is a branch of mathematics whose concepts exist not only in almost all branches of mathematics, but also in many real life applications and concerned with all questions directly or indirectly related to continuity. Many topologists suggested topological models in biology [10–12] and in medicine [28].

Graph theory [5, 6, 42] has recently emerged as a subject in its own right as well as being an important mathematical tool an such diverse subjects as operational research, chemistry, sociology and genetics.

A self-similar set is a set can be decomposed into subsets which are similar copies of the whole set. Cantor set, the Koch curve and the Sierpiński gasket are the first known examples of fractal sets. The basic ideas leading to the analysis of self-similar sets were originated in 1946 by Moran [26], and developed by Mandelbrot et al., in numerous papers [3, 23–25, 39, 40]and Hutchinson [13]. Hata [14] investigated the topological structure of self-similar sets and analyzed many classical sets and curves through the notion of self-similarity. El Atik [8] represented some of self-similar fractals by finite topological spaces. Barnsley; Hutchinson et al., [4] established properties of a new type of fractal which has partial self similarity at all scales. Julia sets of a quadratic polynomial has one critical point. Peitgen, Douady and Hubbard [33] studied the polynomial of degree 2 in a complex variable, specifically, $p_c(z) = z^2 + c$ for z and c in $\mathbb{C}$. For any such polynomial, the filled-in Julia set is defined as the sets of points z with bounded orbits under iteration. The Julia set is the boundary of the filled-in Julia set and denoted by J_c. Julia set and filled-in Julia set are connected if and only if the only critical point 0 has bounded orbit; otherwise, these sets coincide and are a Cantor set. Kameyama [17] proved that the self-similar sets are homeomorphic to quotient spaces of the symbolic dynamics with some equivalent relations. He studied the topology of the quotient spaces of the

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symbolic dynamics, which may not be homeomorphic to self-similar sets. Also, Kameyama [18], proved a self-similar set K is homeomorphic to a Julia set if K is embedded in a sphere S^2 such that the dynamics of K can be extended to a postcritically finite branched covering. El Atik [7] et al., studied and investigated some properties of finite Kolmogorov or T_0 spaces with the existence of an ordered relation between their minimal neighborhoods. Sokół[41] gave a sufficient condition for function to be α-starlike function and some of its applications. García-Arenas and Sánchez-Granero [2] introduced and studied the concept of directed fractal structure which is a generalization of the concept of fractal structure. A subset A of X is said to be α-open [29] if $A \subset Int(Cl(Int(A)))$. The complement of an α-open set is called α-closed [29]. The family of all α-open sets of X is denoted by $\alpha(X)$. The family of all α-open sets of X containing a point $x \in X$ is denoted by $\alpha(X, x)$. The intersection of all α-closed sets of X containing A is called α-closure [29] of A and is denoted by $\alpha Cl(A)$. Each open set in a general topological space is α-open and the converse may not be true. An α-boundary [29] of a set U of a space X (abb. $\alpha B(U)$)is given by $\alpha B(U) = \alpha Cl(U) - \alpha Int(U)$.

Definition 1.1. *[20] Consider Figure 1. Let U be a nonempty finite set of objects called the universe and R be an*

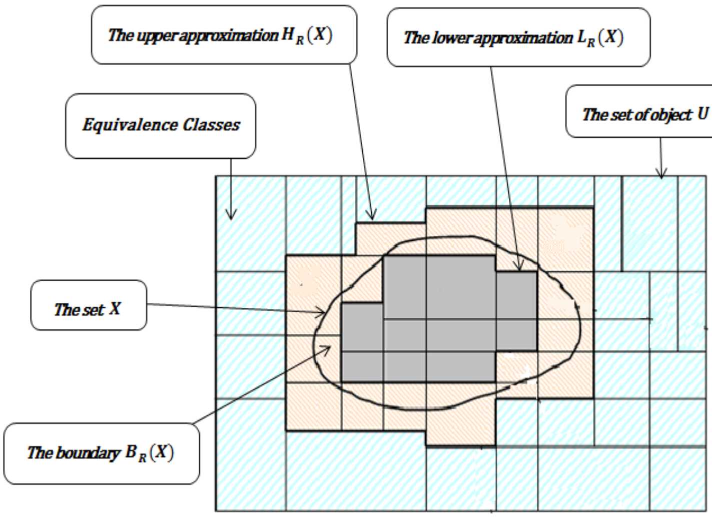

Figure 1: A rough set [34]

equivalence relation on U named the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$,

(i) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$, that is $L_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$ where $R(x)$ denotes the equivalence class determined by x.

(ii) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $H_R(X)$, that is $H_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$.

(iii) The boundary region of X with respect to R is the set of all objects, which can be classified neither as $-X$ nor as not X with respect to R and it is denoted by $B_R(X)$, that is $B_R(X) = H_R(X) - L_R(X)$.

According to Pawlak$^{'}s$ definition, X is called a rough set if $H_R(X) \neq L_R(X)$.

Definition 1.2. *[21] Let $G(V, E)$ be a graph, S be a subgraph of G and $R(v)$ be a relation of v in V. Then we define*

(i) The lower approximation operation induced by a graph as follows: $L : P(V(G)) \to P(V(G))$ such that $L_R(V(S)) = \bigcup_{v\in V(G)}\{v : R(v) \subseteq V(S)\}$;

(ii) The upper approximation operation induced by a graph as follows: $H : P(V(G)) \to P(V(G))$ *such that* $H_R(V(S)) = \bigcup_{v \in V(G)} \{R(v) : R(v) \cap V(S) \neq \phi\}$;

(iii) The boundary region is defines as $B_R(V(S)) = H_R(V(S)) - L_R(V(S))$.

Definition 1.3. *[21] Let* $G(V, E)$ *be a graph,* $R(v)$ *be a relation of* v *in* V *and* S *be a subgraph of* G*. Then* $\tau_R(V(S)) = \{V(G), \phi, L_R(V(S)), H_R(V(S)), B_R(V(S))\}$ *forms a topology on* $V(G)$ *called a nano topology on* $V(G)$ *with respect to* $V(S)$.

We call $(V(G), \tau_R(V(S)))$ a nano topological graph space.

Definition 1.4. *[27] A connected topological space is said to be tree-like if no two of its points are conjugated, that is, if for any two distinct points in the space there is a third point which separates them.*

Definition 1.5. *[16] A space* X *is said to be a connected ordered topological space (abb. COTS) if for every three point subset* Y *in* X*, there exists* $y \in Y$ *such that* Y *meets two connected component of* $X - \{y\}$.

Definition 1.6. *[8] If* $\{a\} = f_0(A) \cap f_1(A)$*, there are* $i_k, j_k \in \{0, 1\}$ *with*

$$\{a\} = f_0 \cdot f_{i_1} \cdots f_{i_n}(A) \cap f_1 \cdot f_{j_1} \cdots f_{j_n}(A) \text{ for } n = 1, 2, \cdots$$

Then two addresses $0\hat{\alpha} = 0i_1 \cdots i_n \cdots$ *and* $1\hat{\beta} = 1j_1 \cdots j_n \cdots$ *of a point* a *determine the topology of* A*. For Julia sets, we get* $\hat{\alpha} = \hat{\beta}$*, and so* $\hat{\alpha}$ *is said to be a kneading sequence.*

Definition 1.7. *[8] For a compact(not necessarily completely regular) space* X*. If* $\sim$ *is an equivalence relation on* X *defined by* $x \sim y \quad \text{iff} \quad f(x) = f(y)$ *for every* $f \in C(X)$ *where* $C(X)$ *is the set of all continuous functions from* X *onto* $\mathbb{R}$*. The quotient space* $X/\sim$ *is called a completely regular modification of* X.

Definition 1.8. *[30] A function* $f : X \to Y$ *is called:*

(i) α*-continuous if* $f^{-1}(U) \in \alpha(X)$*, for each open set* U *in* Y.

(ii) α*-open if* $f(V) \in \alpha(Y)$*, for each open set* V *in* X.

(iii) α*-closed if* $f(V) \in \alpha C(Y)$*, for each closed set* V *in* X.

In the present work, we suggest a new model of fractals in view point of finite topological spaces by the concept of α-open sets which introduced by Njastad [29] and the definition of Lellis Thivagar for nano topological spaces [20]. We study upper(lower) α-continuous multifunctions and its relation with other types of continuous multifunctions. Also, we focus on a self-similar set A with $A = A_0 \cup A_1$ and $A_0 \cap A_1$ is a singleton, specially, for a tree-like [19, 27] set as a special case fractal structure in the sense that it does not topological circles and give an algorithm which approach these types of fractals in the plane. Also, we represent Julia sets J_c as the inverse limit of an inverse system which consist of one dimensional topological spaces $\alpha(X_n)$ with bonding α-continuous functions. We study the dynamics of $\alpha(X_n)$ of Julia sets through upper(lower) α-continuous multifunctions from each space into itself.

2. One dimensional of α-Kolmogorov spaces

Definition 2.1. *[30] A space* X *is said to be:*

(i) α*-Kolmogorov or* αT_0 *if for every* $x, y \in X$, $x \neq y$*, there exist an* α*-open set* U *of* X *such that either* $x \in U$, $y \notin U$ *or* $x \notin U$, $y \in U$.

(ii) αT_2 *if for every* $x, y \in X$, $x \neq y$*, there exist two disjoint* α*-open sets* U *and* V *of* X *such that* $x \in U$, $y \in V$.

Definition 2.2. *In a space* X*, the minimal* α*-neighborhood (*α*-nbd) of a point* $x \in X$ *is given by* $\mathcal{U}_x = \bigcap\{U_x : x \in U_x \in \alpha(X)\}$*. In other words,* α*-nbd of a point* x *is the smallest* α*-open set containing* x.

A topological space $\alpha(X)$ of any space X is defined by the minimal α-nbd of α-closed points.

Lemma 2.3. *Any α-Kolmogorov space contains at least one singular point.*

Proof. Suppose that Y is an α-Kolmogorov space with finite number of points less that k and contains a singular point. Then by induction, we find a space X with $k+1$ of points. Now, let $x, y \in X$, $x \neq y$. Set $Y = U_x$ and $y \notin U_x$. Then by hypothesis, U_x is α-subspace of X and contain a singular point. Therefore X is also contain a singular point. □

Proposition 2.4. *Every α-open set in an α-Kolmogorov space X contains at least one singular point.*

Proof. Let $U \in \alpha(X)$. Then U is an α-open subspace of X. By Lemma 2.3, we find an isolated point x of U in U. Since U an α-open in X, then x is an isolated point in X. □

Lemma 2.5. *Let $X = C \cup V$ in which every $\{c\} \subseteq C$ is an α-closed and $\{v\} \subseteq V$ is an α-open. Then each of C and V is an α-discrete subspace of X.*

Proof. Since C is an α-closed subspace of X, then each $\{c\} \subseteq C$ is an α-closed point in C. Then $C - \{c\}$ is an α-closed subset in C and so $\{c\}$ is an α-open point in C. Therefore C is an α-discrete subspace. Also, V is an α-discrete in the same manner. □

Theorem 2.6. *In an α-Kolmogorov space X, the following are equivalent:*
(i) $\dim X \leq 1$,
(ii) Every singleton in X is either α-open or α-closed.

Proof. (i)⇒ (ii): Let X be an α-Kolmogorov space. By Lemma 2.3, X has an α-open point say x_0 and $U_{x_0} = \{x_0\}$. Since $\dim X \leq 1$, then $\dim \alpha B(\{x_0\}) = 0$ and so $\alpha B(\{x_0\})$ is α-discrete. Then each $y_0 \in \alpha B(\{x_0\})$ is an α-closed in $\alpha B(\{x_0\})$. Since $\alpha B(\{x_0\})$ is an α-closed in X, then $\{y_0\}$ is so in X. set $X' = X - \alpha Cl(\{x_0\})$ which is an α-open α-Kolmogorov subspace of X. By Proposition 2.4, X' has an α-open point set say $U_{x_1} = \{x_1\}$ which is also α-open in X. Also $\dim \alpha B(\{x_1\}) = 0$, then $\alpha B(\{x_1\})$ is an α-discrete. So each $y_1 \in \alpha B(\{x_1\})$ is an α-closed in $\alpha B(\{x_1\})$. Then $\{y_1\}$ is an α-closed in X. Put $X'' = X - \alpha Cl(\{x_0, x_1\})$. By continue, in the same manner, we prove that each singleton is either α-open or α-closed.
(ii)⇒ (i): suppose that each singleton in X is either α-open or α-closed. By Lemma 2.5, we have an α-discrete subspaces of X. So the dimension of each subspace is less than 1 and hence $\dim X \leq 1$. □

3. Mutual relationships

Definition 3.1. *[31] A multifunction $F : X \to Y$ is said to be:*

(i) upper α-irresolute at a point $x \in X$ if for each α-open set V containing $F(x)$, there exists $U \in \alpha(X, x)$ such that $F(U) \subseteq V$.

(ii) lower α-irresolute at a point $x \in X$ if for each α-open set V such that $F(x) \cap V \neq \phi$, there exists $U \in \alpha(X, x)$ such that $F(u) \cap V \neq \phi$ for every $u \in U$.

(iii) upper (lower) α-irresolute if F has this property at every point of X.

Definition 3.2. *A multifunction $F : X \to Y$ is said to be:*

(i) upper precontinuous [35] (resp. upper quasi continuous [36] , upper α-continuous [37] , upper β-continuous [[38], [37]) if for each $x \in X$ and each open set V of Y containing $F(x)$, there exists $U \in PO(X, x)$ (resp. $U \in SO(X, x)$, $U \in \alpha(X, x)$, $U \in \beta(X, x)$) such that $F(U) \subseteq V$.

(ii) lower precontinuous (resp. lower quasi continuous, lower α-continuous , lower β-continuous) if for each $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \phi$, there exists $U \in PO(X, x)$ (resp. $U \in SO(X, x)$, $U \in \alpha(X, x)$, $U \in \beta(X, x)$) such that $F(u) \cap V \neq \phi$ for every $u \in U$.

(iii) upper (lower) precontinuous (resp. upper (lower) quasi continuous , upper (lower) α-continuous , upper (lower) β-continuous) if it has this property at each point of X.

Remark 3.3. *For a multifunction $F : X \to Y$ the following implication hold:*

$$\begin{tikzcd} & & \textbf{upper quasi-cont.} \arrow[dr] & \\ \textbf{upper } \alpha\textbf{-irresolute} \arrow[r] & \textbf{upper } \alpha\textbf{-cont.} \arrow[ur] \arrow[dr] & & \textbf{upper } \beta\textbf{-cont.} \\ & & \textbf{upper precont.} \arrow[ur] & \end{tikzcd}$$

The following examples show that none of these implications are reversible.

Example 3.4. *Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3\}$. Define a topology $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}$ on X and a topology $\sigma = \{\phi, Y, \{1\}\}$ on Y. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is defined as follows:*

$$F(x) = \begin{cases} \{1\}, & \text{if} \quad x = a; \\ Y, & \text{if} \quad x = b \text{ or } c; \\ \{1,2\}, & \text{if} \quad x = d. \end{cases}$$

It can be easily observed that F is upper α-continuous. But F is not upper α-irresolute, since $\{1, 2\} \in \sigma^{\alpha}$ while $F^{+}(\{1, 2\}) = \{a, b\}$ is not α-open in (X, τ).

Example 3.5. *Let $X = \{a, b, c\}$ and $Y = \{y : y \in \{0, \pm 1, \pm 2\}\}$. Define a topology $\tau = \{\phi, X, \{b\}, \{c\}, \{b, c\}$ on X and a topology $\sigma = \{\phi, Y, \{0, 1, -1, -2\}\}$ on Y. Consider the following multifunction $F : (X, \tau) \to (Y, \sigma)$*

$$F(x) = \begin{cases} \{0\}, & \text{if} \quad x = a; \\ \{1,-1\}, & \text{if} \quad x = b; \\ \{2,-2\}, & \text{if} \quad x = c. \end{cases}$$

Then F is upper β-continuous, but not upper precontinuous, since $\{0, 1, -1, -2\} \in \sigma$ but $F^{+}(\{0, 1, -1, -2\}) = \{a, b\}$ is not preopen in (X, τ).

Example 3.6. *Let X and Y be as in Example 3.5 with two topologies $\tau = \{\phi, X, \{b, c\}$ on X and $\sigma = \{\phi, Y, \{1\}, \{-1\}, \{1, -1\}\}$ on Y. Define a multifunction $F : (X, \tau) \to (Y, \sigma)$ as shown in Example 3.5. one can deduce that F is upper precontinuous but not upper α-continuous.*

Example 3.7. *Let X, Y and τ be as in Example 3.4. Define a topology $\sigma = \{\phi, Y, \{1, 3\}\}$ on Y. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is defined as follows: $F(a) = \{1\}$, $F(b) = \{3\}$, $F(c) = \{2, 3\}$ and $F(d) = \{1, 2\}$. Then F is upper β-continuous but not upper quasi-continuous because $\{1, 3\} \in \sigma$ but $F^{+}(\{1, 3\}) = \{a, b\}$ is not open in (X, τ).*

4. Finite topological structures of fractals

We consider each point of the topology $\alpha(J)$ of Julia sets as in figure 2 as a kneading sequence $\hat{\sigma}$ which is the set of all $0-1$ sequences and represents in the following definition. we focus on a self-similar set A with $A = A_0 \cup A_1$ and $A_0 \cap A_1$ is a singleton, specially, for a tree-like [19, 27] set in the sense that it does not topological circles.

Definition 4.1. *Let $X = \{0, 1\}^{\mathbb{N}}$ be a space of kneading sequences with product topology. Each piece of Julia sets J_c can approximate by an α-subspace $X_n = \{0, 1\}^n \cup (\bigcup_{k<n} \{0, 1\}^k \times \{a\})$, where a denote to a connecting α-closed point, such that*

(i) Each $u \in \{0, 1\}^n$ is an α-open point.

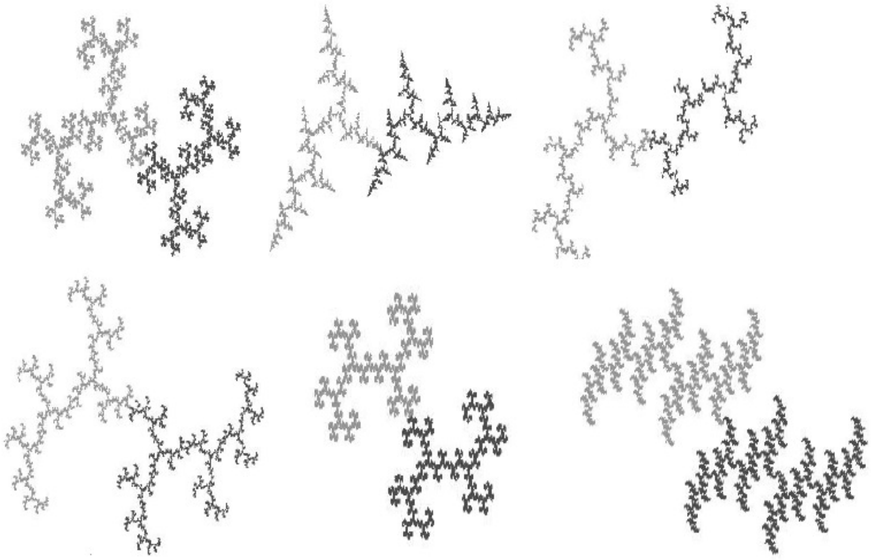

Figure 2: Some types of Julia sets[8]

(ii) Each $v = v_1 v_2 \cdots v_k a \in \bigcup_{k<n} \{0,1\}^k \times \{a\}$ *is an* α*- closed point with a minimal* α*-nbd. With a bonding* α*-continuous function* $h_n : X_n \to X_{n-1}$ *such that*

$h(u_1 u_2 .... u_n) = u_1 u_2 .... u_{n-1}$

$h(v_1 v_2 .... v_m a) = v_1 v_2 .... v_m, \quad m = n - 1$

$h(v_1 v_2 .... v_m a) = v_1 v_2 .... v_m a, \quad m < n - 1$

The one dimensional finite topological spaces $\alpha(X_i)$, for each i, can be illustrated as a structure similar to trees consisting of α-open and α-closed points.

Definition 4.2. *An* α*-continuous function* $f : X \to Y$ *is an* α*-monotone if* $f^{-1}(\{y\})$ *is connected for each* $y \in Y$*.*

Theorem 4.3. *Let* J_c *be a tree-like connected Julia set, for each complex number* $c \in \mathbb{C}$*. Then for each* $n \in \mathbb{N}$*, there exists a space* X_n *with an* α*-continuous function* $h_n : X_{n+1} \to X_n$ *defined by*
(i) $h_n(x_{n+1}) = x_n$*;*
(ii) Each of h_n *is an* α*-monotone relative* X_{n+1}*.*
for each point $x_n \in X_n$*, and the inverse limit* $\varprojlim(X_n, h_n)$ *is a completely regular modification to J.*

Proof. By Definition 4.1, set $X_n = \{0,1\}^n \cup (\bigcup_{j<n} \{0,1\}^j \times \{a\})$. Now for each $n \in \mathbb{N}$, the base of a topology $\alpha(X_n)$ is given by the smallest α-nbd of each point $x_n \in X_n$. The inverse limit $\varprojlim X_n$ is the set of all strings $\tilde{u} = u_1 u_2 \cdots$ which correspond to either $va = h_k(va) = h_k h_{k+1}(va)$ or $u_1 = h_2(u_1 u_2) = h_3(u_1 u_2 u_3) = \cdots$. Then the topology of $\varprojlim X_n$ is generated by the base of α-open sets $\tilde{\mathcal{U}}$ which is given by all α-open sets $U \subset X_k$, for $k = 1, 2, 3, \cdots$ with minimal α-nbd. So $\tilde{\mathcal{U}}$ consists of the all strings with initial part in U and there is some strings $\tilde{u}$ and va which can not be separated by two disjoint α-open sets in the sense of αT_2 spaces. By Definition 1.7, these points can be identified. Therefore the points in $\varprojlim X_n$ having the same α-nbd mapped onto the same piece in J_c. Hence J_c is a completely regular modification of $\varprojlim X_n$. $\square$

Now, we study the dynamics α-Kolmogorov spaces and give some examples.

Definition 4.4. *[22] A function $f : X \to Y$ is called:*
(i) α-irresolute if $f^{-1}(U) \in \alpha(X)$, for each α-open set U in Y.
(ii) Pre-α-open if $f(V) \in \alpha(Y)$, for each α-open set V in X.
(iii) Pre-α-closed if $f(V) \in \alpha C(Y)$, for each α-closed set V in X.

Definition 4.5. *Two spaces X and Y are α-homeomorphic if there exists a bijective, α-irresolute and pre-α-open function $f : X \to Y$.*

Each homeomorphic space is α-homeomorphic, but the converse may not be true, in general.

Example 4.6. *Let $X = \{1,2,3,4,5,6,7,8\}$ and $Y = \{a,b,c,d,x,y,e,f\}$ be spaces of vertices in trees G_1 and G_2, respectively, such that $deg\{3\} = deg\{4\} = deg\{x\} = deg\{4\} = 3$. The function $h : X \to Y$ is an α-homeomorphism when either $h(3) = x, h(4) = y$ or $h(3) = y, h(4) = x$. While there is no α-homeomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(V) = V$ where V is the set of vertices. Because of the embedding of trees G_1 and G_2 in $\mathbb{R}^2$ by connecting vertices 1, 7 in G_1 and a, f in G_2, respectively. So the topological structure for the two planar graphs will be denote by X and X'. Therefore there is no α-homeomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $f(X) = X'$.*

Lemma 4.7. *[27] Every tree-like Julia sets can be embedded into $\mathbb{R}^2$.*

Corollary 4.8. *Any two graphs are α-homeomorphic if they obtained from the same graph by adding vertices of degree 2 into edges.*

Theorem 4.9. *Each tree can be α-homeomorphic to some subset in the plane.*

Proof. Let G be an arbitrary tree with n vertices. It is clear that any vertex is a point in the plane and any edge can be view as an arc in the plane. Now by induction assume that every tree with k vertices can be embed into a subset of the plane and T is a tree with $k+1$ vertices. There exists at least a vertex v_k with $deg(v_k) = 1$ and its edge is $v_k v_{k+1}$. By the connectedness of T and each path must be finish with such vertex after at most k proceeds, then we remove the edge $v_k v_{k+1}$ from T. By assumption, the remainder of T is a subset in the plane. We add an edge, by Corollary 4.8, it does not affect on graph's topology which is a subset in $\mathbb{R}^2$. $\square$

In the following, we approximate the quadratic tree-like Julia sets using the concepts of upper (lower) α-continuous multifunctions.

Theorem 4.10. *For a topology $\alpha(X_n)$ of Julia sets X_n. If a multifunction $F : X_n \to X_n$ defined by:*
(i) $F(u_1u_2\cdots u_n) = \{u_2\cdots u_n0,\ u_2\cdots u_na,\ u_2\cdots u_n1\}$,
(ii) $F(v_1v_2\cdots v_ka) = v_2v_3\cdots v_ka$ for $k < n$.
Then F is lower α-continuous multifunction.

Proof. Let each α-open point $u = u_1u_2\cdots u_n$ has a minimal α-nbd $\mathcal{U}_u = \{u\}$. Then $F(u) = \{u'0, u'a, u'1\}$ where $u' = u_2\cdots u_n$. Then for each $x \in F(u)$, $F(u) \cap \{x\} \neq \phi$. Therefore $F(u) \cap \mathcal{U}_x \neq \phi$. Now for every $va \in X_n$ where $v = v_1\cdots v_k$, $\mathcal{U}_{va} = \{v0\sigma^{(j)}, va, v1\sigma^{(j)}\}$ and $F(\mathcal{U}_{va}) = F(v0\sigma^{(j)}) \cup F(va) \cup F(v1\sigma^{(j)})$. Therefore $F(\mathcal{U}_{v*}) = \{v'0\sigma^{(j)}0, v'0\sigma^{(j)}a, v'0\sigma^{(j)}1, va, v'1\sigma^{(j)}0, v'1\sigma^{(j)}a, v'1a^{(j)}1\}$ where $v' = v_2\cdots v_k$. Since $y \in F(\mathcal{U}_{va})$, then $F(\mathcal{U}_{va}) \cap \{y\} \neq \phi$. So $F(\mathcal{U}_{va}) \cap \mathcal{U}_y \neq \phi$ for each $y \in F(\mathcal{U}_{va})$. This can also be proved for only $a \in X_n$. Therefore F is lower α-continuous multifunction. $\square$

A multifunction in Theorem 4.10 is not upper α-continuous. This can be shown in the following example.

Example 4.11. *Consider an α-closed point $a \in X_3$. Let $F : X_3 \to X_3$ define as in Theorem 4.10. Since $F(a) = \{001\}$, $\mathcal{U}_{001} = \{001\}$ and $\mathcal{U}_a = \{000, a, 100\}$, then $F(\mathcal{U}_a) = F(000) \cup F(a) \cup F(100) = \{000, 00a, 001\}$. Therefore $F(\mathcal{U}_a) \not\subset \bigcup_{y \in F(a)} \mathcal{U}_y$. Then F is not upper α-continuous.*

5. An approach of fractals by nano topological graphs

A tree is a simple graph that contain no cycles [42]. In any tree a vertex v is said to be an end vertex if $\deg v = 1$. The edge which has an end vertex is called a branch. Self similar sets consist of pieces which are similar to each other and similar to the whole structure. Each of which is a compact metric space, say A, consists of pieces A_i which are similar to each other. In other words, $A = A_1 \cup A_2 \cup \cdots \cup A_m$ assigned with a homeomorphism functions $f_i : A \to A_i = f_i(A)$. This setting leads to smaller and smaller parts, so the same maps can be applied on these smaller pieces by other homeomorphism functions $f_j : A_i \to f_j(A_i) = f_j f_i(A_i) = A_{ji}$, where $A_i \subseteq A$ and $A_{ji} \subseteq A_j$, where $i, j \in \mathbb{N}$. In Figures 2, there are some fractal structures with one connected point. In the following, we write the following algorithm which explain how to create nano topological graphs $(G_n, \tau_R(V(G_{n-1})))$ on self similar fractals through the definition of Lellis Thivagar.

Example 5.1. *If we take $V(S_0) = \{v_1\}$ in G_1, then $R(v_1) = \{v_1, v_2\}$ and $R(v_2) = \{v_1, v_2\}$. So $L_R(V(S_0)) = \phi$, $H_R(V(S_0)) = V(G_1)$ and $B_R(V(S_1)) = V(G_1)$. Therefore a nano topology induced by a subgraph S_0 is $\tau_R(V(S_0)) = \{\phi, V(G_1)\}$. It is clear that a subgraph S_0 is homomorphic to any graph with only one vertex.*

Example 5.2. *If we take $V(S_1) = \{v_{11}, v_{21}\}$ in G_2, then $R(v_{12}) = \{v_{12}, v_{11}\}$, $R(v_{11}) = \{v_{12}, v_{11}, v_{21}\}$, $R(v_{21}) = \{v_{11}, v_{21}, v_{22}\}$ and $R(v_{22}) = \{v_{21}, v_{22}\}$. So $L_R(V(S_1)) = \phi$, $H_R(V(S_1)) = V(G_2)$ and $B_R(V(S_1)) = V(G_2)$. Therefore a nano topology induced by a subgraph S_1 is $\tau_R(V(S_1)) = \{\phi, V(G_2)\}$. It is clear that a subgraph S_1 is homomorphic to G_1*

Observation 5.3. *From Example 5.1 and Example 5.2, we observe that the induced topologies coincide with a Pawlak rough topology.*

Example 5.4. *If we take $V(S_2) = \{v_{121}, v_{111}, v_{211}, v_{212}\}$ in G_2, then $R(v_{111}) = \{v_{111}, v_{121}, v_{112}, v_{212}\}$, $R(v_{211}) = \{v_{111}, v_{211}, v_{221}, v_{212}\}$, $R(v_{121}) = \{v_{122}, v_{121}, v_{111}\}$, $R(v_{221}) = \{v_{211}, v_{221}, v_{222}\}$, $R(v_{122}) = \{v_{121}, v_{122}\}$, $R(v_{112}) = \{v_{112}, v_{111}\}$, $R(v_{212}) = \{v_{212}, v_{211}\}$ and $R(v_{222}) = \{v_{211}, v_{222}\}$. So $L_R(V(S_2)) = \{v_{212}, v_{211}\}$, $H_R(V(S_2)) = V(G_3)$ and $B_R(V(S_2)) = \{v_{111}, v_{112}, v_{121}, v_{122}, v_{221}, v_{222}\}$. Therefore the a nano topology induced by a subgraph S_2 is $\tau_R(V(S_2)) = \{\phi, V(G_3), \{v_{212}, v_{211}\}, \{v_{111}, v_{112}, v_{121}, v_{122}, v_{221}, v_{222}\}\}$. It is clear that a subgraph S_2 is homomorphic to G_3.*

6. Dynamics of Julia sets via its topological structures

The branching structure of Julia sets studied by Penrose in [32]. El-Atik [8] defined a prefix tree T of some kneading sequences as $\hat{\sigma_1} = 00100011$, $\hat{\sigma_2} = 010011$ and $\hat{\sigma_3} = 01000011$ and gave their topological space structures. There is an orientation preserving at a definite point in the topology of σ_1 and σ_3. While there is no an orientation preserving at a definite point of σ_2.

The rotation system is used for embedding of each approximation space G_n in the plane. This depend on the branching point in each approximation of degree more than 3. The following proposition give the necessary condition for existence of rotation system at arbitrary approximation.

Definition 6.1. *A surjective function $f : X \to Y$ between two compact spaces X and Y is said to be local α-homeomorphism if for each $x \in X$, there exists an α-open α-nbd U of x such that $f(U)$ is α-open α-nbd.*

Definition 6.2. *Given a kneading sequence $\hat{\sigma} = \sigma_1\sigma_2\cdots$, a prefix tree T of $\hat{\sigma}$ with a vertex set $N = \{1, 2, 3, \cdots\}$ construct as: Each point n is the initial point of an edge. If $\sigma_{n+1} = 1$, the endpoint of the edge is $n+1$. If $\sigma_{n+1} = 0$, set the maximal k such that $\hat{\sigma}_{|k} = \sigma_{n+1}\cdots\sigma_{n+k}$ and $n+k+1$ will be the endpoint of the edge.*

Proposition 6.3. *If G_{n-1} has a rotation system, then there exists a unique rotation system on G_n.*

Proof. Let n be the number of elements in G_{n-1}. Define a relation R^* on G_{n-1} by sR^*t if and only if isR^*it *for every* $s, t \in G_{n-1}$ *and* $i \in \{0, 1\}$. *Consider* $G_n = G_{n0} \cup G_{n1}$, $G_{n0} \cap G_{n1} = \{a\}$ *such that* $G_{n0} = \{0s : s \in G_{n-1}\}$ *and* $G_{n1} = \{1s : s \in G_{n-1}\}$. *This means* G_n *consists of two similar copies of* G_{n-1} *such that* G_{n1} *is the rotation of* G_{n0} *at* $\{a\}$ *with degree* π. *Therefore* G_n *has* $2n$ *elements. Describe the rotation system of* G_n *as follows: begin at* $\hat{\alpha}$ *in* G_{n-1} *and walk through the edges in counter-clockwise sense. If the point* $\hat{\alpha}$ *is of degree 2, then the walk will be in the direction of the endpoint. Secondly, begin with* $0\hat{\alpha}$ *in* G_n *in the same counter-clockwise sense of* G_{n-1}. *Define* $\sigma : G_n \longrightarrow G_{n-1}$ *by* $\sigma(is) = s$ *for* $s \in G_n$ *and* $i \in \{0, 1\}$. *Since* $\sigma(0\hat{\alpha}) = \sigma(1\hat{\alpha}) = \sigma(a) = \hat{\alpha}$, *then* $\sigma^{-1}(\{0\hat{\alpha}, a, 1\hat{\alpha}\}) = \hat{\alpha}$. *When we reach to* $0\hat{\alpha}$ *in the first part of* G_n, *we begin at* $1\hat{\alpha}$ *in the second part. Continue in the same manner is completely define the rotation system of* G_n. □

Theorem 6.4. *The number of disjoint branches at* σ_i *in a prefix tree* T, *for some* i, *is the degree of* $\hat{\sigma}$ *in some topological space* $\alpha(X_i)$ *structure.*

Proof. Let $\hat{\sigma} = \sigma_1\sigma_2\cdots\sigma_n$ be a kneading sequence. Consider at σ_i, there exists k disjoint branches. By the branching structure, $\hat{\sigma}_i = \sigma_1\sigma_2\cdots\sigma_i \in \{0, 1\}^i \subseteq X_i$ which is an α-open point in X_i. The α-closed points which are the α-nbds of $\hat{\sigma}_i$ can be defined from the definition of α-nbds in each $\alpha(X_i)$, the point $\sigma_1\sigma_2\cdots\sigma_{i-1}a$ corresponds to the edge which starts with i. The other α-nbds has the form $\sigma_1\sigma_2\cdots\sigma_{j-1}a$ if $\sigma_{j+1}\sigma_{j+2}\cdots\sigma_i = \sigma_1\sigma_2\cdots\sigma_{i-j}$ each of these α-nbds correspond to the edge which starts in j and goes to some $k \geq i + 1$. □

Theorem 6.5. *Let* $F : \alpha_1(X_i) \to \alpha_2(X_i)$, *for some approximation structure* X_i, *of Julia sets* J_c. *Then all embeddings of* F *can not be extended to an* α*-continuous function. More generally,* F *can not be extended to local* α*-homeomorphism in* $\mathbb{R}^2$.

Proof. By Lemma 6.8, each Julia set has embedding in the plane, we embed it by the kneading sequence $\hat{\sigma}$ which consist of branching points. By Definition 6.2, we generate the set of α-nbds of a branching point $0\hat{\sigma}$. By similarity of embeddings, the degree of the branches of $1\hat{\sigma}$ has the same of branches $0\hat{\sigma}$. We assume that the rotation of α-nbds of all branching points is in anticlockwise sense. In a prefix tree T, we begin with a branching point $0\hat{\sigma}_{|_k}$ in some $\alpha(X_k)$. By recursion, we continue to generate more branching points and their α-nbds until having one embedding with $0\hat{\sigma}_{|_l}$ of degree ≤ 2 such that $l < k$. Now it is enough to investigate a local α-homeomorphism function $F : \alpha_1(X_i) \to \alpha_2(X_i)$ for these embeddings. Let F_0 and F_1 are the inverse branches of the doubling map or quadratic map for Julia sets and $E_{\hat{\sigma}_{|_j}}$ be a subspace of $\alpha(X_j)$ which consists of a branching point and its α-nbd. Define a function $h = F_{\sigma_1\sigma_2\cdots\sigma_k} = F_{\sigma_1}F_{\sigma_2}\cdots F_{\sigma_k}$ from a branch $E_{\hat{\sigma}_{|_k}}$ onto a branch $E_{\hat{\sigma}_{|_l}}$ define by $h(x) = \sigma_1\sigma_2\cdots\sigma_k x$ for each $x \in E_{\hat{\sigma}_{|_k}}$. h is an α-homeomorphism, since the branching points of the same degree. Then h is a graph α-homeomorphism. We extend $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $h(E_{\hat{\sigma}_{|_k}}) = E_{\hat{\sigma}_{|_l}}$. There are two cases: Case 1, if the rotation system of images of $E_{\hat{\sigma}_{|_k}}$ have the same sense, then h is still α-homeomorphism. Case 2, if images have reverse direction, then h is not aα-homeomorphism. In case 2, F is not local α-homeomorphism. Therefore in Case 2, the kneading sequence can not be realized a tree-like Julia set in the plane. □

We use our given approximations to some physical properties of fractal structures via topological properties of its topological space induced by its graph. In the following, we give some characterizations simple graphs, specially, in trees.

Proposition 6.6. *(i) The homomorphism between two trees maps endpoints into endpoints and each branching point of degree* ≥ 3 *into branching point of the same degree.*

(ii) The degree of a vertex x *for graphs is preserved under a homomorphism.*

Proof. (i) Let $f : G_1 \longrightarrow G_2$ be a homomorphism function between graphs G_1 and G_2. Then for every a point $x \in G_1$ and a relation $R(f(x))$ of $f(x)$. By continuity condition, there exists a relation $R(x)$ of x in G_1 such that $f(R(x)) \subset R(f(x))$. Then $R(x)$ and $f(R(x))$ are homomorphic subgraphs. Also, $R(x) - \{x\}$ and $f(R(x)) - \{f(x)\}$ are also homomorphic subgraphs and each of them have the same number of components. That means x and $f(x)$ have the same number of degree.

(ii) Let $f : G_1 \longrightarrow G_2$ be a homeomorphism and for $x \in G_1$. Since f is homomorphism, then there exists $R(x)$ with $f(R(x)) = R(f(x))$. Then $R(x)$ and $f(R(x))$ are homeomorphic subgraphs. That means $R(x) - \{x\}$ and $f(R(x)) - \{f(x)\}$ are also homomorphic subgraphs and so they must have the same number of components. $\square$

From Proposition 6.6, one can deduce that any two graphs are homomorphic if they can made isomorphic by inserting (contracting) vertices of degree 2 into edges. Also, if any graph has no vertices of degree 2, then the homomorphic is a graph isomorphic.

Proposition 6.7. *Every tree is homomorphic to a subspace of the topology on* $\mathbb{R}^2$.

Proof. We use the mathematical induction to prove this theorem. Let T be a tree with n vertices. The theorem is true for $n = 1$, for any vertex is a point in the plane. Also, at $n = 2$ give an edge which can be marked as an arc in the plane. Assume that every tree with k vertices embed in a subspace of $\mathbb{R}^2$. Now consider a tree T with $k + 1$ vertices. Since there exists at least a vertex v_k of degree 1 with an edge $v_k v_{k+1}$. This vertex must be exist since every tree is connected and each path must finish with such vertex after at most k steps. We remove the edge $v_k v_{k+1}$ from T. By assumption, the remaining is a subspace of $\mathbb{R}^2$. Finally, we add an edge to this embedding does not change the graph's topology and give also a subspace of $\mathbb{R}^2$. Therefore the theorem is true for all $k \in \mathbb{N}$. $\square$

Lemma 6.8. *Every self similar structure J can be embedded into $\mathbb{R}^2$ through a nano topological spaces which defined on it.*

Proof. It is straightforward through Propositions 6.6 and 6.7. $\square$

Theorem 6.9. *Let $(V(G_{n-1}), \tau_R(V(G_{n-1})))$ be a nano topological graph induced by a subgraph of $V(G_n)$. Let $F : V(G_n) \to V(G_n)$. Then all embeddings of F can be extended to a continuous function. More generally, F can not be extended to a local homomorphism.*

Proof. By Lemma 6.8, each self similar structure J can be embedded into $\mathbb{R}^2$ through a nano topological spaces trees which consist of branching points. We evaluate the set of relations of each branch point and assume the rotation of all branching points is in anticlockwise sense. Begin with a branching point ,say, $v_{i_1 i_2 i_3 \cdots i_n}$ in some G_n generated by algorithm in Section 7. By recursion, we continue to generate more branching points and their relations until having one embedding with $v_{i_1 i_2 i_3 \cdots i_m}$ in G_{n-1} of degree ≤ 2 such that $m < n$. This means there are two similar subspaces in G_n, each of them is homeomorphic to G_{n-1}. Each topology on G_n is topologically homeomorphic with a nano topological graph of $V(G_n)$. Now, we find a local homeomorphism function $F : V(G_n) \to V(G_n)$, for these approximation structures G_n. Let $F_1(v_{i_1 i_2 i_3 \cdots i_k})$ and $F_2(v_{i_1 i_2 i_3 \cdots i_j})$, where $i_1 i_2 i_3 \cdots i_j \in \prod\{1, 2\}^j$ are the inverse branches of the doubling map and $R(v_{i_1 i_2 i_3 \cdots i_j})$ is a subspace of $V(G_j)$. Define a function $h = F_{i_1 i_2 \cdots i_k} = F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_k}$ from a branch $R(v_{i_1 i_2 i_3 \cdots i_k})$ onto a branch $R(v_{i_1 i_2 i_3 \cdots i_l})$ defined by $h(x) = F_{i_1 i_2 \cdots i_k} = F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_k}(x)$ for each $x \in R(v_{i_1 i_2 i_3 \cdots i_k})$. If the rotation of images of $R(v_{i_1 i_2 i_3 \cdots i_k})$ have the same sense, then h is still homeomorphism. Otherwise, if images have reverse direction, then h is not a homeomorphism. $\square$

Corollary 6.10. *G_i and G_j are isomorphic graphs if and only if their nano topological graphs of $V(G_i)$ and $V(G_j)$ are homeomorphism for $i, j \in \mathbb{N}$.*

Proof. Clearly by Theorem 6.9 $\square$

Observation 6.11. *We observe that if a self similar fractal has a rotation, then fourth approximation of G_4 will be G_5 as shown in Figure 3. It is clear that G_4 is homomorphic to G_5.*

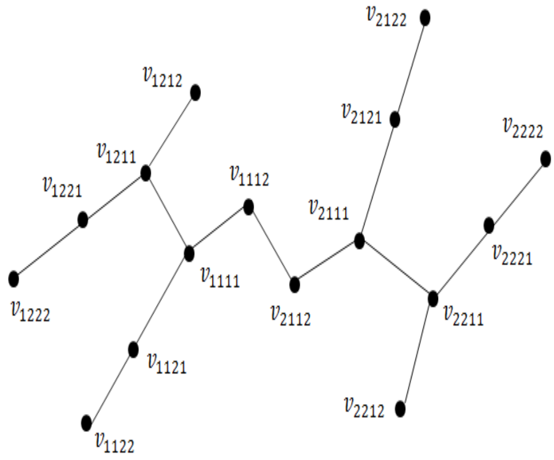

Figure 3: Graph G_5

7. Algorithm

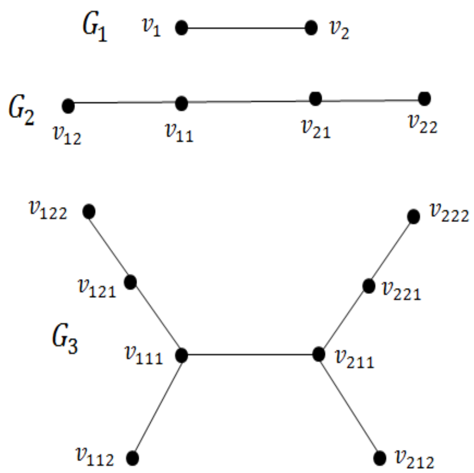

Figure 4: Graph G_1, G_2 and G_3

For finite topological spaces and nano topological spaces, we introduce the following algorithm.

Step (1): Represent a fractal part A_1 with a vertex v_1 and part A_2 with v_2. The connected point between A_1 and A_2 will be represented by an edge v_1v_2. This can be shown in a Figure 4, where $V(G_1) = \{v_1, v_2\} = \bigcup\{v_i : i \in \{1, 2\}\}$ and $|E(G_1)| = |V(G_1)| - 1 = 1$.

Step (2): Part A_1 is divided into similar parts A_{11} and A_{12}. We represent A_{11} with v_{11} and A_{12} with v_{12}. Connect between v_{11} and v_{12} by a vertex $v_{11}v_{12}$. By a similar way, represent A_{21} and A_{22} with vertices v_{21} and v_{22} and connect them by $v_{21}v_{22}$ as shown in Figure 4, where $V(G_2) = \{v_{11}, v_{12}, v_{21}, v_{22}\} = \bigcup\{v_{i_1i_2} : i_1i_2 \in \prod\{1, 2\}^2\}$ and $|E(G_2)| = |V(G_2)| - 1 = 3$.

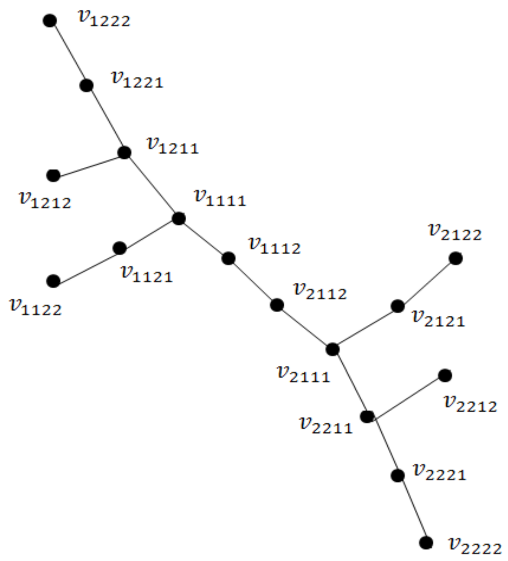

Figure 5: Graph G_4

Step (3): Part A_{11} is divided into similar parts A_{111} and A_{112}. We represent A_{111} with v_{111} and A_{112} with v_{112}. Connect between v_{111} and v_{112} by a vertex $v_{111}v_{112}$. Also, represent A_{121} and A_{122} with vertices v_{121} and v_{122} and connect them by $v_{121}v_{122}$. By a similar way, part A_{21} is divided into similar parts A_{211} and A_{212}. We represent A_{211} with v_{211} and A_{212} with v_{212}. Connect between v_{211} and v_{212} by a vertex $v_{211}v_{212}$. Also, represent A_{221} and A_{222} with vertices v_{221} and v_{222} and connect them by $v_{221}v_{222}$. This can be shown in Figure 4, where $V(G_3) = \{v_{111}, v_{112}, v_{121}, v_{122}, v_{211}, v_{212}, v_{221}, v_{222}\} = \bigcup\{v_{i_1i_2i_3} : i_1i_2i_3 \in \prod\{1,2\}^3\}$ and $|E(G_3)| = |V(G_3)| - 1 = 2^3 - 1 = 7$.
Step (4): In the same manner, we represent Figure 5, where $V(G_4) = \{v_{1111}, v_{1112}, v_{1121}, v_{1122}, v_{1211}, v_{1212}, v_{1221}, v_{1222}, v_{2111}, v_{2112}, v_{2121}, v_{2122}, v_{2211}, v_{2212}, v_{2221}, v_{2222}\} = \bigcup\{v_{i_1i_2i_3i_4} : i_1i_2i_3i_4 \in \prod\{1,2\}^4\}$ and $|E(G_4)| = |V(G_3)| - 1 = 2^4 - 1 = 17$, and so on.
Step (5): By n procedures, we have $V(G_n) = \bigcup\{v_{i_1i_2i_3\cdots i_n} : i_1i_2i_3\cdots i_n \in \prod\{1,2\}^n\}$ with $|E(G_n)| = |V(G_n)| - 1 = 2^n - 1$.

8. Conclusion

The field of mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. Therefore, the theory of graphs and topological spaces became the most important mathematical subjects. On the other hand, a topology plays a significant rule in quantum physics, high energy physics and superstring theory [9] . Thus, we study the approximations of self-similar sets by a relation which may have possible applications in quantum physics and superstring theory. Moreover, the concepts proposed in this paper can be extended in fuzzy topological structures [1] and thus one can get a more affirmative solution in decision making problems [15, 43–47] in real life solutions.

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