Filomat 34:1 (2020), 143–152
https://doi.org/10.2298/FIL2001143Q

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Review of INMA Integer-valued Model Class, Application and Further Development

A.M.M. Shahiduzzaman Quoreshi[a], Reaz Uddin[a], Naushad Ali Mamode Khan[b]

[a]*Department of Industrial Economics, Blekinge Institute of Technology, 371 79 Karlskrona, Sweden*
[b]*Department of Economics and Statistics, University of Mauritius*

Abstract. In this paper, we review INMA time series of integer-valued model class, and discuss its further development. These models have been developed for analyzing high frequency financial count data. A vivid description of high frequency data in the context of market micro structure is given. The most distinguishing feature that makes the INMA model class different from its continuous variable MA counterpart is that multiplication of variables with real valued parameters no longer remains a viable operation when the result is to be integer-valued. In the estimation of these models, no underlying distributions are assumed. Hence, the discussion of estimations is limited to CLS, FGLS and GMM. A further development of estimation procedures for these models have also been reviewed. We suggest that the models could be estimated with Quasi Maximum Likelihood and propose in addition a Generalized Method of Moment of Quasi Maximum Likelihood. We have also discussed how INMA model class can be extended with different underlying distributions for innovations.

1. Introduction

A time series of count data is an integer-valued non-negative sequence of count observations observed at equidistant instants of time. There is a growing literature on various aspect of how to model, estimate and use such data. Jacobs and Lewis [1, 2, and 3] develop discrete ARMA (DARMA) models that introduce time dependence through a mixture process. McKenzie [4] and Al-Osh and Alzaid [5] introduce independently the integer-valued autoregressive moving average (INARMA) model for pure time series data, while Brännäs [6] extends the INAR model to incorporate explanatory variables. Zheng et al. [7, 8] propose an first-order respective pth-order random coefficient integer-valued autoregressive RCINAR(1) respective RCINAR(p) model while Nastić et al. [9] introduces a mixed thinning Based Geometric INAR(1) model. The regression analysis of count data is relatively new, though the statistical analysis of count data has a long and rich history. The increased availability of count data in recent years has stimulated the development of models for both panel and time series count data. For reviews of these and other models, see, e.g., Cameron and Trivedi [10] and McKenzie [11]. In INARMA, the parameters are interpreted as probabilities and hence restricted to unit intervals. Some empirical applications of INAR are due to Blundell, Griffith and Windmeijer [12], who studied the number of patents in firms, while Rudholm [13] studied competition

2010 *Mathematics Subject Classification*. 62H20, 62E15, 60E15
Keywords. INMA, Integer-valued, count data, Estimation
Received: 24 January 2019; Revised: 02 April 2019; Accepted: 15 May 2019
Communicated by Biljana Popović
Email addresses: shahiduzzaman.quoreshi@bth.se (A.M.M. Shahiduzzaman Quoreshi), mreaz.uddin@bth.se (Reaz Uddin), n.mamodekhan@uom.ac.mu (Naushad Ali Mamode Khan)

in the generic pharmaceuticals market, and Brännäs, Hellström and Nordström [14] estimated a nonlinear INMA(1) model for tourism demand.

In this paper, we review INMA integer-valued model class and discuss its application and further development. The integer-valued moving average model of order q [INMA(q)], i.e. a special case of the INARMA model class, is developed for analyzing high frequency financial data in the form of stock transactions data aggregated over one, two or five minute intervals of time Brännäs, and Quoreshi [15]. Quoreshi [16, 17, and 18] proposes a bivariate integer-valued moving average (BINMA) model, a vector integer-valued moving average (VINMA) model and an integer-valued autoregressive fractionally integrated moving average (INARFIMA) model. The BINMA model is developed to capture the covariance between stock transactions data due to macroeconomic news or rumors, while the VINMA Model is more general than the BINMA model and enables the study of the spillover effects of news from one stock to other. The INARFIMA model is developed to study the long memory property of high frequency count data. The models can also be used to measure the reaction times to shocks or news. The INMA, BINMA, VINMA and INARFIMA models have been described below together with 'long memory' and 'high frequency data'.

2. Models

2.1. The INMA, BINMA and VINMA Models

The INMA model is a special case of the INARMA model. The INMA model of order 1, INMA (1), is introduced by Al-Osh and Alzaid [19] and in a slightly different form by McKenzie [4]. These studies assumed Poisson distribution for the time series. For a time series $\{y_t\}$, the INMA (1) of Al-Osh and Alzaid is

$$y_t = u_t + \alpha \circ u_{t-1}$$

where $\alpha \in [0, 1]$ which is a binomial thinning operator Steutel and van Harn [20]. The single thinning operator makes the INMA model visibly different from its continuous variable MA counterpart. The multiplication of variables with real valued parameters is no longer a viable operation, when the result is to be integer-valued. Multiplication is therefore replaced by the binomial thinning operator

$$\alpha \circ u = \sum_{j}^{u} z_j \tag{1}$$

with $\{z_j\}_{j=1}^{u}$ an iid sequence of 0-1 random variables, such that $\Pr(z_j = 1) = \alpha = 1 - \Pr(z_j = 0)$. Conditionally on the integer-valued u, $\alpha \circ u$ is binomially distributed with

1. $E(\alpha \circ u \mid u) = \alpha u$ and
2. $V(\alpha \circ u \mid u) = \alpha(1-\alpha)u$.

Unconditionally it holds that

1. $E(\alpha \circ u) = \alpha\lambda$ and
2. $V(\lambda \circ u) = \alpha^2\sigma^2 + \alpha(1-\alpha)\lambda$,

where $E(u) = \lambda$ and $V(u) = \sigma^2$. Obviously, $\alpha \circ u \in \{0, 1, \ldots, u\}$[5, 9, 20, 21].
Employing this binomial thinning operator, an INARMA(p, q) model can be written

$$y_t - \alpha_1 \circ y_{t-1} - \cdots - \alpha_p \circ y_{t-p} = u_t + \beta_{11} \circ u_{t-1} + \cdots + \beta_q \circ u_{t-q} \qquad (2a)$$

with α_j, $\beta_i \in [0, 1]$, j = 1, . . . , p − 1 and i = 1, . . . , q − 1, and α_p, $\beta_q \in (0, 1]$. Setting all $\alpha_j = 0$ we obtain the INAR(q) model

$$y_t = \alpha_1 \circ y_{t-1} - \cdots - \alpha_p \circ y_{t-p} + u_t \qquad (2b)$$

and setting all $\beta_i = 0$ we obtain the INMA(q) model

$$y_t = u_t + \beta_1 \circ u_{t-1} + \cdots + \beta_q \circ u_{t-q} \qquad (2c).$$

Brännäs and Hall [22] discuss model generalizations and interpretations resulting from different thinning operator structures, and an empirical study and approaches to estimation are reported by Brännäs et al. [14]. McKenzie [4], Joe [23], Jørgensen and Song [24] and others stress exact distributional results for y_t, while Brännäs & Quoreshi [15] emphasize only the first two conditional and unconditional moments of the model. Moreover, they discuss and introduce more flexible conditional mean and heteroskedasticity specifications for y_t than implied by the above equation. There is an obvious connection between the introduced count data model and the conditional duration model of, e.g., Engle and Russell [25] in the sense that long durations in a time interval correspond to a small count and vice versa. Hence, one main use of the count data models discussed here is also one of measuring reaction times to shocks or news.

Quoreshi [16] focuses on the modelling of bivariate time series of count data that are generated from stock transactions. The data are aggregates over five minute intervals and computed from tick-by-tick data. One obvious advantage of the BINMA model over the conditional duration model is that there is no synchronization problem between the time series. Hence, the spread of shocks and news is more easily studied in the present framework. Moreover, the bivariate count data models can easily be extended to multivariate models without much complication. The bivariate time series count data model also allows for negative correlation between the counts and the integer-value property of counts is taken into account. The model is employed to capture covariance between stock transactions time series and to measure the reaction time for news or rumors. Moreover, this model is capable of capturing the conditional heteroskedasticity.

Quoreshi [17] extends the INMA model to a vector INMA (VINMA) model. The VINMA enables the study of the spillover effects of transactions from one stock to the other. A large number of studies have considered the modelling of bivariate or multivariate count data assuming an underlying Poisson distribution (e.g., Gourieroux, Monfort and Trognon [26]). Heinen and Rengifo [27] introduce multivariate time series count data models based on the Poisson and the double Poisson distribution. Other extensions to traditional count data regression models are considered by, e.g., Brännäs and Brännäs [28] and Rydberg and Shephard [29].

Sunecher et al. [30] recently introduce a first-order bivariate integer-valued moving average process (BINMA(1)) where the respective innovation series are marginally COM-Poisson distributed under non-stationary moments. The purpose of this process is to model inter-related INMA (1) time series that are known to exhibit different levels and types of dispersion. Ristic et al.[31] introduces a new bivariate integer-valued moving average of the first order (BINMA(1)) process with independent Negative Binomial (NB) innovations under non-stationary moment conditions.

2.2. Long Memory and the INARFIMA Model

Hurst [32, 33] considered first the long memory phenomenon in time series. He explained the long term storage requirements of the Nile River. He showed that the cumulated water flows in a year had a persistent impact on the water flows in the later years. By employing fractional Brownian motion, Mandelbrot and van Ness [34] explain and advance the Hurst's studies. In analogy with Mandelbrot and van Ness [34], Granger [35], Granger and Joyeux [36] and Hosking [37] develop Autoregressive Fractionally Integrated Moving Average (ARFIMA) models to account for the long memory in time series data. According to Ding and Granger [38], a number of other processes can also have the long memory property. In a recent empirical study, Bhardwaj and Swanson [39] found strong evidence in favor of ARFIMA in absolute, squared and log-squared stock index returns. Granger and Joyeux [36] and Hosking [37] introduce the ARFIMA(p, d, q) class of models of the discrete time real-valued series x_t

$$\alpha(L)(1-L)^d x_t = \beta(L)\, u_t. \tag{3}$$

where L is a lag operator and d is any real number. The $\{u_t\}$ is a white noise process of random variables with mean $E(u_t) = 0$ and variance $V(u_t) = \sigma_a^2$. Employing binomial series expansion, we can write

$$(1-L)^d = \Delta^d = 1 - \sum_{i=1}^{\infty} \frac{(i-1-d)!}{i!\,(-d-1)!} L^i = 1 - \sum_{i=1}^{\infty} \frac{\Gamma(i-d)}{\Gamma(i-d)\,\Gamma(d-1)} L^i \tag{4}$$

and correspondingly

$$\Delta^{-d} = 1 + dL + \frac{1}{2}d(1+d)L^2 + \frac{1}{6}d(1+d)(2+d)L^3 + \ldots$$
$$= 1 + \sum_{i=1}^{\infty} \frac{(i+d-1)!}{i!\,(d-1)!}L^i = 1 - \sum_{i=1}^{\infty} \frac{\Gamma(i+d)}{\Gamma(i+d)\,\Gamma(d)}L^i \tag{5}$$

The moving average representation of ARFIMA (0,d,0) of the series x_t is
$$x_t = u_t + d_1 u_{t-1} + d_2 u_{t-2} + d_3 u_{t-3} \ldots$$

or

$$x_t = (1+L)^{-d} u_t. \tag{6}$$

Note that $x_t$has long memory in a sense that the variable has a slow decaying autocorrelation function and the parameters $d_j = \Gamma(j+d)/[\Gamma(j+1)\Gamma(d)],\ j = 0,\ 1, 2, \ldots$ where $d_0 = 1$ and where u_t is zero-mean serially uncorrelated process. Approximating $d_j \doteq Aj^{-d}$, for $j \geq 1$, Granger and Joyeux [36] propose the following representation of fractionally integrated MA(∞) model

$$y_t = A \sum_{j=1}^{\infty} j^{-d} u_{t-j} + u_t. \tag{7a}$$

According to Granger and Jouex [36], the series has the following variance

$$V(y) = A\sigma_u^2 \sum_{j=1}^{\infty} \left(1 + j^{2(d-1)}\right). \tag{7b}$$

From the theory of infinite series it is known that $\sum_{i=1}^{\infty} j^{-d}$ converges for $d > 1$, otherwise it diverges. They conclude that the variance of $x_t$and$y_t$differ only in finite quantity. Hence the variance for x_t is finite provided $d < \frac{1}{2}$, but infinite if $d \geq \frac{1}{2}$. Geweke and Porter-Hudak [40] show that $\sum_{i=1}^{\infty} d_i < \infty$ if and only if $d < 0$.

Quoreshi [18] concentrates on modeling the long memory property of time series of count data and its application in financial analysis. However, combining the ideas of the INARMA model (2a) with fractional integration is complicated. Direct use of (4) or (5) will not provide integer-values since multiplying an integer-valued variable with a real-valued d can not produce an integer-valued result and this alternative is hence ruled out. Instead, he departs from the binomial expansion expression and proposes in analogy with Granger and Joyeux [36] and Hosking [37] INARFIMA models that accounts for integer-valued counts and long memory. Quoreshi [18] considers the following representation of the INARFIMA (p, d, q)

$$\alpha(L^\circ)\, x_t = \beta(L^\circ)(1 - L^\circ)^{-d} u_t. \tag{8}$$

In (8), $\alpha(L^\circ) = 1 - \alpha_1 \circ L - \alpha_2 \circ L^2 \cdots - \alpha_p \circ L^p$ and $\beta(L^\circ) = 1 + \beta_1 \circ L + \beta_2 \circ L^2 \cdots + \beta_q \circ L^q$ are lag polynomials of orders p and q, respectively. Note that we require $\alpha_i, \beta_j, d \in [0,1]$, for $i > 0$ and $j \geq 0$ for an INARFIMA (p, d, q) model.

Quoreshi [18] applies the INARFIMA $(0, d, 0)$ models to stock transactions data for AstraZeneca and Ericsson B. The paper presents evidence for long memory for the AstraZeneca series while the series for Ericsson B indicates a process that has a mean reversion property.

Like INARFIMA (0,d,0), Quoreshi [41] introduces BINFIMA (d_1, d_2) in its simplest form which can be defined as follows
$$y_{1t} = u_{1t} + d_{11} \circ u_{1t-1} + d_{12} \circ u_{1t-2} + d_{13} \circ u_{1t-3} \ldots$$

$$y_{2t} = u_{2t} + d_{21} \circ u_{2t-1} + d_{22} \circ u_{2t-2} + d_{23} \circ u_{1t-3} \ldots$$

or

$$y_{1t} = \left(1 + d_{11} \circ L + d_{12} \circ L^2 + d_{13} \circ L^3 \ldots\right) u_{1t}$$

$$y_{2t} = \left(1 + d_{21} \circ L + d_{22} \circ L^2 + d_{23} \circ L^3 \ldots\right) u_{2t}$$

or

$$y_{it} = (1 + L^{\circ})^{-d_i} u_{it}. \tag{9}$$

where L is a lag operator and the notation $L^{\circ} = (\circ L)^i$ for $i > 0$. It's assumed that here is no cross-lag dependence among u_{it}. Note that y_{1t} and y_{2t} have long memory in a sense that the variables have slow decaying autocorrelation functions and the parameters$d_{ij} = \Gamma(j + d_i)/\left[\Gamma(j + 1)\Gamma(d_i)\right]$, $i = 1, 2$ and$j = 0,\ 1, 2, \ldots$ where$d_0 = 1$. Note that d_{ij} are considered thinning probabilities and hence$d_{ij} \in [0, 1]$. The macro-economic news are assumed to be captured by $\{u_{it}\}$, $i = 1, 2$andfiltered by $\{d_{ij}\}$ through the system.

2.3. *Further Development of Model*

The INMA model class may be developed in a fashion similar to MA model class. In general, the multiplication shall be replaced with thinning operators. Further restriction is needed so that the innovations u_t are integer-valued observation that are generated from some underlying distribution. The estimation procedure needs to be developed based on known and unknown underlying distribution. In the context of high frequency data and data generated from social media, different applications and other sources of computed communications may not have known underlying distribution. In our knowledge, there are no corresponding integer-valued models for, e.g., AR representation of ARFIMA, seasonal ARMA, Markov-Switching Model and Threshold Autoregressive (TAR) models. Another aspect of developing INMA model class is within the framework of long memory. In equation (9), it is shown that the long memory parameters are not independent and follow a gamma function. It would be of interest to study long memory properties for the parameters under other functional forms, e.g., sines, cosines and wavelet functions. These functions have cyclical behavior. Employing these functions may encounter the problem of stationarity and predictability of the models. These are open questions to study which may have of academic interest.

Consider the following data set (Figure 1). As we see that there is sudden change at observation number 6000. We may consider this as regime shift and hence a suitable integer-valued model is required.

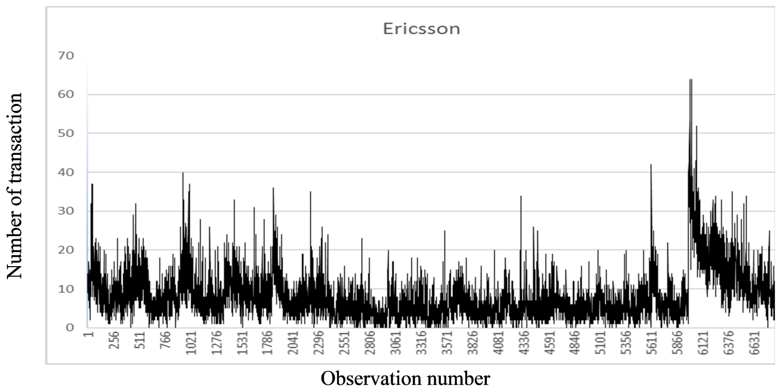

Figure 1: Time series plots for Ericsson (Brännäs and Quoreshi, 2010).

Employing the thinning operator in Threshold Autoregressive (TAR) model, we may introduce Integer-valued TAR (INTAR) model. For two regime,p_1, p_2 , case and the threshold k, the INTAR(P) model can be written

$$y_t = \begin{cases} \theta_{0,1} + \theta_{1,1} \circ y_{t-1} + \cdots + \theta_{p1,1} \circ y_{t-1} + u_t & if\ y_{t-1} \leq k \\ \theta_{1,0} + \theta_{1,2} \circ y_{t-1} + \cdots + \theta_{p2,2} \circ y_{t-1} + u_t & if\ y_{t-1} \geq k. \end{cases}$$

Similarly, the INTMA(q) can be written

$$y_t = \begin{cases} u_t + \theta_{1,1} \circ u_{t-1} + \cdots + \theta_{q1,1} \circ u_{t-1} & if\ y_{t-1} \leq k \\ u_t + \theta_{1,2} \circ u_{t-1} + \cdots + \theta_{q2,2} \circ u_{t-1} & if\ y_{t-1} \geq k. \end{cases}$$

The possible application of these models are discussed in section 3.2

3. Application

3.1. Application of INMA Model Class on High Frequency Data

The INMA, BINAM, VINMA, INARFIMA and BINFIMA models are applied on financial market tick-by-tick data. Each tick represents a change in, e.g., a quote or corresponds to a transaction. For a liquid stock or a currency, these tick-by-tick data generate high frequency data. Such financial data are also characterized by lack of synchronization, in the sense that only rarely there is more than one transaction at a given instant of time. For reviews of high frequency data and their characteristics, see, e.g., Tsay [42], Dacorogna et al. [43] and Gourieroux and Jasiak [45]. Accessibility to and affordability of high frequency data by individual researchers are fueling studies on many issues related to the trading process and the market microstructure. Transactions data are collected from an electronic limited order book for each stock. Incoming orders are ranked according to price and time of entry and are continuously updated. Hence, new incoming buy and sell orders and the automatic match of the buy and sell orders are recorded. The automatic match of a buy and a sell order generates a transaction. In Figure 2, we see that the transactions in the two stocks are not synchronized, i.e. the transactions appear at different points of time. The counts in the intervals are the number of transactions for corresponding intervals. Brännäs and Quoreshi [15] employ one minute time scale while Quoreshi [16, 17] uses a five minute scale. The collection of the number of transactions over a time period makes up a time series of count data. The time series of transactions or count data are synchronized between stocks in the sense that all the numbers of transactions are aggregated transactions over the same time interval. An example of real transactions data over a 30 minute period for the stock AstraZeneca is exhibited in Figure 3. Each observation number corresponds to one minute of time. This type of data series comprises frequent zero frequencies and motivates a count data model. The time series of transactions or count data may have a long memory property.

The long memory implies the long range dependence in the time series of counts, i.e. the present information has a persistent impact on future counts. Note that the long memory property is related to the sampling frequency of a time series. A manifest long memory may be shorter than one hour if observations are recorded every minute, while stretching over decades for annual data. The time series containing long memory has a very slowly decaying autocorrelation function. The autocorrelation function for stock transactions data aggregated over one minute interval of time for AstraZeneca is illustrated in Figure 4. The autocorrelation function decays sharply in the first few lags but thereafter decay is very slow. Hence, we may expect long memory in stock transactions data for AstraZeneca. Models for long memory and continuous variable time series are not appropriate for integer-valued counts. Therefore, long memory models developed for continuous variables are not automatically of relevance neither with respect to interpretation nor to efficient estimation.

3.2. Further Application

The INMA models class discussed in section 2 has been used on stock transaction data. The possible area of use of these models and the other models proposed in section 2 may also be in stock transaction data in connection with unexpected arrival of news or downloading of Apps or activities in social media. Due to unexpected news on a company, the trading behavior of the stock may be switched. Similarly, downloading a particular apps or a particular activities in a social media may be changed due to breaking out of unexpected news on the Apps or on the issue related to the particular activities in social medias.

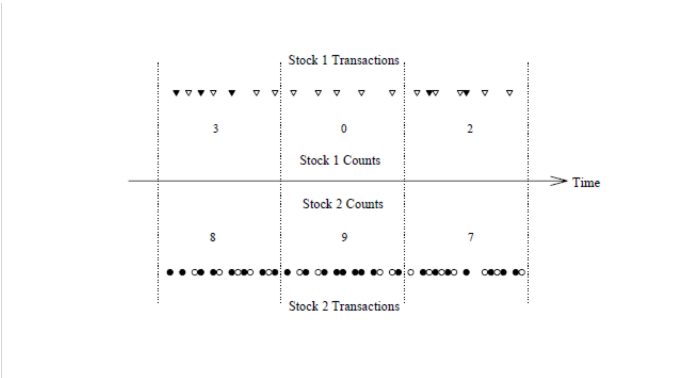

Figure 2: An illustration of how transactions data are generated. The black triangles and circles represent transactions for stock 1 and stock 2, respectively, while the white triangles and circles represent all other activities in an order book. The stock counts record the number of black triangles/circles falling into a time interval, i.e. falling between vertical lines.

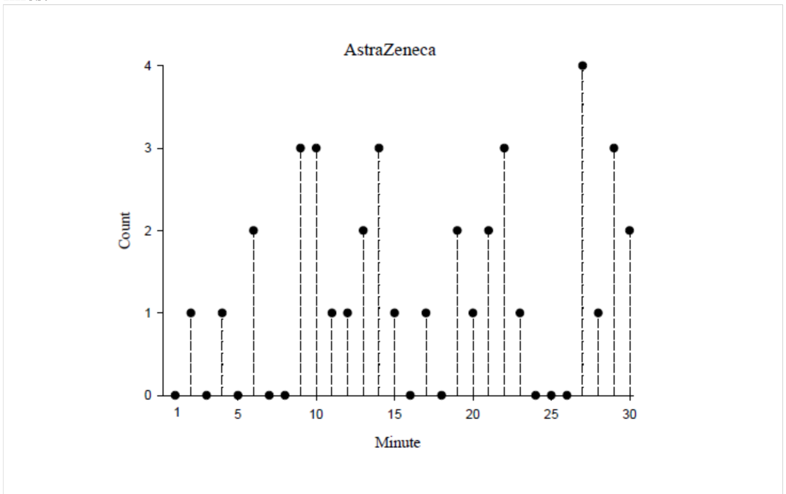

Figure 3: The number of transactions data over minute long intervals for 30 minutes of trading in AstraZeneca.

4. Estimation

4.1. The INMA, BINMA and VINMA Models Estimation

Brännäs and Quoreshi [15] provide conditional mean and variance properties and applied the properties in estimations of conditional least squares (CLS), the feasible generalized least square (FGLS), and the generalized method of moments (GMM) estimators. In a limited Monte Carlo experiment they study the bias and MSE properties of the CLS, FGLS and GMM estimators for finite-lag specifications, when data is generated according to an infinite-lag INMA model. In addition, they study the serial correlation properties of estimated models by the Ljung-Box statistic as well as the properties of forecasts one and two steps ahead. In this Monte Carlo study, the feasible least squares estimator comes out as the best choice. However, the CLS estimator which is the simplest to use of the three considered estimators is not far behind. The GMM performance is weaker than that of the CLS estimator. It also shows that the lag length should be large and both under and over-parameterization give rise to detectable serial correlation. Quoreshi [16] discusses the conditional least squares CLS, FGLS and GMM estimator for BINMA model. Quoreshi [17, 18] consider CLS and FGLS estimators for VINMA and long memory model. The authors did not consider the maximum likelihood due to unknown underlying distributions of the innovations. The above literatures dealt with

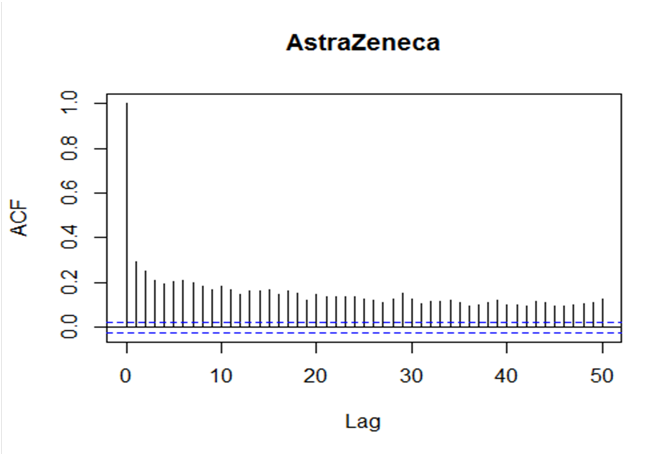

Figure 4: The autocorrelation function for stock transactions data aggregated over five minute interval of time for AstraZeneca

the inferential part mainly by the Conditional Least squares approach (CLS) which provides consistent estimators. However, it cannot be ignored that estimating function in the CLS approach in INMA modeling depends on the knowledge of the error term which is latent. Notably, to compensate for this in the CLS, the random error component in the estimating function was approximated by the condition on the previous observation. Recently, some other research in INMA or bivariate INMA modelling has illustrated that an alternative and robust estimation approach termed as Generalized Quasi-likelihood (GQL) may be used to estimate the unknown parameters. The GQL estimating equation is sourced from the likelihood estimating equation based on the exponential dispersion family [45], consisting of three components: The score vector and its corresponding mean, the auto-covariance function and the derivative component. Under the correct specification of the expected score and auto-covariance function, the GQL approach is shown to yield asymptotically equally efficient estimates as the maximum-likelihood based approach which is as expected [46]. Besides, in Mamode Khan et al. [46] and Sunecher et al. [47], it is proved that GQL yields more efficient estimates than CLS.

4.2. Further Development

An important issue in the proposed MA-based models is the estimation of the parameters or the contributory effects. Since the distribution of the counting series in these sophisticated set-ups is generally unknown, the application of likelihood-based approach becomes restricted. Brännäs and Quoreshi [16] and Quoreshi [17] have developed the inferential procedures for the INMA, BINMA and VINMA models via the CLS, FGLS and GMM approaches while Zheng et al. [7] also discussed Maximum likelihood estimation for RCINAR(p)]. Mamode Khan et al.[46], Sunecher et al. [47], Ristic et al. [31] have proposed a Generalized Quasi-likelihood (GQL) estimation technique based on the correct score specification for the BINMA(1) model only. Under these techniques, Sunecher et al. [47] demonstrated through the Monte Carlo simulation BINMA(1) experiments that both CLS, GMM and GQL yield consistent parameter estimates but with GQL yielding estimates with lower bias and superior standard errors. However, at this stage, the performance of the GQL approach has not yet been investigated on the VINMA, long memory and INARFIMA models. Thus, the plan of this paper is to re-propose the BINMA, VINMA, INARFIMA and long memory models under some prominent generalized-type Poisson distributions and focus on developing and comparing the GQL and approach with the other estimation techniques for these models. Further, we may develop Maximum likelihood and Quasi-Maximum likelihood for these models in a similar fashion like Zheng et al. [7]. These techniques should then be applied to simulation experiments and real-life application to assess their performance.

5. Summary and Conclusion

Brännäs and Quoreshi [15] discuss and introduce more flexible conditional mean and heteroskedasticity specifications for INMA model and applied the model on financial market data tick-by-tick data. These tick-by-tick data generate high frequency data. Such financial data are also characterized by lack of synchronization, in the sense that only rarely is there more than one transaction at a given instant of time. Later, Quoreshi [16, 17, and 18] develops BINMA and VINMA models to deal with the synchronization problem and INARFIMA model to capture the long memory phenomenon. Brännäs and Quoreshi discussed CLS, FGLS and GMM estimators, while they did not consider Maximum-Likelihood estimator since the underlying distributions of the counts were unknown. Similar argument was discussed by Mamode Khan et al. [46] and Sunecher et al. [47] and recommended a GQL approach to estimate the parameters. Besides, the innovation terms are unobserved and therefore likelihood computations become unfeasible. To summarize, in INMA-related processes, there are different classes of models that can be referred and developed further but the inferential part has to be based on robust non-likelihood methods such as GQL. Further, we may develop Maximum likelihood and Quasi-Maximum likelihood for these models as done by Zheng et al. [7]. These techniques should then be applied to simulation experiments and real-life application to assess their performance.

References

[1] Jacobs, P.A. and Lewis, P.A.W. (1978a). Discrete Time Series Generalized by Mixtures I: Correlational and Runs Properties. Journal of the Royal Statistical Society B40, 94-105.

[2] Jacobs, P.A. and Lewis, P.A.W. (1978b). Discrete Time Series Generalized by Mixtures II: Asymptotic Properties. Journal of the Royal Statistical Society B40, 222-228.

[3] Jacobs, P.A. and Lewis, P.A.W. (1983). Stationary Discrete Autoregressive Moving Average Time Series Generated by Mixtures. Journal of Time Series Analysis 4, 19-36.

[4] McKenzie, E. (1986). Autoregressive Moving-Average Processes with Negative Binomial and Geometric Marginal Distributions. Advances in Applied Probability 18, 679-705.

[5] Al-Osh, M.A. and Alzaid, A.A. (1987). First Order Integer-Valued Autoregressive (INAR(1)) Process. Journal of Time Series Analysis 8, 261-275.

[6] Brännäs, K. (1995). Explanatory Variables in the AR(1) Model. Umeå Economic Studies 381.

[7] Zheng, Haitao and Basawa, Ishwar V. and Datta, Somnath (2007).First-order random coefficient integer-valued autoregressive processes. Journal of Statistical Planning and Inference, Volume 137, Issue 1, 1 January 2007, Pages 212-229.

[8] Zheng, Haitao and Basawa, Ishwar V. and Datta, Somnath (2006). Inference for Pth-Order Random Coefficient Integer-Valued Autoregressive Processes. Journal of Time Series Analysis, Vol. 27, No. 3, pp. http://dx.doi.org/10.1111/j.1467-9892.2006.00472.

[9] Nastić, A. S., Ristić, M. M., and Janjic, A. (2017). A Mixed Thinning Based Geometric INAR(1) Model. Filomat, Vol 31, No 13.

[10] Cameron, A.C. and Trivedi, P.K. (1998). Regression Analysis of Count Data. Cambridge: Cambridge University Press.

[11] McKenzie, E. (2003). Discrete Variate Time Series. In Shanbhag, D.N. and 14 Introduction and Summary Rao, C.R. (eds.) Handbook of Statistics, Volume 21, pp. 573-606. Amsterdam: Elsevier Sciences.

[12] Blundell, R., Griffith, R. and Windmeijer, F. (2002). Individual Effects and Dynamics in Count Data Models. Journal of Econometrics 108, 113-131.

[13] Rudholm, N. (2001). Entry and the Number of Firms in the Swedish Pharmaceuticals Market. Review of Industrial Organization 19, 351-364.

[14] Brännäs, K., Hellström, J. and Nordström, J. (2002). A New Approach to Modelling and Forecasting Monthly Guest Nights in Hotels. International Journal of Forecasting 18, 19-30.

[15] Brännäs, K. and Quoreshi, A.M.M.S., 2010. Integer-value moving average modeling of the transactions in stocks. Applied Financial Economics 20, 1429-1440.

[16] Quoreshi, A.M.M.S., Bivariate time series modelling of financial count data. Communication in Statistics: Theory and Method, 2006, 35, 1343–1358.

[17] Quoreshi, A.M.M.S., A vector integer-valued moving average model for high frequency financial count data. Economics Letters 101 (2008) 258–261.

[18] Quoreshi, A.M.M.S., 2014. A long-memory integer-valued time series model, INARFIMA, for financial application. Quantitative Finance Vol 12, 2225-2235.

[19] Al-Osh M. and Alzaid A. (1988). Integer-Valued Moving Average (INMA) Process. Statistical Papers 29, 281-300.

[20] Steutel, F. W. and van Harn, K. (1979). Discrete analogues of self-decomposability and stability. The Annals of Probability, 7(5), 893-899.

[21] Weiß,C. (2008). Thinning operations for modeling time series of counts—a survey. Advances in StatisticalAnalysis, 92: 319 Springer-Verlag.

[22] Brännäs, K. and Hall, A., 2001. Estimation in integer-value moving average models. Applied Stochastic Models in Business and Industry 17, 277-291.

[23] Joe, H. (1996). Time SeriesModels with UnivariateMargins in the Convolution- Closed Infinitely Divisible Class. Journal of Applied Probability 33, 664-677.
[24] Jørgensen, B. and Song, P.X-K. (1998). Stationary Time Series Models with Exponential Dispersion Model Margins. Journal of Applied Probability 35, 78-92.
[25] Engle, R. F., Russell, J. R. (1998). Autoregressive conditional duration: a new model for irregularly spaced transaction data. Econometrica 66:1127–1162.
[26] Gourieroux, C., Monfort, A. and Trognon, A. (1984). Pseudo Maximum Likelihood Methods: Application to Poisson Models. Econometrica 52, 701-720.
[27] Heinen, A. and Rengifo, E. (2003). Multivariate Modelling of Time Series Count Data: An Autoregressive Conditional Poisson Model. CORE Discussion Paper 2003/25, UniversitéCatholique de Louvain.
[28] Brännäs, K. and Brännäs, E. (2004). Conditional Variance in Count Data Regression. Communication in Statistics; Theory and Methods 33, 2745-2758.
[29] Rydberg, T.H. and Shephard, N. (1999). BIN Models for Trade-by-trade Data. Modelling the Number of Trades in a Fixed Interval of Time. Working Paper Series W23. Nuffield College, Oxford.
[30] Sunecher,Y., Mamode Khan, N. and Jowaheer, V. (2018). BINMA(1) Model with COM Poisson Innovations: Estimation and Application, Communication in Statistics: Simulation and Computation, https://doi.org/10.1080/03610918.2018.1502780.
[31] Ristic, M.M., Sunecher, Y., Mamode Khan, N and Jowaheer, V. (2018). A GQL-based Inference in non-stationary BINMA(1) Time Series, TEST, https://doi.org/10.1007/s11749-018-0615-1.
[32] Hurst, H.E. (1951). Long-Term Storage Capacity of Reservoirs. Transactions of the American Society of Civil Engineers 116, 770-808.
[33] Hurst, H.E. (1956). Methods of Using Long-Term Storage in Reservoirs. Proceedings of the Institute of Civil Engineers 1, 519-543.
[34] Mandelbrot, B.B. and Van Ness, J.W. (1968). Fractional Brownian Motions, Fractional Noises and Applications. SIAM Review 10, 422-437.
[35] Granger, C.W.J., 1980. Long Memory Relationships and the Aggregation of Dynamic Models. Journal of Econometrics 14, 227-238.
[36] Granger, C.W.G., Joyeux, R., 1980.An introduction to long memory time series models and fractional differencing. Journal of Time Series Analysis 1, 15-29.
[37] Hosking, J., 198l. Fractional differencing, Biometrika 68 (l), 165-176.
[38] Ding, Z. and Granger, C.W.J. (1996). Varieties of Long Memory Models. Journal of Econometrics 73, 61-77.
[39] Bhardwaj, G., and Swanson, N.R., 2006. An Empirical Investigation of the usefulness of ARFIMA models for predicting macroeconomic and financial time series. Journal of Econometrics 131(1,2),539-578.
[40] Geweke, J. and Porter-Hudak, S., 1983. The Estimation and Application of Long Memory Time Series Models. Journal of Time Series Analysis Vol 4, No. 4.
[41] Quoreshi, A.M.M.S., 2017. A bivariate integer-valued long-memory model for high-frequency financial count data. Communications in Statistics - Theory and Methods Volume 46, - Issue 3.
[42] Tsay, R.S. (2002). Analysis of Financial Time Series. New York: John Wiley and Sons, Inc.
[43] Dacorogna, M.M., Gencay, R., Muller, U., Olsen, R.B. and Pictet, O.V. (2001). An Introduction to High-Frequency Finance. San Diego: Elsevier.
[44] Gourieroux, C and Jasiak, J. (2001). Financial Econometrics. Princeton: Princeton University Press.
[45] McCullagh, P. and Nelder, J.A, (1989). Generalized Linear Models.2nd Edition. Chapman and Hall.
[46] Mamode Khan, N. Sunecher, Y.,Jowaheer, V., Ristic, M. and Heenaye-Mamode Khan, M. (2018), Investigating GQL-based inferential approaches for non-stationary BINAR(1)model under different quantum of over- dispersion with application, Computational Statistics, 10.1007/s00180-018-0836-5
[47] Sunecher, Y., Mamode Khan, N. and Jowaheer, V. (2017). Estimating the parameters of a BINMA Poisson model for a non-stationary bivariate time series, Communication in Statistics: Simulation and Computation, 47:9,6803-6827.