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On Mostar Index of Trees with Parameters

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Abstract. The Mostar index of a graph G is defined as the sum of absolute values of the differences between n_u and n_v over all edges uv of G, where n_u and n_v are respectively, the number of vertices of G lying closer to vertex u than to vertex v and the number of vertices of G lying closer to vertex v than to vertex u. We identify those trees with minimum and/or maximum Mostar index in the families of trees of order n with fixed parameters like the maximum degree, the diameter, number of pendent vertices by using graph transformations that decrease or increase the Mostar index.

1. Introduction

All graphs considered in this paper are finite and simple. Let G be a connected graph on n vertices with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, let $N_G(v)$ be the set of all neighbors of v in G. The degree of $v \in V(G)$, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. A vertex is said to be pendent if its degree is one, and an edge is said to be pendent if one end vertex is pendent. The graph formed from G by deleting a vertex $v \in V(G)$ (and its incident edges) is denoted by $G - v$. A connected graph with n vertices is a tree if $|E(G)| = n - 1$. A caterpillar is a tree, the deletion of whose pendent vertices outside a diametral path produces a path. A vertex having degree greater than two is called branch vertex. As usual, by S_n and P_n we denote the star and path on n vertices, respectively. For $e = uv \in E(G)$, let $N_u(e|G)$ and $N_v(e|G)$ be respectively the set of vertices of G lying closer to vertex u than to vertex v and the set of vertices of G lying closer to vertex v than to vertex u, i.e.,

$$N_u(e|G) = \{x \in V(G) : d_G(u, x) < d_G(v, x)\},$$
$$N_v(e|G) = \{x \in V(G) : d_G(v, x) < d_G(u, x)\}.$$

The numbers of vertices of $N_u(e|G)$ and $N_v(e|G)$ are denoted by $n_u(e|G)$ and $n_v(e|G)$, respectively.

Let G be a connected graph. The Szeged index of G, proposed by Gutman [9], is defined as

$$Sz(G) = \sum_{e=uv \in E(G)} n_u(e|G) n_v(e|G).$$

Now the Szeged index and its variants have been studied extensively, see, e.g., [1–3, 5, 6, 8, 10, 15, 16, 23, 24, 27]. We mention that the Szeged index is a particular case of the modified Wiener index, defined as

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$\sum_{e=uv\in E(G)}(n_u(e|G)n_v(e|G))^\lambda$ for a nonzero real λ, see, e.g., [11, 25, 26]. If G is a tree, then $Sz(G)$ coincides with the Wiener index of G [22]. Actually, the Wiener index of a connected graph G is defined as the sum of distances between all pairs of vertices, see, e.g., [4, 19]. To measure the peripherality in a graph (i.e., how far a graph is from being distance-balanced [13, 17]), Doslić et al. [7] introduced a novel graph invariant, named as Mostar index of a graph. For a connected graph G, the Mostar index of G is defined as

$$Mo(G) = \sum_{e=uv\in E(G)} |n_u(e|G) - n_v(e|G)|.$$

Doslić et al. [7] studied the Mostar index of trees and unicyclic graphs, and showed how the Mostar index can be efficiently computed for various classes of chemically interesting graphs using a variant of the cut method proposed by Klavžar et al. [14]. In particular, they showed that among n-vertex trees, the star S_n and path P_n are the unique ones with maximum and minimum Mostar index, respectively. Tepeh [20] studied the Mostar index of bicyclic graphs. Hayat and Zhou [12] determined all the n-vertex cacti with the largest Mostar index, and obtained a sharp upper bound for the Mostar index for cacti of order n with k cycles, and characterize the extremal case.

In this note, we give some further properties of Mostar index of trees. We identify those trees with minimum and/or maximum Mostar index in the families of trees of order n with fixed parameters like the maximum degree, the diameter, number of pendent vertices using graph transformations that decrease or increase the Mostar index.

2. Preliminaries

For simplicity, we set $\psi_G(e) = |n_u(e|G) - n_v(e|G)|$ for a connected graph G with $e = uv \in E(G)$.

Let G be a connected graph with a cut edge $e = uv$, and let G/e be the graph obtained from G by contracting the edge e into a new vertex w_e such that it is adjacent to each vertex in $N_G(u) \cup N_G(v) \setminus \{u, v\}$ and then attaching a pendent edge at w_e.

Lemma 2.1. *[7] Let G be a connected graph with a cut edge e. If e is not a pendent edge, then $Mo(G/e) > Mo(G)$.*

A path $v_0 \dots v_s$ in a graph G is a pendent path of length s (at v_0) if $d_G(v_0) \geq 2$, $d_G(v_s) = 1$, and if $s \geq 2$, $d_G(v_i) = 2$ for $i = 1, \dots, s-1$. Evidently, a pendent path of length one is a pendent edge. Let H be a nontrivial connected graph with $u \in V(H)$. For two nonnegative integers ℓ and m, let $H_{u;\ell,m}$ be the graph obtained from H by attaching two pendent paths of length ℓ and m, respectively at u. In particular, $H_{u;0,0} = H$ and $H_{u;\ell,0}$ is obtained from H by attaching a pendent path of length ℓ. From the proof of Theorem 5 in [7], we have

Lemma 2.2. *[7] Let H be a nontrivial connected graph with $u \in V(H)$. If $\ell \geq m \geq 1$, then $Mo(G_{u;\ell,m}) > Mo(G_{u;\ell+1,m-1})$.*

Let $A_n(a, b)$ be the n-vertex tree obtained from a path by attaching a and b pendent vertices respectively to the two terminal vertices, where $n - a - b \geq 2$ and $a \geq b \geq 0$.

Lemma 2.3. *Let a and b be positive integers with $a \geq b + 2$. Then $Mo(A_n(a - 1, b + 1)) < Mo(A_n(a, b))$.*

Proof. If $a \leq \frac{n}{2}$, then $b + 1 < a \leq \frac{n}{2}$, and otherwise, $b + 1 < n - a < \frac{n}{2}$. Thus $Mo(A_n(a, b)) > Mo(A_n(a - 1, b + 1))$, as $Mo(A_n(a, b)) - Mo(A_n(a - 1, b + 1)) = |b + 1 - (n - b - 1)| - |a - (n - a)| > 0$. □

For a graph G with $E_1 \subseteq E(G)$, $G - E_1$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \setminus E_1$. Similarly, if $E_2 \subseteq E(\overline{G})$, then $G + E_2$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \cup E_2$, where $\overline{G}$ is the complement of G. In particular, if $E_1 = \{e\}$ ($E_2 = \{f\}$, respectively), then we write $G - e$ ($G + f$, respectively) instead of $G - \{e\}$ ($G + \{f\}$, respectively).

Lemma 2.4. *Let G be a tree with $x, y \in V(G)$. Suppose that $x_1, \dots, x_s$ and $y_1, \dots, y_t$ are pendent vertices adjacent to x and y respectively. Let x' and y' be respectively the neighbors of x and y in the path connecting x and y. Let*

$$G' = G - \{xx_i : i = 1, \dots, s\} + \{x'x_i : i = 1, \dots, s\},$$

$$G'' = G - \{yy_i : i = 1, \dots, t\} + \{y'y_i : i = 1, \dots, t\}.$$

Then $Mo(G) < \max\{Mo(G'), Mo(G'')\}$.

Proof. Let $e' = xx'$ and $e'' = yy'$. Note that, if x and y are adjacent, then $x' = y$ and $y' = x$. From the constructions of G' and G'', we have $\psi_G(e) = \psi_{G'}(e)$ for all $e \in G \setminus \{e'\}$, and $\psi_G(e) = \psi_{G''}(e)$ for all $e \in G \setminus \{e''\}$. Let $n_z(f) = n_z(f|G)$ where $z \in \{x, x'\}$ if $f = e'$ and $z \in \{y, y'\}$ if $f = e''$. Then

$$\begin{aligned} Mo(G) - Mo(G') &= \psi_G(e') - \psi_{G'}(e') \\ &= |n_x(e') - n_{x'}(e')| - |n_x(e') - s - (n_{x'}(e') + s)|, \end{aligned}$$

$$\begin{aligned} Mo(G) - Mo(G'') &= \psi_G(e'') - \psi_{G''}(e'') \\ &= |n_y(e'') - n_{y'}(e'')| - |n_y(e'') - t - (n_{y'}(e'') + t)|. \end{aligned}$$

Obviously, $n_{x'}(e') \geq n_y(e'')$ and $n_{y'}(e'') \geq n_x(e')$. If $n_x(e') > n_{x'}(e')$ and $n_y(e'') > n_{y'}(e'')$, then $n_x(e') > n_{x'}(e') \geq n_y(e'') > n_{y'}(e'') \geq n_x(e')$, which is a contradiction. Thus we have either $n_x(e') \leq n_{x'}(e')$ or $n_y(e'') \leq n_{y'}(e'')$. In the former case, $Mo(G) - Mo(G') = n_{x'}(e') - n_x(e') - (n_{x'}(e') - n_x(e') + 2s) = -2s < 0$, and in the latter case, $Mo(G) - Mo(G'') = n_{y'}(e'') - n_y(e'') - (n_{y'}(e'') - n_y(e'') + 2t) = -2t < 0$. Thus, $Mo(G) < Mo(G')$ or $Mo(G) < Mo(G'')$, as desired. □

For integer k, let $P_{n,d,k}$ be the tree obtained from the path $P_{d+1} = v_0v_1 \dots v_d$ by attaching $n - d - 1$ pendent edges at v_k, where $1 \leq k \leq \lfloor \frac{d}{2} \rfloor$.

Let $S(a_1, \dots, a_r)$ be the tree consisting of r pendent paths of lengths $a_1, \dots, a_r$ respectively at a common vertex u, where $a_1 \geq \cdots \geq a_r \geq 1$. If $a_i - a_j = 0, 1$ for any i and j with $1 \leq i < j \leq r$, then we call $S(a_1, \dots, a_r)$ a balanced starlike tree and denoted by $BS_{n,r}$.

3. Results

The maximum degree of a graph is the the maximum degree of its vertices.

Theorem 3.1. *Among all trees of order n with maximum degree Δ, $P_{n,n-\Delta+1,1}$ is the unique tree with minimum Mostar index, where $3 \leq \Delta \leq n - 2$.*

Proof. Let T be a tree of order n with maximum degree Δ such that $Mo(T)$ is as small as possible. We only need to show that $T \cong P_{n,n-\Delta+1,1}$.

Choose a vertex $v \in V(T)$ with degree Δ. Let $N_T(v) = \{v_1, \dots, v_\Delta\}$. Let T_i be the component of $T - v$ containing v_i, where $i = 1, \dots, \Delta$. Suppose that for some i, T_i is not a path with one terminal vertex v_i. Then there is a vertex in T_i such that its degree in T is at least three. So there is a vertex w in T_i such that $d_T(v, w)$ is as large as possible. That is to say, there are two pendent paths, say with lengths ℓ and m respectively at w in T. Assume that $\ell \geq m$. So $T \cong G_{w;\ell,m}$, where G is the graph obtained from T by deleting the vertices of degree two and one in the two pendent paths. By Lemma 2.2, we have $Mo(T) = Mo(G_{w;\ell,m}) > Mo(G_{w;\ell+1,m-1})$, a contradiction. Therefore, for each $i = 1, \dots, \Delta$, T_i is a path with one terminal vertex v_i. So T consist of Δ pendent paths at v. By Lemma 2.2, $T \cong P_{n,n-\Delta+1,1}$. □

The diameter of a graph is the largest distance between any pair of vertices.

Theorem 3.2. *Among all trees of order n with diameter d, $P_{n,d,\lfloor \frac{d}{2} \rfloor}$ is the unique tree with maximum Mostar index, where $3 \leq d \leq n - 2$.*

Proof. Let T be a tree of order n with diameter d such that $Mo(T)$ is as large as possible. We only need to show that $T \cong P_{n,d,\lfloor \frac{d}{2} \rfloor}$.

Let $P = v_0 \dots v_d$ be a diametric path of T. Suppose that T is not a caterpillar. Then there exist some vertex say w outside P with $d_T(w) \geq 2$ such that $wv_i \in E(T)$ for some $1 \leq i \leq d-1$. Let N be the set of neighbour of w except v_i. Let $T' = T - \{ws : s \in N\} + \{v_i s : s \in N\}$. It is obvious that T is a tree of order n with diameter d. But by Lemma 2.1, $Mo(T) < Mo(T')$, a contradiction to the maximality of T. Thus, T is a caterpillar.

Next, we show that all pendent edges outside the path P are adjacent to a single vertex on the path P. Otherwise, we may choose two vertices v_i and v_j ($1 \leq i < j \leq d-1$) on the path P with degree at least three in T. Let Q be the sub-path of P from v_i to v_j. By Lemma 2.4, we may find a tree of order n with diameter d whose Mostar index is larger than $Mo(T)$, which is a contradiction. Thus, all pendent edges outside the path P are adjacent to a single vertex on the path P. That is, $T \cong P_{n,d,k}$ for some k with $1 \leq k \leq \lfloor \frac{d}{2} \rfloor$. Then $n - (2k+2) \geq 0$. Suppose that $k < \lfloor \frac{d}{2} \rfloor$. Let $T' = P_{n,d,k+1}$. As $k - 1 < \min\{d-k, n-(d-k)\} \leq \frac{n}{2}$, we have $Mo(T) - Mo(T') = \psi_T(v_k v_{k+1}) - \psi_{T'}(v_k v_{k+1}) = |n - (d-k) - (d-k)| - |k+1-(n-k-1)| < 0$, implying that $Mo(T) < Mo(T')$, a contradiction. Therefore, $k = \lfloor \frac{d}{2} \rfloor$, i.e., $T \cong P_{n,d,\lfloor \frac{d}{2} \rfloor}$. □

Next we consider the Mostar index of trees when the number of pendent vertices is fixed. We use the techniques from [25].

Recall that a starlike tree is a tree with a unique vertex of degree at least three. For $3 \leq r \leq n-2$, by $BS_{n,r}$, we denote the starlike tree of order n with maximum degree r such that the r pendent paths have almost equal lengths, i.e., for any two pendent paths with length ℓ and s, $|\ell - s| = 0, 1$. Let $n - 1 = rs + t$, where $0 \leq t \leq r-1$. Then $BS_{n,r}$ consists of t pendent paths of length $s+1$ and $r-t$ pendent paths of length s at a common vertex.

Theorem 3.3. *Among all trees of order n with r pendent vertices, $BS_{n,r}$ is the unique tree with maximum Mostar index, where* $3 \leq r \leq n-2$.

Proof. Let T be a tree of order n with r pendent vertices such that $Mo(T)$ is as large as possible.
Claim. T contains exactly one branch vertex.

Suppose on contrary that T contains at least two branch vertices. Obviously, we may choose two branch vertices, say x and y, such that $d_T(x,y)$ is as small as possible. Let P be the path connecting x and y. If $d_T(x,y) > 1$, then each internal vertex of P is of degree 2. Let n_x (n_y, respectively) be the order of the component of $T - E(P)$ containing x (y, respectively). Assume that $n_x \geq n_y$. Obviously, $n_y \leq \frac{n}{2}$. Let z be the neighbor of y in P and w be any other neighbor of y. Let $n_w = n_w(yw|T)$. Let $T' = T - yw + zw$. Obviously, T' is a tree of order n with r pendent vertices. Note that $\psi_T(e) = \psi_{T'}(e)$ for $e \in E(T) \setminus \{zy, yw\} = E(T') \setminus \{zy, zw\}$ and $\psi_T(yw) = \psi_{T'}(zw)$. Thus

$$Mo(T) - Mo(T') = \psi_T(zw) - \psi_{T'}(zw) = |n_y - (n - n_y)| - |(n_y - n_w) - n - (n_y - n_w)| < 0,$$

implying that $Mo(T) < Mo(T')$, a contradiction. This proves the claim.

By the claim, T consists of r some pendent paths at a common vertex. Let $a_1, \dots, a_r$ be the lengths of these pendent paths, where $a_1 \geq \cdots \geq a_r \geq 1$.

Suppose that $a_i - a_j \geq 2$ for some pair of i and j with $1 \leq i < j \leq r$. Let u be the vertex with maximum degree r. Then $T \cong G_{u;a_i,a_j}$, where G is the graph obtained from T by deleting vertices of degree two or one in two pendent paths with lengths a_i and a_j, respectively. Obviously, $G_{u;a_i-1,a_j+1}$ is a tree of order n with r pendent vertices. By Lemma 2.2, $Mo(T) < Mo(G_{u;a_i-1,a_j+1})$, a contradiction. Therefore, $a_i - a_j = 0, 1$ for any i and j with $1 \leq i < j \leq r$. That is, $T \cong BS_{n,r}$. □

The matching number of a graph is the number of edges in a maximum matching (i.e., set of disjoint edges with maximum number of edges). The domination number of a graph is the number of vertices in a minimum dominating set (a set of vertices with minimum number of vertices such that every vertex outside this set is adjacent to at least one member of the set).

For $1 \leq m \leq \frac{n}{2}$, let $A_{n,m}$ be the tree consists of $m-1$ pendent paths of length two and $n - 2(m-1)$ pendent edges at a common vertex. Obviously, $A_{n,m} = BS_{n,n-m}$ for $1 \leq m \leq \frac{n}{2}$.

Corollary 3.4. *Among trees of order n with matching number s (domination number t, respectively), $A_{n,s}$ ($A_{n,t}$, respectively) is the unique tree with maximum Mostar index, where $1 \le s \le \frac{n}{2}$ ($1 \le t \le \frac{n}{2}$, respectively).*

Proof. It is trivial if $n = 2$ as $A_{2,1} = P_2$. Suppose that $n \ge 3$. Let T be a tree of order n with matching number s and domination number t. By König's theorem, s is equal to the minimum cardinality of a covering of T. As a covering of T is also a dominating set of T. So $t \le s$. Then $n - s \le n - t$. Denote by r the number of pendent vertices of T. Note that $r \le n - s \le n - t$.

We claim that $Mo(B_{n,r}) < Mo(B_{n,r+1})$. This is clearly true if $r = 2$ as $B_{n,2} = P_n$. Suppose that $r \ge 3$. For $3 \le r \le n-2$, let u be the vertex of degree r in $BS_{n,r}$ and v a neighbor of u in a pendent path of length at least two. By Lemma 2.1 or 2.2, we have $Mo(B_{n,r}) < Mo(B_{n,r}/uv)$. Note that $B_{n,r}/uv$ consists of $r+1$ pendent paths at a common vertex u. Now, by Lemma 2.2, $Mo(B_{n,r}/uv) < Mo(BS_{n,r+1})$ if $B_{n,r}/uv \ncong BS_{n,r+1}$. It follows that $Mo(B_{n,r}) < Mo(B_{n,r+1})$ for $2 \le r \le n-2$.

If T maximizes the Mostar index among trees of order n with matching number s, then, by Theorem 3.3 and the above claim, $T \cong BS_{n,r}$ with $r = n - s$, i.e., $T \cong BS_{n,n-s} = A_{n,s}$.

If T maximizes the Mostar index among trees of order n with domination number t, then, by Theorem 3.3 and the above claim, $T \cong BS_{n,r}$ with $r = n - t$, i.e., $T \cong BS_{n,n-t} = A_{n,t}$. □

Theorem 3.5. *Among trees of order n with r pendent vertices, $A_n(\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor)$ is the unique tree with minimum Mostar index, where $3 \le r \le n-2$.*

Proof. Let T be a tree of order n with r pendent vertices such that $Mo(T)$ is as small as possible.
Claim. T has at most two branch vertices.

Suppose on contrary that T contains at least three branch vertices. Obviously, we may choose two branch vertices, say x and y, such that $d_T(x, y)$ is as large as possible. Let P be the path connecting x and y. Let n_x (n_y, respectively) be the order of the component of $T - E(P)$ containing x (y, respectively). Obviously, some internal vertex of P is a branch vertex of T. So we may choose branch vertices w and z in P such that both $d_T(x, w)$ and $d_T(z, y)$ are as small as possible. Assume that $n_x + d_T(x, w) \ge n_y + d_T(z, y)$. Then $n_y + d_T(z, y) \le \frac{n}{2}$. Let $s = d_T(z, y)$ and let $z_0 \dots z_s$ be the path from z to y, where $z_0 = z$ and $z_s = y$. Let $u_1, \dots u_p$ be the neighbors of z outside P in T, where $p = d_T(z) - 2$. Let $T' = T - \{zu_i : i = 1, \dots, p\} + \{yu_i : i = 1, \dots, p\}$. Evidently, T' is a tree of order n with r pendent vertices. Let n'_z be the total number of vertices of the components of $T - z$ containing one of $u_1, \dots, u_p$. For $i = 0, \dots, s$, we have $n_y + s - i < \min\{n_y + n'_z + s - i, n - (n_y + n'_z + s - i)\} \le \frac{n}{2}$, implying that $\psi_T(z_{i-1}z_i) = |(n_y + s - i) - (n - (n_y + s - i))| > |n_y + n'_z + s - i - (n - (n_y + n'_z + s - i))| = \psi_{T'}(z_{i-1}z_i)$. Therefore, $Mo(T) - Mo(T') = \sum_{i=1}^{s}(\psi_T(z_{i-1}z_i) - \psi_{T'}(z_{i-1}z_i)) > 0$, i.e., $Mo(T) > Mo(T')$, a contradiction. This proves the claim.

By the claim, T has at most two branch vertices. If T has exactly one branch vertex, then T consists of r pendent paths at a common vertex. By Lemma 2.2, $T \cong A_n(r-1, 1)$. By Lemma 2.3, we have $r = 3$ and $T \cong A_n(\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor)$.

Suppose that T has exactly two branch vertices, say x and y. Let $a = d_T(x) - 1$ and $b = d_T(y) - 1$. Obviously, $a, b \ge 2$, and there are a pendent paths at x and b pendent paths at y. By Lemma 2.2, among the a (b, respectively) pendent paths at x (y, respectively), all except one are of length one. As early, let P be the path connecting x and y, and let n_x (n_y, respectively) be the order of the component of $T - E(P)$ containing x (y, respectively). Assume that $n_x \ge n_y$. Then $n_y \le \frac{n}{2}$.

Suppose that there is a pendent path at y whose length is at least two. Let $y_0 \dots y_\ell$ be this path, where $y_0 = y$. Let $T'' = T - \{yv : v \in N\} + \{y_1 v : v \in N\}$, where N is the set of pendent neighbors of y. Obviously, T'' is a tree of order n with r pendent vertices. As $\ell < n_y - 1 < \frac{n}{2}$, we have

$$Mo(T) - Mo(T') = \psi_T(y_0y_1) - \psi_{T''}(y_0y_1) = |\ell - (n - \ell)| - |n_y - 1 - (n - n_y + 1)| > 0,$$

i.e., $Mo(T) > Mo(T'')$, a contradiction. Thus, all pendent paths at y are of length one.

Suppose that $n_x > \frac{n}{2}$ and there is a pendent path at x whose length is at least two. Let $P = z_0 \dots z_s$, where $z_0 = x$ and $z_s = y$. Let u be a pendent neighbor of x. Let $T^* = T - xu + yu$. Note that T^* is a tree of order n with r pendent vertices, and that

$$Mo(T) - Mo(T^*) = \psi_T(z_{s-1}z_s) - \psi_{T^*}(z_0z_1) = |n_y - (n - n_y)| - |n_x - 1 - (n - n_x + 1)|.$$

If $n_x > \frac{n+1}{2}$, i.e., $n_x \geq \frac{n}{2}+1$ then, as $n_y < n-n_x+1 \leq \frac{n}{2}$, we have $Mo(T) - Mo(T^*) > 0$, i.e., $Mo(T) > Mo(T')$, a contradiction. Thus, $n_x = \frac{n+1}{2}$ and then $n_y = \frac{n+1}{2} - r$. If $s \geq 2$, then $n_y < n_x - 1 < \frac{n}{2}$, implying that $Mo(T) > Mo(T^*)$, also a contradiction. Therefore, we have $s = 1$, and then $n_y = n_x - 1$, implying that $Mo(T) = Mo(T^*)$. Now consider the tree T^*. Suppose that $a \geq 3$. Then the order of the component of $T^* - xy$ containing x is smaller than $\frac{n}{2}$. By similar argument as above by deleting all pendent edges at x in T^* and adding the same number of pendent edges at the neighbor of x in the pendent path with length at least two to get a tree T^{**} of order n with r pendent vertices such that $Mo(T^*) > Mo(T^{**})$. Then $Mo(T) > Mo(T^{**})$, a contradiction. Therefore, we are left with the case $a = 2$, and then $T^* \cong A_n(1, b+1)$ with $b+2 = r \geq 4$. By Lemma 2.3, $Mo(T) = Mo(T^*) > Mo(A_n(\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor))$, a contradiction. Therefore, all pendent paths at x are of length one (which follows by similar argument as above if $n_x \leq \frac{n}{2}$). Then $T \cong A_n(a, b)$, where $a + b = r$ and $a \geq b \geq 2$. By Lemma 2.3, we have $T \cong A_n(\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor)$. □

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