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On Characterizations of Finite Topological Spaces with Granulation and Evidence Theory

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Abstract. The theory of finite topological spaces can be used to investigate deep well-known problems in Topology, Algebra, Geometry and Artificial Intelligence. To represent uncertainty knowledge of a finite topological space, two kinds of measurement of a finite topological space are first introduced. Firstly, a kind of granularity of a finite topological space is defined, and properties of the granularity are explored. Secondly, relationships between the belief and plausibility functions in the Dempser-Shafer theory of evidence and the interior and closure operators in topological theory are established. The probabilities of interior and closure of sets construct a pair of belief and plausibility functions and its belief structure. And, for a belief structure with some properties, there exists a probability and a finite topology such that the belief and plausibility functions defined by the given belief structure are, respectively, the belief and plausibility functions by the topology. Then a necessary and sufficient condition for a belief structure to be the belief structure induced by a finite topology is presented.

1. Introduction

The theory of finite topological spaces is interesting, which can be used to investigate deep well-known problems in Topology, Algebra and Geometry. In 1937, Alexandroff paid attention to the finite topological spaces, and presented the relationships between finite topological spaces and finite partially ordered sets (posets) [1]. In 1966, Stong discussed the combinatorics of finite topological spaces and explained their homotopy types [27]. In the same year, McCord discovered the relationships between finite topological spaces and compact polyhedra [22]. Then, a few interesting papers on finite topological spaces appeared [12, 15, 28]. In 2003, May synthesized the most important ideas on finite topological spaces known until that time, who also noted that Stong's combinatorial point of view and the bridge constructed by McCord could be used together to attack problems in Algebraic Topology using finite spaces [19–21]. Afterward, more researchers explored the theory of finite topological spaces [2–6, 8, 23]. Moreover, the results about

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finite topological spaces can be applied to graph theory [9], and finite topological spaces have relationships with toric varieties [7].

The definition of granulation was first introduced by Zadeh [31] in 1979. To present characterizations of the granulation, granular computing was introduced [32, 33]. The granular computing contains all the results on granulations. The granulation presents a more visual and easily understandable description for a partition or covering on the universe. The definitions of granulation of partition and covering have been presented to evaluate uncertainty of a knowledge from an information system [10, 16–18, 26, 29]. Finite topology is a covering with topological structure, which can be used to characterize the knowledge from an information system [30]. Then it is necessary to define a granularity of a finite topology to evaluate uncertainty of the knowledge by the topology.

The concept of lower and upper probabilities was presented by Dempster in 1967 [25]. In 1976, it was extended by Shafer as a theory, i.e. the Dempster-Shafer theory of evidence or the theory of belief function, which is a generalization of Bayesian theory of subjective judgment. The Dempster-Shafer theory is one of the methods used to model and manipulate uncertain information. In Dempster-Shafer theory of evidence, there exists a dual pair of uncertainty measures, the plausibility and belief functions. In the theory of general topology, there exists a pair of operators, topological interior and closure operators. It seems that there are some natural correspondences between the theory of evidence and the theory of general topology, and it is interesting to construct relationships between the two theories.

The purpose of this paper is to characterize finite topological spaces by granulations and evidence theory. In Section 2, we introduce a kind of granularity of a finite topological space, and present properties of the granularity. In Section 3, we establish relationships between the plausibility and belief functions in the Dempser-Shafer theory of evidence and the interior and closure operators in topological theory. The probabilities of interior and closure of sets construct a pair of belief and plausibility functions and its belief structure. Properties of the belief structure are also discussed. Conversely, for a belief structure with some properties, there exists a probability and a finite topology such that the belief and plausibility functions defined by the given belief structure are, respectively, the belief and plausibility functions by the finite topology. Then a necessary and sufficient condition for a belief structure to be the belief structure induced by a finite topology is presented.

2. On Measurement of a Finite Topological Space Based on Granules

A finite topological space is a topological space having only a finite number of points. A finite topological space can be considered as a knowledge with topological structure in information field. Then it is meaningful to discuss granulations of a finite topological space to evaluate the uncertainty of the knowledge by the topological space. In this section, we discussed a kind of knowledge granulation of a finite topological space. Firstly, we introduced a proposition about the minimal base in a finite topological space, which is presented by Stong [27].

Proposition 2.1. ([27]) *Let (X, τ) be a finite topological space. Then, there exists a unique minimal base $\mathcal{B} = \{(x)_\tau | x \in X\}$ for the topology, where $(x)_\tau = \cap\{G \in \tau | x \in G\}$.*

Now, we give a definition of granularity of a finite topological space based on the minimal base.

Definition 2.2. Let (X, τ) be a finite topological space. Define a granularity of τ by

$$Gr(\tau) = \sum_{x \in X} \frac{|(x)_\tau|^2}{|X|^3}.$$

According to Definition 2.2, we can obtain

Proposition 2.3. *Let (X, τ) be a finite topological space.*

(1) If (X, τ) is discrete, then $Gr(\tau) = \frac{1}{|X|^2}$.

(2) If (X, τ) is trivial, then $Gr(\tau) = 1$.

Proof. (1) If (X,τ) is discrete, then $(x)_\tau = \{x\}$ for all $x \in X$. Thus

$$Gr(\tau) = \sum_{x\in X} \frac{|\{x\}|^2}{|X|^3} = \frac{1}{|X|^2}.$$

(2) If (X,τ) is trivial, then $(x)_\tau = X$ for all $x \in X$. Hence

$$Gr(\tau) = \sum_{x\in X} \frac{|X|^2}{|X|^3} = 1. \quad \square$$

As a consequence of Proposition 2.3, we get the next corollary.

Corollary 2.4. *Let (X,τ) be a finite topological space. Then $\frac{1}{|X|^2} \le Gr(\tau) \le 1$.*

Proof. For each $x \in X$, $\{x\} \subseteq (x)_\tau \subseteq X$. It follows from Proposition 2.3 that $\frac{1}{|X|^2} \le Gr(\tau) \le 1$. $\square$

Proposition 2.5. *Let τ_1, τ_2 be two topologies of X. If $\tau_1 \subseteq \tau_2$, then $Gr(\tau_2) \le Gr(\tau_1)$.*

Proof. Since $\tau_1 \subseteq \tau_2$, we deduce that $(x)_{\tau_2} \subseteq (x)_{\tau_1}$ for all $x \in X$. It follows that $|(x)_{\tau_2}| \le |(x)_{\tau_1}|$. Therefore,

$$Gr(\tau_2) = \sum_{x\in X} \frac{|(x)_{\tau_2}|^2}{|X|^3} \le \sum_{x\in X} \frac{|(x)_{\tau_1}|^2}{|X|^3} = Gr(\tau_1). \quad \square$$

From Proposition 2.3, one can see that the granularity of τ has close relationships with the separability of τ. By Proposition 2.5, if a topology is finer, then the granularity of the topology is greater.

Definition 2.6. Let (X,τ) be a finite topological space. Define a distinctive degree of τ by

$$GE(\tau) = \sum_{x\in X} \frac{|(x)_\tau|}{|X|^2}\left(\frac{|X|}{|(x)_\tau|} - \frac{|(x)_\tau|}{|X|}\right).$$

We have properties of $GE(\tau)$ as following

Corollary 2.7. *Let (X,τ) be a finite topological space. Then*
(1) $Gr(\tau) + GE(\tau) = 1$.
(2) $0 \le GE(\tau) \le 1 - \frac{1}{|X|^2}$.
(3) If (X,τ) is discrete, then $GE(\tau) = 1 - \frac{1}{|X|^2}$.
(4) If (X,τ) is trivial, then $GE(\tau) = 0$.

Proof. (1) By Definitions 2.2 and 2.6, we have that

$$\begin{aligned} Gr(\tau) + GE(\tau) &= \sum_{x\in X} \frac{|(x)_\tau|^2}{|X|^3} + \sum_{x\in X} \frac{|(x)_\tau|}{|X|^2}\left(\frac{|X|}{|(x)_\tau|} - \frac{|(x)_\tau|}{|X|}\right) \\ &= \sum_{x\in X} \frac{|(x)_\tau|}{|X|^2}\frac{|X|}{|(x)_\tau|} = 1. \end{aligned}$$

(2) According to (1) and Corollary 2.4, we get that $0 \le GE(\tau) \le 1 - \frac{1}{|X|^2}$.
(3)-(4) Combining (1) and Proposition 2.1, we have the conclusions. $\square$

Corollary 2.8. *Let τ_1, τ_2 be two topologies of X. If $\tau_1 \subseteq \tau_2$, then $GE(\tau_1) \le GE(\tau_2)$.*

Proof. Since $\tau_1 \subseteq \tau_2$, we have that $Gr(\tau_2) \le Gr(\tau_1)$. Then, by Corollary 2.7(1), we obtain that $GE(\tau_1) = 1 - Gr(\tau_1) \le 1 - Gr(\tau_2) = GE(\tau_2)$. $\square$

Example 2.9. Let $X = \{a,b,c,d,e\}$. There exist four topologies on X as follows:
$\tau_1 = \{\emptyset, X\}$ is a trivial topology,
$\tau_2 = \{\emptyset, \{a,b,c\}, \{c\}, \{c,d,e\}, X\}$,
$\tau_3 = \{\emptyset, \{a,b\}, \{a,b,c\}, \{c\}, \{c,d\}, \{e\}, \{a,b,c,d\}, \{a,b,e\}, \{a,b,c,e\}, \{c,e\}, \{c,d,e\}, X\}$,
τ_4 is a discrete topology.
It is easy to have $\tau_1 \subseteq \tau_2 \subseteq \tau_3 \subseteq \tau_4$. By Definition 2.2, we obtain
$Gr(\tau_1) = \frac{5^2}{5^3} + \frac{5^2}{5^3} + \frac{5^2}{5^3} + \frac{5^2}{5^3} + \frac{5^2}{5^3} = 1$,
$Gr(\tau_2) = \frac{3^2}{5^3} + \frac{3^2}{5^3} + \frac{1^2}{5^3} + \frac{3^2}{5^3} + \frac{3^2}{5^3} = \frac{37}{125}$,

$Gr(\tau_3) = \frac{2^2}{5^3} + \frac{2^2}{5^3} + \frac{1^2}{5^3} + \frac{2^2}{5^3} + \frac{1^2}{5^3} = \frac{14}{125}$,
$Gr(\tau_4) = \frac{1}{5^3} + \frac{1}{5^3} + \frac{1}{5^3} + \frac{1}{5^3} + \frac{1}{5^3} = \frac{1}{25}$.
According to Definition 2.6, we have
$GE(\tau_1) = 0, GE(\tau_2) = \frac{88}{125}, GE_{\tau_3} = \frac{111}{125}, GE(\tau_4) = \frac{24}{25}$.
Then, it is clear that
$Gr(\tau_1) \geq Gr(\tau_2) \geq Gr(\tau_3) \geq Gr(\tau_4)$,
$GE(\tau_1) \leq GE(\tau_2) \leq GE(\tau_3) \leq GE(\tau_4)$.

3. On Measurement of a Finite Topology Based on Evidence Theory

In this section, we introduce the basic definitions of evidence theory, and construct relationships between evidence theory and the theory of finite topological spaces.

3.1. Basic Notions Related to Evidence Theory

In this subsection, some basic definitions about evidence theory will be recalled.

Definition 3.1. ([25]) Let X be a non-empty finite set. A set function $m : \mathcal{P}(X) \to [0, 1]$ is referred to as a basic probability assignment or mass distribution if it satisfies
$(M1)$ $m(\emptyset) = 0$, $(M2)$ $\sum\limits_{A \subseteq X} m(A) = 1$.
A set $A \subseteq X$ with $m(A) > 0$ is referred to as a focal element. Denote $M = \{A \subseteq X | m(A) > 0\}$. (M, m) is called a belief structure on X.

A pair of belief and plausibility functions can be derived in terms of the mass distribution.

Definition 3.2. ([11, 25]) A set function $Bel : \mathcal{P}(X) \to [0, 1]$ is referred to as a belief function if
$$Bel(A) = \sum_{B \subseteq A} m(B), \forall A \subseteq X.$$
A set function $Pl : \mathcal{P}(X) \to [0, 1]$ is referred to as a plausibility function,
$$Pl(A) = \sum_{B \cap A \neq \emptyset} m(B), \forall A \subseteq X.$$

Belief and plausibility functions based on the same belief structure are connected by the dual property $Pl(A) = 1 - Bel(A^c)$. Furthermore, $Bel(X) \leq Pl(A)$ for $A \subseteq X$.

Definition 3.3. Let Ω be a sample space, $\mathcal{F}$ is a σ-algebra on Ω. Then, a real-valued function $P : \mathcal{F} \to [0, 1]$ is referred to as a probability on $(\Omega, \mathcal{F})$ if it satisfies
(1) for any $X \in \mathcal{F}$, $0 \leq P(X) \leq 1$,
(2) $P(\Omega) = 1$,
(3) for any $X_i \in \mathcal{F}$ $(i = 1, 2, \cdots)$, if $X_i \cap X_j = \emptyset (i \neq j)$, then $P(\cup_{i=1}^{\infty} X_i) = \sum_{i=1}^{\infty} P(X_i)$.
Moreover, $(\Omega, \mathcal{F}, P)$ is a probability space.

3.2. Measure Interior Operator and Closure Operator by Evidence Theory

In this subsection, the interior and closure operators in a finite topological space will be measured by belief and plausibility functions. We firstly present three lemmas.

Lemma 3.4. *Let (X, τ) be a finite topological space. Then, for any $A \subseteq X$:*
$i(A) = \{x | (x)_\tau \subseteq A\}$,
$c(A) = \{x | (x)_\tau \cap A \neq \emptyset\}$.

Proof. For any $x_0 \in i(A)$, there exists an open set $B \in \tau$ such that $x_0 \in B \subseteq A$. Then $(x_0)_\tau \subseteq B \subseteq A$. It follows that $x_0 \in \{x|(x)_\tau \subseteq A\}$. Hence $i(A) \subseteq \{x|(x)_\tau \subseteq A\}$. Conversely, for any $x_0 \in \{x|(x)_\tau \subseteq A\}$, we have that $(x_0)_\tau \subseteq A$. By Proposition 2.1, $(x_0)_\tau$ is the minimal open set containing x_0, which implies that $x_0 \in i(A)$. Therefore, $\{x|(x)_\tau \subseteq A\} \subseteq i(A)$. Consequently, $i(A) = \{x|(x)_\tau \subseteq A\}$.

For any $y \in c(A)$, note that $(y)_\tau$ is an open set containing y, we have that $(y)_\tau \cap X \neq \emptyset$. Then $c(A) \subseteq \{x|(x)_\tau \cap A \neq \emptyset\}$. Conversely, for any $y \in \{x|(x)_\tau \cap A \neq \emptyset\}$, we obtain that $(y)_\tau \cap A \neq \emptyset$. Hence, for any open set B containing y, we get that $(y)_\tau \subseteq B$. It follows that $\emptyset \neq (y)_\tau \cap A \subseteq B \cap A$. Therefore, $y \in c(A)$, which implies that $\{x|(x)_\tau \cap A \neq \emptyset\} \subseteq c(A)$. We can conclude that $c(A) = \{x|(x)_\tau \cap A \neq \emptyset\}$. □

In Lemma 3.4, we give interior and closure of a set by the neighborhoods in the minimal based.

Lemma 3.5. *Let (X, τ) be a finite topological space. For any $A \subseteq X$, define $j(A) = \{x \in X|(x)_\tau = A\}$. Then:*
(1) $j(\emptyset) = \emptyset$,
(2) $\cup_{A\in\mathcal{P}(X)} j(A) = X$,
(3) $A \neq B \Rightarrow j(A) \cap j(B) = \emptyset$.

Proof. (1) For any $x \in U$, $x \in (x)_\tau \neq \emptyset$. Then $x \notin j(\emptyset)$, which follows that $j(\emptyset) = \emptyset$.

(2) For any $x \in X$, $x \in j((x)_\tau) \subseteq \cup_{A\in\mathcal{P}(X)} j(A)$. Hence $X \subseteq \cup_{A\in\mathcal{P}(X)} j(A)$. Clearly, $\cup_{A\in\mathcal{P}(X)} j(A) \subseteq X$. Thus $\cup_{A\in\mathcal{P}(X)} j(A) = X$.

(3) For any $x \in j(A)$, $(x)_\tau = A \neq B$, then $x \notin j(B)$. We deduce that $j(A) \cap j(B) = \emptyset$. □

From Lemma 3.5, in a finite topological space (X, τ), $\{j(A) \neq \emptyset|A \subseteq X\}$ constructs a partition of X. In the following, we character interior and closure of a set by the family of set $\{j(A)|A \subseteq X\}$.

Lemma 3.6. *Let (X, τ) be a finite topological space. Then, for any $A \subseteq X$,*
(1) $i(A) = \cup_{B\subseteq A} j(B)$,
(2) $c(A) = \cup_{B\cap A\neq\emptyset} j(B)$.

Proof. (1) For any $x \in i(A)$, by Lemma 3.4, we get that $(x)_\tau \subseteq A$. Then $x \in j((x)_\tau) \subseteq \cup_{B\subseteq A} j(B)$. It follows that $i(A) \subseteq \cup_{B\subseteq A} j(B)$. Conversely, for any $x \in \cup_{B\subseteq A} j(B)$, there exists a $B_0 \subseteq A$ such that $x \in j(B_0)$. Hence $(x)_\tau = B_0 \subseteq A$. According to Lemma 3.4, $x \in i(A)$, which implies that $\cup_{B\subseteq A} j(B) \subseteq i(A)$. Consequently, $i(A) = \cup_{B\subseteq A} j(B)$.

(2) For any $x \in c(A)$, it follows from Lemma 3.4 that $(x)_\tau \cap A \neq \emptyset$. Hence $x \in j((x)_\tau) \subseteq \cup_{B\cap A\neq\emptyset} j(B)$. Then $c(A) \subseteq \cup_{B\cap A\neq\emptyset} j(B)$. Conversely, for any $x \in \cup_{B\cap A\neq\emptyset} j(B)$, there exists a $B_0 \subseteq X$ such that $B_0 \cap A \neq \emptyset$ and $x \in j(B_0)$. Hence $(x)_\tau = B_0$. By Lemma 3.4, $x \in c(A)$, which implies that $\cup_{B\cap A\neq\emptyset} j(B) \subseteq c(A)$. We conclude that $\cup_{B\cap A\neq\emptyset} j(B) = c(A)$. □

According to Lemmas 3.5 and 3.6, we can measure interior and closure of a set by belief and plausibility functions.

Theorem 3.7. *Let $(X, \mathcal{P}(X), P)$ be a probability space, and τ a topology on X. For any $A \subseteq X$, define*
$Bel_\tau(A) = P(i(A))$, $Pl_\tau(A) = P(c(A))$.
Then Bel_τ and Pl_τ are the belief and plausibility functions, respectively.

Proof. Define a set function $m_\tau : \mathcal{P}(X) \to [0, 1]$ by

$$m_\tau(A) = P(j(A)),\ A \in \mathcal{P}(X).$$

Then $m_\tau(\emptyset) = P(j(\emptyset)) = P(\emptyset) = 0$. And, by Lemma 3.5, we have

$$\sum_{A\in\mathcal{P}(X)} m_\tau(A) = \sum_{A\in\mathcal{P}(X)} P(j(A)) = P(\cup_{A\in\mathcal{P}(X)} j(A)) = P(X) = 1,$$

Hence m_τ is a mass distribution.

By Lemmas 3.5 and 3.6, we get

$$Bel_\tau(A) = P(i(A)) = P(\cup_{B\subseteq A} j(B)) = \sum_{B\subseteq A} P(j(B)) = \sum_{B\subseteq A} m_\tau(B),$$

$$Pl_\tau(A) = P(c(A)) = P(\cup_{B\cap A\neq\emptyset} j(B)) = \sum_{B\cap A\neq\emptyset} P(j(B)) = \sum_{B\cap A\neq\emptyset} m_\tau(B).$$

Thus Bel_τ and Pl_τ are the belief and plausibility functions, respectively. □

In particularly, in Theorem 3.7, if we take $P(A) = \frac{|A|}{|X|}$, then we can obtain

Corollary 3.8. *Let (X, τ) be a finite topological space. For any $A \subseteq X$, define*
$Bel_\tau(A) = \frac{|i(A)|}{|X|}$, $Pl_\tau(A) = \frac{|c(A)|}{|X|}$.
Then Bel_τ and Pl_τ are the belief and plausibility functions, respectively.

Proof. Define a set function $P : \mathcal{P}(X) \to [0, 1]$ by: for any $A \subseteq X$,
$P(A) = \frac{|A|}{|X|}$.
Then it is clear that P is a probability measurement. By Theorem 3.7, it is easy to get the corollary. □

In the following, we employ an example to illustrate Theorem 3.7.

Example 3.9. Let $X = \{a, b, c, d\}$. $\tau = \{\emptyset, \{a, b, c\}, \{c\}, \{c, d\}, X\}$ is a topology on X. A probability measurement $P : \mathcal{P}(X) \to [0, 1]$ is defined as follows:
$P(\emptyset) = 0$, $P(\{a\}) = \frac{1+\sqrt{2}}{8}$, $P(\{b\}) = \frac{3}{8}$, $P(\{c\}) = \frac{2-\sqrt{2}}{8}$, $P(\{d\}) = \frac{2}{8}$,
$P(A) = \sum_{\{x \in A\}} P(\{x\})$ for all other $A \subseteq X$.
Then we have $(a)_\tau = (b)_\tau = \{a, b, c\}$, $(c)_\tau = \{c\}$, $(d)_\tau = \{c, d\}$. We also get $i(\emptyset) = \emptyset$, $i(\{a\}) = i(\{b\}) = i(\{d\}) = \emptyset$, $i(\{c\}) = \{c\}$, $i(\{a, b\}) = i(\{a, d\}) = i(\{b, d\}) = \emptyset$, $i(\{a, c\}) = i(\{b, c\}) = \{c\}$, $i(\{c, d\}) = \{c, d\}$, $i(\{a, b, c\}) = \{a, b, c\}$, $i(\{a, b, d\}) = \emptyset$, $i(\{a, c, d\}) = i(\{b, c, d\}) = \{c, d\}$. $i(\{a, b, c, d\}) = \{a, b, c, d\}$.

Therefore, $Bel_\tau(\emptyset) = P(i(\emptyset)) = P(\emptyset) = 0$,
$Bel_\tau(\{a\}) = P(i(\{a\})) = P(\emptyset) = 0$,
$Bel_\tau(\{b\}) = P(i(\{b\})) = P(\emptyset) = 0$,
$Bel_\tau(\{c\}) = P(i(\{c\})) = P(\{c\}) = \frac{2-\sqrt{2}}{8}$,
$Bel_\tau(\{d\}) = P(i(\{d\})) = P(\emptyset) = 0$,
$Bel_\tau(\{a, b\}) = P(i(\{a, b\})) = P(\emptyset) = 0$,
$Bel_\tau(\{a, c\}) = P(i(\{a, c\})) = P(\{c\}) = \frac{2-\sqrt{2}}{8}$,
$Bel_\tau(\{a, d\}) = P(i(\{a, d\})) = P(\emptyset) = 0$,
$Bel_\tau(\{b, c\}) = P(i(\{c\})) = P(\{c\}) = \frac{2-\sqrt{2}}{8}$,
$Bel_\tau(\{a, b, c\}) = P(i(\{a, b, c\})) = P(\{a, b, c\}) = \frac{6}{8}$,
$Bel_\tau(\{a, b, d\}) = P(i(\{a, b, d\})) = P(\emptyset) = 0$,
$Bel_\tau(\{a, c, d\}) = P(i(\{a, c, d\})) = P(\{c, d\}) = \frac{4-\sqrt{2}}{8}$,
$Bel_\tau(\{b, c, d\}) = P(i(\{b, c, d\})) = P(\{c, d\}) = \frac{4-\sqrt{2}}{8}$,
$Bel_\tau(X) = P(i(X)) = P(X) = 1$.

We present properties of the belief structure of the belief function Bel_τ and plausibility function Pl_τ in the following Proposition 3.10.

Proposition 3.10. *Let $(X, \mathcal{P}(X), P)$ be a probability space, and τ a topology on X. If $P(A) > 0$ for all $A \neq \emptyset$, then the belief structure $(\mathcal{M}, m_\tau)$ of the belief function $Bel_\tau(A) = P(i(A))$ and plausibility function $Pl_\tau(A) = P(c(A))$ has properties as follows:*
(1) $\cup\mathcal{M} = X$,
(2) for any $A, B \in \mathcal{M}$, any $x \in A \cap B$, there exists a $G \in \mathcal{M}$ such that $x \in G \subseteq A \cap B$,
(3) for any $B \in \mathcal{M}$, B could not be represented by the union of some elements in $\mathcal{M} \setminus \{B\}$.

Proof. $\mathcal{M} = \{(x)_\tau | x \in X\} = \mathcal{B}$. In deed, for any $x \in X$, $x \in j((x)_\tau) \neq \emptyset$. Hence $m_\tau((x)_\tau) = P(j((x)_\tau)) > 0$. Then, $(x)_\tau \in \mathcal{M}$. It follows that $\{(x)_\tau | x \in U\} \subseteq \mathcal{M}$. Conversely, for any $A \in \mathcal{M}$, $m_\tau(A) = P(j(A)) > 0$. Then $j(A) = \{x | (x)_\tau = A\} \neq \emptyset$. It follows that there exists an $x_0 \in U$ such that $(x_0)_\tau = A$. Hence $A \in \{(x)_\tau | x \in X\}$. We can conclude that $\mathcal{M} \subseteq \{(x)_\tau | x \in X\}$.

Since $\mathcal{M}$ is the minimal base of the topology τ, it is easy to obtain (1) and (2).

(3) If not, there exist $x_0, x_1, x_2, \cdots, x_m \in X$ such that $\cup_{i=1}^m (x_i)_\tau = (x_0)_\tau$ and $(x_i)_\tau \neq (x_0)_\tau (i = 1, 2, \cdots, m)$. It is clear that $x_0 \in (x_0)_\tau = \cup_{i=1}^n (x_i)_\tau$. Then there exists an $i_0 \in \{1, 2, \cdots, m\}$ such that $x_0 \in (x_{i_0})_\tau$. Hence $(x_0)_\tau \in (x_{i_0})_\tau$ and $(x_{i_0})_\tau \subseteq (x_0)_\tau$. Thus $(x_{i_0})_\tau = (x_0)_\tau$, which contradicts the fact $(x_{i_0})_\tau \neq (x_0)_\tau$. □

3.3. Construct a Probability Space with a Topology from a Belief Structure

From Theorem 3.7, the probabilities of interior and closure of sets construct a pair of belief and plausibility functions and its belief structure. Conversely, for a belief structure, there exist a probability and a finite topology such that the belief and plausibility functions defined by the given belief structure are, respectively, the belief and plausibility functions by the topology.

Theorem 3.11. *Let m be a mass distribution on X, its belief structure be $(\mathcal{M}, m)$, and belief function and plausibility function be Bel and Pl. Then the following are equivalent.*

(1) $\mathcal{M}$ satisfies that (1a) $\cup\mathcal{M} = X$, (1b) for any $A, B \in \mathcal{M}$, any $x \in A \cap B$, there exists a $G \in \mathcal{M}$ such that $x \in G \subseteq A \cap B$, (1c) for any $B \in \mathcal{M}$, B could not be represented by the union of some elements in $\mathcal{M}\backslash\{B\}$.

(2) There exist a topology τ and a probability $P : \mathcal{P}(X) \to [0, 1]$ with $P(A) > 0$ for all $A \neq \emptyset$ such that $Bel_\tau(A) = P(i(A)) = Bel(A)$ and $Pl_\tau(A) = P(c(X)) = Pl(X)$ for all $A \subseteq X$.

Proof. (2) ⇒ (1). According to (2), $Bel_\tau = Bel$ and $Pl_\tau = Pl$. Then $\mathcal{M}$ is the family of all the focal elements of Bel_τ. By Proposition 3.10, we can obtain (1).

(1) ⇒ (2). First, let $\mathcal{M} = \{B_1, B_2, \cdots, B_n\}$. By (1a) and (1b), $\mathcal{M}$ is a base for a topology, which is denoted by τ.

Second, for any $A \subseteq X$, define $j(A) = \{x \in X | (x)_\tau = A\}$. Then we have

(a1) $j(\emptyset) = \emptyset$,
(a2) $\cup_{i=1}^n j(B_i) = X$,
(a3) $B_i \neq B_j \Rightarrow j(B_i) \cap j(B_j) = \emptyset$,
(a4) for any $B \notin \mathcal{M}$, $j(B) = \emptyset$,
(a5) for any $B_i \in \mathcal{M}$, $j(B_i) \neq \emptyset$.

Now, we present the proof of (a1)-(a5).

(a1) For any $x \in X$, $(x)_\tau \neq \emptyset$, then $x \notin j(\emptyset)$. It follows that $j(\emptyset) = \emptyset$.

(a2) For any $x \in X$, $(x)_\tau$ is an open set containing x. Then, there exists a $B_i \in \mathcal{M}$ such that $x \in B_i \subseteq (x)_\tau$. Since B_i is an open set and $x \in B_i$, we have $(x)_\tau \subseteq B_i$. It follows that $(x)_\tau = B_i$. Then $x \in j(B_i)$, which implies that $X \subseteq \cup_{i=1}^n j(B_i)$. Consequently, $X = \cup_{i=1}^n j(B_i)$.

(a3) For any $x \in X$, if $x \in j(B_i)$, then $(x)_\tau = B_i \neq B_j$. Hence $x \notin j(B_j)$. Similarly, for any $x \in X$, if $x \in j(B_j)$, then $x \notin j(B_i)$. Therefore, $j(B_i) \cap j(B_j) = \emptyset$.

(a4) For any $B \notin \mathcal{M}$, suppose that $j(B) \neq \emptyset$. Then there exists an $x \in X$ such that $x \in j(B)$. By (a2), there exists a $B_i \in \mathcal{M}$ such that $x \in j(B_i)$. It follows that $B_i = (x)_\tau = B$, which contradicts the fact $B \notin \mathcal{M}$.

(a5) For any $B_i \in \mathcal{M}$, there exists an $x \in B_i$ such that $(x)_\tau = B_i$. Or else, for any $x \in B_i$, $(x)_\tau \subseteq B_i$ and $(x)_\tau \neq B_i$. By (a2), we know that there exists a $B_j \in \mathcal{M}$ such that $x \in j(B_j)$. It implies that $(x)_\tau = B_j \subsetneq B_i$. Then B_i is the union of some elements in $\mathcal{M}\backslash\{B_i\}$, which contradicts the condition (1c). Hence $j(B_i) \neq \emptyset$.

Third, define a real-value function $P : \mathcal{P}(X) \to [0, 1]$ by:

$$P(\emptyset) = 0;$$
$$P(\{x\}) = \frac{m(B_i)}{|j(B_i)|}, \text{ for any } x \in X, x \in j(B_i);$$
$$P(A) = \textstyle\sum_{x\in A} P(\{x\}), \text{ for any } A \subseteq X.$$

Then P is a probability on X. Initially, for any $A \subseteq X$, $P(A) \geq 0$. Next, according to (a2), (a3) and (a5), we have

$$P(X) = P(\cup_{i=1}^n j(B_i)) = \textstyle\sum_{i=1}^n P(j(B_i))) = \sum_{i=1}^n \sum_{x\in j(B_i)} P(\{x\})$$
$$= \textstyle\sum_{i=1}^n \sum_{x\in j(B_i)} \frac{m(B_i)}{|j(B_i)|} = \sum_{i=1}^n m(B_i) = 1.$$

Finally, for any $A, B \subseteq X$, if $A \cap B = \emptyset$, then

$$P(A \cup B) = \textstyle\sum_{x\in A\cup B} P(\{x\}) = \sum_{x\in A} P(\{x\}) + \sum_{x\in B} P(\{x\}) = P(A) + P(B).$$

4) For any $B \notin \mathcal{M}$, according to (a4), $m_\tau(B) = P(j(B)) = P(\emptyset) = 0 = m(B)$. For any $B \in \mathcal{M}$, by (a5), $m_\tau(B) = P(j(B)) = \sum_{x\in j(B)} P(\{x\}) = \sum_{x\in j(B)} \frac{m(B)}{|j(B)|} = m(B)$.

Then we can have that

$$Bel_\tau(A) = P(i(A)) = P(\cup_{B\subseteq A} j(B)) = \textstyle\sum_{B\subseteq A} P(j(B)) = \sum_{B\subseteq A} m(B) = Bel(A),$$
$$Pl_\tau(A) = P(c(A)) = P(\cup_{B\cap A\neq\emptyset} j(B)) = \textstyle\sum_{B\cap A\neq\emptyset} P(j(B)) = \sum_{B\cap A\neq\emptyset} m(B) = Pl(A). \quad \square$$

By Theorem 3.11, we can know that a belief structure with some properties, if and only if there exist a probability and a finite topology such that the belief and plausibility functions defined by the given belief structure are, respectively, the belief and plausibility functions by the finite topology.

Example 3.12. Let $X = \{a, b, c, d, e, f\}$. A mass function m on X is defined as follows:
$m(\{a, b\}) = \frac{1+\sqrt{2}}{20}$, $m(\{b\}) = \frac{2-\sqrt{2}}{10}$, $m(\{b, c\}) = \frac{2+\sqrt{2}}{20}$, $m(\{b, c, d\}) = \frac{2}{5}$, $m(\{e, f\}) = \frac{1}{4}$,
$m(A) = 0$ for all other $A \subseteq X$.
Then the set of all the focal elements of m is $\mathcal{M} = \{\{a, b\}, \{b\}, \{b, c\}, \{b, c, d\}, \{e, f\}\}$. Clearly, $\mathcal{M}$ satisfies the conditions (1)-(3) in Theorem 3.11. We can have the topology τ with $\mathcal{M}$ as a base. We can deduce that $(a)_\tau = \{a, b\}$, $(b)_\tau = \{b\}$, $(c)_\tau = \{b, c\}$, $(d)_\tau = \{b, c, d\}$, $(e)_\tau = (f)_\tau = \{e, f\}$. Hence $j(\{a, b\}) = \{a\}$, $j(\{b\}) = \{b\}$, $j(\{b, c\}) = \{c\}$, $j(\{b, c, d\}) = \{d\}$, $j(\{e, f\}) = \{e, f\}$, $j(A) = \emptyset$ for all other $A \subseteq U$.

Thus a probability on X is defined as:
$P(\emptyset) = \emptyset$, $P(\{a\}) = \frac{m(\{a,b\})}{|j(\{a,b\})|} = \frac{1+\sqrt{2}}{20}$, $P(\{b\}) = \frac{m(\{b\})}{|j(\{b\})|} = \frac{2-\sqrt{2}}{10}$, $P(\{c\}) = \frac{m(\{b,c\})}{|j(\{b,c\})|} = \frac{2+\sqrt{2}}{20}$, $P(\{d\}) = \frac{m(\{b,c,d\})}{|j(\{b,c,d\})|} = \frac{2}{5}$, $P(\{e\}) = P(\{f\}) = \frac{m(\{e,f\})}{|j(\{e,f\})|} = \frac{1}{8}$, $P(A) = \sum_{x \in A} P(\{x\})$.

Therefore,
$m_\tau(\{b\}) = P(j(\{b\})) = P(\{b\}) = \frac{2-\sqrt{2}}{10} = m(\{b\})$,
$m_\tau(\{b, c\}) = P(j(\{b, c\})) = P(\{c\}) = \frac{2+\sqrt{2}}{20} = m(\{b, c\}$,
$m_\tau(\{b, c, d\}) = P(j(\{b, c, d\})) = P(\{d\}) = \frac{2}{5} = m(\{b, c, d\})$,
$m_\tau(\{a, b\}) = P(j(\{a, b\})) = P(\{a\}) = \frac{1+\sqrt{2}}{20} = m(\{a, b\})$,
$m_\tau(\{e, f\}) = P(j(\{e, f\})) = P(\{e, f\}) = P(\{e\}) + P(\{f\}) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} = m(\{e, f\})$,
$m_\tau(A) = P(j(A)) = P(\emptyset) = 0 = m(A)$ for all other $A \subseteq U$.
Then we get that for any $A \subseteq U$,
$Bel_\tau(A) = P(i(A)) = \sum_{B \subseteq A} m_\tau(B) = \sum_{B \subseteq A} m(B) = Bel(A)$,
$Pl_\tau(A) = P(c(A)) = \sum_{B \cap A \neq \emptyset} m_\tau(B) = \sum_{B \cap A \neq \emptyset} m(B) = Pl(A)$.

4. Conclusion

In this paper, we have explored two kinds of measurement of a finite topological space. We have first introduced a granularity of a finite topological space, and have presented properties of the granularity of a finite topological space. We have also presented that the probability of interior and closure of sets construct a pair of belied and plausibility functions. And for an belief structure with some properties, there exist a probability and a finite topology such that the belief and plausibility functions defined by the given belief structure are, respectively, the belief and plausibility functions by the finite topology. Then a necessary and sufficient condition for a belief structure to be the belief structure induced by a finite topology have been presented. In our future work, we will use the results in this paper to present a new algorithm for knowledge discovery in information systems.

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