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Schrödinger Perturbations with Concave Control Function

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Abstract. We study Schrödinger perturbations series of integral kernels on space-time assuming concavity of the majorant of the first nontrivial term of the series. We give explicit estimates of the perturbation series in terms of the original kernel and the majorant.

1. Introduction

Schrödinger perturbation consists in adding to a given operator, say the Laplacian Δ, an operator of multiplication by a function, say q. Estimates of the Green function and the heat kernel of Schrödinger operators $\Delta + q$ were widely studied, for example in [10, 11, 19, 20]. Local integral smallness of the function q, formulated by Kato-type conditions [10, 20], played an important role in these considerations. Similar Schrödinger-type operators based on the fractional Laplacian $\Delta^{\alpha/2}$ were studied in [1, 2, 8] (see also [9]), with focus on *comparability* of the resulting Green functions. The heat kernel estimates for $\Delta^{\alpha/2} + q$, in fact Schrödinger-type perturbations of general transition densities were then studied in [3] under the following integral condition on q,

$$\int_s^t \int_X p(s,x,u,z)|q(u,z)|p(u,z,t,y)dzdu \le [\eta + \gamma(t-s)]p(s,x,t,y).$$

Here p is a finite positive jointly measurable transition density, γ and η are fixed nonnegative numbers, while times $s < t$ and states x, y are arbitrary. Given the above assumption, the following explicit estimate was obtained in [3] when $\eta < 1$,

$$\tilde{p}(s,x,t,y) \le \frac{1}{1-\eta} \exp\left(\frac{\gamma}{1-\eta}(t-s)\right) p(s,x,t,y).$$

Here $\tilde{p}$ is the Schrödinger perturbation series defined by p and q (see e.g. [6]).

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Inspired by [3], combinatorial arguments based on Stirling numbers were used in [16] to refine the above result of [3] by (a) skipping the Chapman-Kolmogorov condition on p, (b) relaxing the assumptions on q, and (c) strengthening the estimate. Namely, if $0 < \eta < 1$ and $Q \geq 0$ is superadditive, then the relative boundness condition

$$\int_s^t \int_X p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu \leq [\eta + Q(s,t)]p(s,x,t,y), \tag{1}$$

implies the following main estimate of [16]:

$$\widetilde{p}(s,x,t,y) \leq \left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta} p(s,x,t,y) \tag{2}$$

Meanwhile, a more straightforward method was proposed in [17] for *gradient* perturbations of the transition density of $\Delta^{\alpha/2}$. It was also suggested in [17, p. 321] that the technique may also be applied to Schrödinger perturbations to reproduce the main results of [16]. In [6], this observation is developed in considerable generality: the authors estimate Schrödinger-type perturbations of Markovian semigroups, potential kernels, and general *forward* integral kernels on space-time by rather singular functions q. Forward kernels reflect directionality, or transience of time, and their perturbation series have a distinctly exponential flavor. We obtain local in time and global in space comparability of the original and perturbed kernels under suitable smallness conditions on the first non-trivial term of the perturbation series.

Our reasoning in this paper is motivated by recent estimates of Schrödinger perturbations of integral kernels in [4, 6]. The estimates reflect interdependencies of multiple integrations on time symplexes of different dimensions in the perturbations series, but the multiple integrations do not explicitely show in the arguments. In this paper we reexamine the integrations in the case of concave majorization. Our new approach works, e.g., for $Q(s,t) = (t-s)_+^\beta$ with $0 < \beta < 1$ in (1), and leads to better exponents in estimates of the resulting perturbation series. Here is a typical result (see Section 2 for definitions).

Theorem 1.1. *Let $q \geq 0$ be a measurable function on space-time $E = \mathbb{R} \times X$, K be a forward kernel on E and $K_n = (Kq)^n K$, $n = 0, 1, \ldots$. Let $\beta \in (0,1)$. If (3) below holds for $n = 1$ then it holds for all $n = 1, 2, \ldots$,*

$$K_n(s,x,dtdy) \leq K_{n-1}(s,x,dtdy)\left[\eta + \frac{(\mu(t-s))^\beta}{n^\beta}\right], \quad (s,x) \in E. \tag{3}$$

If $\eta = 0$ and $\mu > 0$, then there exists $C = C(\beta)$ such that for all $(s,x) \in E$,

$$\widetilde{K}(s,x,dtdy) \leq C \max\{1, \mu(t-s)\}^{\frac{1-\beta}{2}} \exp(\mu\beta(t-s))\, K(s,x,dtdy). \tag{4}$$

If, furthermore, $0 \leq \eta < 1$ and $\mu = 1$, then for all $(s,x) \in E$, we have

$$\sum_{n=0}^{\infty} K_n(s,x,dtdy) \leq K(s,x,dtdy) \times \tag{5}$$

$$\times\left(1 + \frac{8((t-s)+1)}{(2^\beta - 1)(1-\eta)^{1/\beta}} \cdot \exp\left[(t-s)\beta(1-\eta)^{1-\frac{1}{\beta}} \int_0^1 \frac{dr}{\eta(r^\beta - 1) + 1}\right]\right).$$

The paper is composed as follows. The main theorem is formulated above. Section 2 discusses the definitions and main assumptions. In Section 3 we give the estimates of the terms of the perturbations series. In Section 4 we prove Theorem 1.1 and give some examples.

2. Preliminaries

We recall basic properties of integral kernels [12].

Definition 2.1. *Let $(E, \mathcal{E})$ be a measurable space. A kernel on E is a map K from $E \times \mathcal{E}$ to $[0, \infty]$ with the following properties:*

1. *$x \mapsto K(x, A)$ is $\mathcal{E}$-measurable for all $A \in \mathcal{E}$,*
2. *$A \mapsto K(x, A)$ is countably additive for all $x \in E$.*

Consider kernels K and q on E. The map from $(E \times \mathcal{E})$ to $[0, \infty]$ given by

$$(x, A) \mapsto \int_E K(x, dy) q(y, A)$$

is another kernel on E, called the *composition* of K and q, and denoted Kq. Here and below we alternatively write $\int f(x)\mu(dx) = \int \mu(dx) f(x)$. We let

$$K_n = (Kq)^n K, \qquad n = 0, 1, \ldots.$$

The composition of kernels is associative [12], which yields the next lemma.

Lemma 2.2. $K_n = K_{n-1-m} q K_m$ *for all $n \in \mathbb{N}$ and $m = 0, 1, \ldots, n-1$.*

We define the *perturbation* $\widetilde{K}$, of K by q, via the *perturbation series*,

$$\widetilde{K} = \sum_{n=0}^{\infty} K_n = \sum_{n=0}^{\infty} (Kq)^n K.$$

Of course, $K \le \widetilde{K}$, and the following *perturbation formula* holds,

$$\widetilde{K} = K + \widetilde{K} q K.$$

We write $q \in \mathcal{E}^+$ if $q : E \to [0, \infty]$ is $\mathcal{E}$-measurable. Function q defines the *multiplication kernel*,

$$q(x, A) = q(x) \mathbb{1}_A(x),$$

where $\mathbb{1}_A$ is the indicator function of A. Below we always interpret Kq and $K_n = (Kq)^n K$ as composition of kernels. $\widetilde{K}$ is called *Schrödinger* perturbation of K, if $q(x, A) = q(x)\mathbb{1}_A(x)$, i.e. q is a *multiplication*.

Consider a set X (the state space) with σ-algebra $\mathcal{M}$ of subsets of X, the real line $\mathbb{R}$ (the time) equipped with the Borel sets $\mathcal{B}_{\mathbb{R}}$, and the space-time

$$E := \mathbb{R} \times X,$$

with the product σ-algebra $\mathcal{E} = \mathcal{B}_{\mathbb{R}} \times \mathcal{M}$. Let $q \in \mathcal{E}_+$ be a function, to wit, a multiplication kernel. Let $\eta \in [0, \infty)$, and let K be a *forward* kernel on E, that is for $A \in \mathcal{E}$, $s \in \mathbb{R}$, $x \in X$, we assume

$$K(s, x, A) = 0 \ \text{ for } \ A \subseteq (-\infty, s] \times X.$$

In the language of [4], the sets $(s, \infty) \times X$ are *absorbing* for K.

It may be useful to realize that forward kernels may be localized in time as follows. For $r < t$ we consider the strip $S = (r, t] \times X$, and the restriction of K to S, i.e. $K(s, x, A)$ for $(s, x) \in S$ and $A \subset S$. We note that the restriction of Kq to S depends only on the restrictions of K and q to S. In fact we can consider $E = (r, t] \times X$ as in the setting of Definition 2.1.

Let $Q : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ and $\eta \in [0, \infty)$. We assume

$$KqK(s, x, A) \le \int_A K(s, x, dtdy) \left[\eta + Q(s, t)\right], \qquad (s, x) \in E, A \in \mathcal{E}. \tag{6}$$

In short $KqK(s, x, dtdy) \le K(s, x, dtdy)\left[\eta + Q(s, t)\right]$, thus $\eta + Q(s, t)$ serves as majorant of the Radon-Nikodym derivative of KqK with respect to K. Below we shall often assume some concavity properties of Q.

3. Iterated suprema

By the principle of iterated supremum:

$$\sup_{x_1,\dots,x_n} f(x_1,\dots,x_n) = \sup_{x_1}\dots\sup_{x_n} f(x_1,\dots,x_n),$$

we obtain

$$\begin{aligned}&\sup_{s\le u_1\le\dots\le u_n\le t}[Q(s,u_1)+\dots+Q(u_n,t)] \\ &= \sup_{s\le u_n\le t}\left[\sup_{s\le u_1\le\dots\le u_{n-1}\le u_n}[Q(s,u_1)+\dots+Q(u_{n-1},u_n)]+Q(u_n,t)\right].\end{aligned} \tag{7}$$

Here is the main result of this section.

Theorem 3.1. *If* (6) *holds, then for $n \ge 1$, we have*

$$\begin{aligned}&K_n(s,x,dtdy) \\ &\le K_{n-1}(s,x,dtdy)\left[\eta+\frac{1}{n}\sup_{s\le u_1\le\dots\le u_{n-1}\le t}[Q(s,u_1)+\dots+Q(u_{n-1},t)]\right].\end{aligned} \tag{8}$$

Proof. For $n = 1$, (8) means (6). For $n \ge 1$ by induction and (7),

$$\begin{aligned}&(n+1)K_{n+1}(s,x,A) = nK_nqK(s,x,A) + K_{n-1}qK_1(s,x,A) \\ &= \int_A\int_E nK_n(s,x,dudz)q(u,z)K(u,z,dtdy) \\ &\quad + K_{n-1}(s,x,dudz)q(u,z)K_1(u,z,dtdy) \\ &\le \int_A\int_E nK_{n-1}(s,x,dudz)\left[\eta+\frac{1}{n}\sup_{s\le u_1\le\dots\le u_{n-1}\le u}[Q(s,u_1)+\dots+Q(u_{n-1},u)]\right] \\ &\quad q(u,z)K(u,z,dtdy) + K_{n-1}(s,x,dudz)q(u,z)K(u,z,dtdy)[\eta+Q(u,t)] \\ &\le \int_A K_n(s,x,dtdy)\left[(n+1)\eta+\sup_{s\le u_1\le\dots\le u_{n-1}\le u_n\le t}[Q(s,u_1)+\dots+Q(u_n,t)]\right].\end{aligned}$$

□

Lemma 3.1 generalizes Theorem 1 in [6], as we now shall see.

Example 3.2. If $Q : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ is super-additive, i.e.

$$Q(u,r) + Q(r,v) \le Q(u,v), \qquad u < r < v,$$

then,

$$\sup_{s\le u_1\le\dots\le u_{n-1}\le t}[Q(s,u_1)+\dots+Q(u_{n-1},t)] \le Q(s,t),$$

If (6) also holds, then, for $n \ge 1$, we get

$$K_n(s,x,dtdy) \le K_{n-1}(s,x,dtdy)\left[\eta+\frac{Q(s,t)}{n}\right].$$

In comparison with [6, 7], we do not generally require superadditivity of Q in our present considerations.

4. Concave control functions

Function $g : \mathbb{R} \to \mathbb{R}$ is concave if for all $x, y \in \mathbb{R}$ and $t \in [0,1]$,

$$tg(x) + (1-t)g(y) \leq g(tx + (1-t)y).$$

Lemma 4.1. *If $g \geq 0$ is concave and $Q(s,t) = g(t-s)$, then,*

$$\sup_{s \leq u_1 \leq \ldots \leq u_{n-1} \leq t} [Q(s,u_1) + \ldots + Q(u_{n-1},t)] = ng((t-s)/n). \tag{9}$$

Proof. If $s \leq u \leq t$, then, by concavity,

$$\frac{Q(s,u) + Q(u,t)}{2} = \frac{g(u-s) + g(t-u)}{2} \leq g\left(\frac{t-s}{2}\right).$$

Actually, for $u = (t-s)/2$, we have equality, hence (9) holds for $n = 2$. If $n \geq 2$, then by induction and (7), we similarly get

$$\begin{aligned}
&\sup_{s \leq u_1 \leq \ldots \leq u_n \leq t} [Q(s,u_1) + \ldots + Q(u_n,t)] \\
&= \sup_{s \leq u_n \leq t} \left[\sup_{s \leq u_1 \leq \ldots \leq u_n} [Q(s,u_1) + \ldots + Q(u_{n-1},u_n)] + Q(u_n,t) \right] \\
&= (n+1) \sup_{s \leq u \leq t} \left[\frac{n}{n+1} g((u-s)/n) + \frac{1}{n+1} g(t-u) \right] = (n+1) g\left(\frac{t-s}{n+1}\right).
\end{aligned}$$

□

We call g a (time-homogenous) control function. By Lemma 3.1 and Lemma 4.1, we have the following result.

Corollary 4.2. *If $Q(s,t) = g(t-s)$, where $g \geq 0$ is concave, then* (6) *implies*

$$K_n(s,x,dtdy) \leq K_{n-1}(s,x,dtdy) \left[\eta + g\left(\frac{t-s}{n}\right) \right], \qquad n \geq 1.$$

Example 4.3. We consider $g(x) = \min(x,1)$ and $Q(s,t) = \min(t-s,1)$, which arise, e.g, if $q(u,z) \leq \mathbb{1}_{[0,1]}(u)$ and K satisfies the Chapman-Komogorov equation. Thus, if

$$K_1(s,x,dtdy) \leq \min(t-s,1) K(s,x,dtdy),$$

then Corollary 4.2 yields

$$\begin{aligned}
K_n(s,x,dtdy) &\leq \min\left(\frac{t-s}{n}, 1\right) K_{n-1}(s,x,dtdy) \\
&\leq \prod_{k=1}^{n} \min\left(\frac{t-s}{k}, 1\right) K(s,x,dtdy).
\end{aligned}$$

For fixed $s < t$, we denote $k_0 = \lfloor t-s \rfloor$, the largest integer not exceeding $t-s$. Then,

$$\prod_{1 \leq k \leq n} \min\left(\frac{t-s}{k}, 1\right) = \prod_{k_0 < k \leq n} \frac{t-s}{k} = \frac{(t-s)^n}{n!} \frac{k_0!}{(t-s)^{k_0}}.$$

If $t-s < 1$, then

$$\begin{aligned}
\widetilde{K}(s,x,dtdy) &\leq \sum_{n=0}^{\infty} \frac{(t-s)^n}{n!} K(s,x,dtdy) = e^{t-s}\, K(s,x,dtdy) \\
&\leq [1 + e(t-s)] K(s,x,dtdy).
\end{aligned}$$

If $t-s \geq 1$, then, by Stirling's formula, we obtain

$$\sum_{n=0}^{\infty}\prod_{k=1}^{n}\min\left(\frac{t-s}{k},1\right) \leq k_0+1+\frac{k_0!}{(t-s)^{k_0}}\sum_{n>k_0}\frac{(t-s)^n}{n!}$$
$$\leq k_0+1+e^{13/12}\sqrt{2\pi k_0} \leq 1+9(t-s).$$

Summarizing, we always have $\widetilde{K}(s,x,dtdy) \leq [1+9(t-s)]K(s,x,dtdy)$. This linear bound of $\widetilde{K}/K$ is clearly an improvement over the exponential bound offered in Theorem 1 in [6], i.e. (2) above.

Proof. [Proof of Theorem 1.1] We use the control function $g(x) = (\mu x)^{\beta}$ with $\beta \in (0,1)$ and $\mu > 0$. By Corollary 4.2 for $n \geq 1$, we have

$$K_n(s,x,dtdy) \leq K_{n-1}(s,x,dtdy)\left[\eta+\frac{(\mu(t-s))^{\beta}}{n^{\beta}}\right]$$
$$\leq K(s,x,dtdy)\prod_{k=1}^{n}\left[\eta+\frac{(\mu(t-s))^{\beta}}{k^{\beta}}\right].$$

Let $q_0(\eta,\beta,v) = 1$, and for $n \geq 1$,

$$q_n(\eta,\beta,v) = \prod_{k=1}^{n}\left[\eta+\frac{v^{\beta}}{k^{\beta}}\right]. \tag{10}$$

We denote

$$F^{\eta,\beta}(v) = 1+\sum_{n=1}^{\infty}\prod_{k=1}^{n}\left[\eta+\frac{v^{\beta}}{k^{\beta}}\right] = \sum_{n=0}^{\infty}q_n(\eta,\beta,v),$$

and for $(s,x) \in E$, we have

$$\sum_{n=0}^{\infty}K_n(s,x,dtdy) \leq K(s,x,dtdy)F^{\eta,\beta}(\mu(t-s)).$$

We write $f \sim g$ if $f(v)/g(v) \to 1$ as $v \to \infty$.

We have $F^{\eta,\beta}(0) = 1/(1-\eta)$. We shall estimate $F^{\eta,\beta}(v)$ for all $v \geq 0$. We start with $\eta = 0$. By [13, Theorem 1] (see also [15, page 55]), we have

$$F^{0,\beta}(v) = \sum_{n=0}^{\infty}\frac{v^{n\beta}}{n!^{\beta}} \sim \frac{1}{\sqrt{\beta}}(2\pi)^{\frac{1-\beta}{2}}v^{\frac{1-\beta}{2}}\exp(\beta v).$$

We note that

$$v^{\frac{1-\beta}{2}}\exp(\beta v) \leq \max\{1,v\}^{\frac{1-\beta}{2}}\exp(\beta v), \qquad v \geq 0. \tag{11}$$

The right side of (11) is continuous and positive for $v \in [0,\infty)$, and so it majorizes $F^{0,\beta}(v)$ up to a multiplicative constant depending on β. This proves (4).

For general η,β and v, the sequence $q_n(\eta,\beta,v)$ is initially nondecreasing and then it decreases for $n > v/(1-\eta)^{1/\beta}$, because then $\eta+v^{\beta}/n^{\beta} < 1$, cf. (10). Let $N = \min\{n = 1,2,\ldots : n \geq v/(1-\eta)^{1/\beta}\}$. We either have $q_N(\eta,\beta,v) = \max_{n\geq 1}q_n(\eta,\beta,v)$ or $N \geq 2$ and $q_{N-1}(\eta,\beta,v) = \max_{n\geq 1}q_n(\eta,\beta,v)$. In the latter case, we have

$(N-1)^\beta < v^\beta/(1-\eta) \le N^\beta$, and $q_N/q_{N-1} = \eta + v^\beta/N^\beta \ge \eta + (1-\eta)/2^\beta \ge (1+\eta)/2 \ge 1/2$. In either case,

$$\begin{aligned}
F^{\eta,\beta}(v) &= 1 + \sum_{n=1}^{2N-1} q_n(\eta,\beta,v) + \sum_{n=2N}^{\infty} q_n(\eta,\beta,v)\\
&\le 1 + 4Nq_N(\eta,\beta,v) + q_{2N}(\eta,\beta,v)\sum_{n=0}^{\infty}\left[\eta + \frac{v^\beta}{(2N)^\beta}\right]^n\\
&= 1 + 4Nq_N(\eta,\beta,v) + q_{2N}(\eta,\beta,v)\frac{1}{1-\eta-v^\beta/(2N)^\beta}\\
&\le 1 + q_N(\eta,\beta,v)\left[\frac{4v}{(1-\eta)^{1/\beta}} + 4 + \frac{2^\beta}{(1-\eta)(2^\beta-1)}\right]\\
&\le 1 + q_N(\eta,\beta,v)\frac{8(v+1)}{(2^\beta-1)(1-\eta)^{1/\beta}}.
\end{aligned}$$

To estimate $q_N(\eta,\beta,v)$, we integrate by parts (recall that $v^\beta/N^\beta \le 1-\eta$) and change variables: $t = rv(1-\eta)^{-1/\beta}$,

$$\begin{aligned}
&\log q_N(\eta,\beta,v) = \sum_{n=1}^{N}\log\left(\eta + \frac{v^\beta}{n^\beta}\right) \le \log(\eta + v^\beta) + \int_1^N \log\left(\eta + \frac{v^\beta}{t^\beta}\right)dt\\
&= \log(\eta+v^\beta) + \left[N\log\left(\eta + \frac{v^\beta}{N^\beta}\right) - \log(\eta+v^\beta) + \beta\int_1^N \frac{v^\beta t^{-\beta}}{\eta + v^\beta/t^\beta}dt\right]\\
&\le \beta\int_1^N \frac{v^\beta t^{-\beta}}{\eta + v^\beta/t^\beta}dt = \beta v^\beta\int_1^N \frac{dt}{\eta t^\beta + v^\beta} = \beta v^\beta \int_{(1-\eta)^{1/\beta}/v}^{N(1-\eta)^{1/\beta}/v} \frac{v(1-\eta)^{-1/\beta}dr}{v^\beta\left(\frac{\eta}{1-\eta}r^\beta + 1\right)}\\
&\le \beta v(1-\eta)^{1-1/\beta}\int_0^1 \frac{dr}{\eta(r^\beta-1)+1}.
\end{aligned}$$

This yields (5). □

We shall compare Theorem 1.1 with the estimate provided by Theorem 1 in [6]. Suppose that

$$\begin{aligned}
K_1(s,x,dtdy) &\le (\mu(t-s))^\beta K(s,x,dtdy) &(12)\\
&\le [\eta + \gamma(t-s)]K(s,x,dtdy). &(13)
\end{aligned}$$

We note that Example 2 in [6] yields kernels satisfying (12). In the next lemma we clarify for which μ,β,η and γ, (12) implies (13).

Lemma 4.4. *Let $\eta \in (0,1)$, $\beta \in (0,1)$ and $\mu,\gamma > 0$. Then,*

$$(\mu t)^\beta \le \eta + \gamma t \qquad \text{for all } \; t \ge 0,$$

if and only if

$$(\mu\beta)^\beta(1-\beta)^{1-\beta} \le \eta^{1-\beta}\gamma^\beta.$$

Proof. We note that (the line) $t \mapsto \eta + \gamma t$ has the same slope as $t \mapsto (\mu t)^\beta$ at $t = \frac{1}{\mu}\left(\frac{\mu\beta}{\gamma}\right)^{1/(1-\beta)}$. Therefore, $(\mu t)^\beta \le \eta + \gamma t$ holds for all $t \ge 0$ if and only if

$$\left(\frac{\mu\beta}{\gamma}\right)^{\frac{\beta}{1-\beta}} \le \eta + \frac{\gamma}{\mu}\left(\frac{\mu\beta}{\gamma}\right)^{\frac{1}{1-\beta}}.$$

We transform the last inequality into

$$(\mu\beta)^{\frac{\beta}{1-\beta}}(1-\beta) \le \eta\gamma^{\frac{\beta}{1-\beta}},$$

and we obtain the result. □

We now compare the estimates provided by Theorem 1 from [6] and Theorem 1.1. To this end, we assume that (12) holds with some $\beta \in (0,1)$ and $\mu > 0$. According to (13), we also assume that $\eta \in (0,1)$ and $\gamma > 0$ are such that $(\mu t)^\beta \le \eta + \gamma t$ for all $t \ge 0$, cf. (13). By Lemma 4.4, $(\mu\beta)^\beta(1-\beta)^{1-\beta} \le \eta^{1-\beta}\gamma^\beta$. Theorem 1.1 gives power-exponential upper bounds for $\widetilde{K}/K$ with exponent $\mu\beta(t-s)$, see either (5) or (4). Theorem 1 in [6] yields

$$\widetilde{K}(s,x,dtdy) \le K(s,x,dtdy)\frac{1}{1-\eta}\exp\left[\frac{\gamma(t-s)}{\eta}\log\frac{1}{1-\eta}\right].$$

We shall prove that

$$\mu\beta \le \frac{\gamma}{\eta}\log\frac{1}{1-\eta}. \tag{14}$$

Taking into account Lemma 4.4, it is enough to verify that

$$\mu\beta \le \frac{\mu\beta(1-\beta)^{\frac{1-\beta}{\beta}}}{\eta^{\frac{1}{\beta}}}\log\frac{1}{1-\eta},$$

or

$$\eta^{\frac{1}{\beta}} \le (1-\beta)^{\frac{1-\beta}{\beta}}\log\frac{1}{1-\eta}. \tag{15}$$

Clearly, (15) holds for small η. We also have inequality between derivatives:

$$\frac{1}{\beta}\eta^{\frac{1}{\beta}-1} \le (1-\beta)^{\frac{1-\beta}{\beta}}\frac{1}{1-\eta}. \tag{16}$$

Indeed, by calculus, $\eta \mapsto \eta^{1-\beta}(1-\eta)^\beta$ has maximum at $\eta = 1-\beta$, and (16) follows from this. This proves (14). This means that for large times the estimate in the consider case from Theorem 1.1 is better than from Theorem 1 in [6].

Below, we give two applications, in which power-type concave majorants appear.

Example 4.5. (compare [6, Example 2]) Let $X = \{x_0\}$ consist of one point and dz is the Dirac measure at x_0, so we can skip them from the notation. Let $\beta \in (0,1)$ and $p(s,t) = \Gamma(\beta)^{-1}(t-s)^{\beta-1}$, where $s < t$. For $q(u) \in L^\infty$, we have

$$\begin{aligned} p_1(s,t) &= \frac{1}{\Gamma(\beta)^2}\int_s^t p(s,u)q(u)p(u,t)du \\ &\le \frac{\|q\|_\infty}{\Gamma(\beta)^2}\int_s^t p(s,u)p(u,t)du \le \frac{\|q\|_\infty}{\Gamma(\beta)^2}(t-s)^\beta p(s,t). \end{aligned}$$

Hence, we get

$$p_n(s,t) \le p_{n-1}(s,t)\frac{\|q\|_\infty}{\Gamma(\beta)^2}\frac{(t-s)^\beta}{n^\beta},$$

and consequently, by Themorem 1.1, for $\mu = (\|q\|_\infty/\Gamma(\beta)^2)^{1/\beta}$ we have

$$\tilde{p}(s,t) \le C(\beta)\max\{1, \mu(t-s)\}^{(1-\beta)/2}e^{\mu\beta(t-s)}p(s,t).$$

Example 4.6. Let $p(t,x,y)$ be the time-homogeneous transition density function of the isotropic stable process in $\mathbb{R}^d$ (see e.g., [2]) and let $q(z)=|z|^{-\beta}$ with $\beta\in(0,\alpha\wedge d)$. Then,

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^d}p(s,x,z)\frac{1}{|z|^\beta}dzds &= \int_0^t\int_{\mathbb{R}^d}s^{-d/\alpha}p(1,s^{-1/\alpha}x,s^{-1/\alpha}z)\frac{1}{|z|^\beta}dzds\\ &= \int_0^t\int_{\mathbb{R}^d}p(1,s^{-1/\alpha}x,w)\frac{1}{|w|^\beta}s^{-\beta/\alpha}dwds\\ &\le \int_0^t\int_{\mathbb{R}^d}p(1,0,w)\frac{1}{|w|^\beta}s^{-\beta/\alpha}dwds = c_\beta t^{\frac{\alpha-\beta}{\alpha}}. \end{aligned} \tag{17}$$

Denote $p(s,x,t,y)=p(t-s,x,y)$. From (17) and 3P Theorem [5], we have

$$\int_0^t\int_{\mathbb{R}^d}p(s,x,u,z)\frac{1}{|z|^\beta}p(u,z,t,y)dzdu\le(\mu(t-s))^{\frac{\alpha-\beta}{\alpha}}p(s,x,t,y),$$

for all $s<t$ and $x,y\in\mathbb{R}^d$. Here, $\mu=\left(c_{d,\alpha}c_\beta\right)^{\frac{\alpha}{\alpha-\beta}}$, where $c_{d,\alpha}$ is a constant from 3P inequality. Therefore, Theorem 1.1 applies.

$$\tilde{p}(s,x,t,y)\le C\max\{1,\mu(t-s)\}^{\frac{\beta}{2\alpha}}e^{\frac{\mu(\alpha-\beta)}{\alpha}(t-s)}p(s,x,t,y).$$

We note here that we can calculate the constant c_β above. Namely,

$$c_\beta=\frac{\alpha}{\alpha-\beta}\int_{\mathbb{R}^d}p(1,0,w)\frac{1}{|w|^\beta}dw=\frac{2^{1-\beta}}{\alpha-\beta}\frac{\Gamma(\frac{d-\beta}{2})\Gamma(\frac{\beta}{\alpha})}{\Gamma(\frac{d}{2})\Gamma(\frac{\beta}{2})}. \tag{18}$$

Indeed, first, we note that

$$\begin{aligned}p(1,0,0)&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-|z|^\alpha}dz=\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\frac{1}{(2\pi)^d}\int_0^\infty t^{d-1}e^{-t^\alpha}dt\\ &=\frac{1}{\Gamma\left(\frac{d}{2}\right)\pi^{\frac{d}{2}}2^{d-1}}\int_0^\infty s^{\frac{d-1}{\alpha}}e^{-s}\frac{1}{\alpha}s^{\frac{1}{\alpha}-1}ds=\frac{\Gamma\left(\frac{d}{\alpha}\right)}{\alpha\Gamma\left(\frac{d}{2}\right)\pi^{\frac{d}{2}}2^{d-1}}.\end{aligned} \tag{19}$$

Next, by [18, (2.10)]

$$|x|^{-\beta}=2^{d-\beta}\pi^{d/2}\frac{\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{d-\beta}{\alpha}\right)\Gamma\left(\frac{\beta}{2}\right)}\int_0^\infty p(t,x)t^{\frac{d-\alpha-\beta}{\alpha}}dt,\quad x\in\mathbb{R}^d.$$

Hence, by semigroup property, scaling and (19),

$$\begin{aligned}c_\beta&=2^{d-\beta}\pi^{d/2}\frac{\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{d-\beta}{\alpha}\right)\Gamma\left(\frac{\beta}{2}\right)}\int_{\mathbb{R}^d}p(1,0,w)\int_0^\infty p(t,w,0)t^{\frac{d-\alpha-\beta}{\alpha}}dtdw\\ &=2^{d-\beta}\pi^{d/2}\frac{\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{d-\beta}{\alpha}\right)\Gamma\left(\frac{\beta}{2}\right)}\int_0^\infty p(1+t,0,0)t^{\frac{d-\alpha-\beta}{\alpha}}dt\\ &=2^{d-\beta}\pi^{d/2}\frac{\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{d-\beta}{\alpha}\right)\Gamma\left(\frac{\beta}{2}\right)}\int_0^\infty\frac{\Gamma\left(\frac{d}{\alpha}\right)}{\alpha\Gamma\left(\frac{d}{2}\right)\pi^{\frac{d}{2}}2^{d-1}}(1+t)^{-\frac{d}{\alpha}}t^{\frac{d-\alpha-\beta}{\alpha}}dt\end{aligned}$$

Finally, by [14, 3.194.3],

$$\int_0^\infty (1+t)^{-\frac{d}{\alpha}} t^{\frac{d-\alpha-\beta}{\alpha}}\, dt = B\left(\frac{d-\beta}{\alpha}, \frac{d}{\alpha} - \frac{d-\beta}{\alpha}\right) = \frac{\Gamma\left(\frac{d-\beta}{\alpha}\right)\Gamma\left(\frac{\beta}{\alpha}\right)}{\Gamma\left(\frac{d}{\alpha}\right)},$$

and we get (18).

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