Filomat 33:19 (2019), 6381–6383
https://doi.org/10.2298/FIL1919381P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Fixed Point Theorem for Mappings Satisfying a New Common Range Property

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Abstract. In this paper a general fixed point theorem for two pairs of mappings satisfying a new type of common range property without limit of sequences in metric spaces are proved.

1. Introduction and Preliminaries

Let X be a non empty set and $A, S : X \to X$ two self mapping on X. A point $x \epsilon X$ is a coincidence point of A and S if $w = Ax = Sx$ for some $x \epsilon X$.

The set of all coincidence points of A and S is denoted by $C(A, S)$, and w is said to be a point of coincidence of A and S.

Definition 1.1. [7] Let X be a nonempty set and A and S be two self mappings on X. A and S are weakly compatible if $ASu = SAu$ for all $u \epsilon C(A, S)$.

In 2011, Sintunavarat and Kumam [12] introduced the notion of common limit range property in metric spaces.

Definition 1.2. [12] A pair of self mappings A and S on a metric space (X,d) is said to satisfy common limit range property with respect to S, denoted $CLR_{(S)}$ property if there exists a sequence $x_n \epsilon X$ such that

$$\lim_{n \longrightarrow \infty} Ax_n = \lim_{n \longrightarrow \infty} Sx_n = t \epsilon S(X).$$

Recently, Imdad et all. [3] extend this notion of common limit range property for two pairs of mappings.

Definition 1.3. [3]. Two pairs (A, S) and (B, T) of self mappings on a metric space (X,d) satisfy common limit range property with respect to (ST), denoted $CLR_{(S,T)}$ property if there exist two sequences x_n and $y_n \epsilon X$ such that

$$\lim_{n \longrightarrow \infty} Ax_n = \lim_{n \longrightarrow \infty} Sx_n = \lim_{n \longrightarrow \infty} By_n = \lim_{n \longrightarrow \infty} Ty_n = u \epsilon S(X) \cap T(X).$$

Some fixed point results for two pairs of mappings with theorems with $CLR_{(S)}$ and $CLR_{(S,T)}$ - properties are obtained in [4],[5],[6] and other papers. Quite recently, a new type of common limit range property is introduced in [11].

Definition 1.4. [11] Let A , S and T be self mappings of a metric space (X,d). The pair (A,S) is said to satisfy a common limit range property with respect to T, denoted by $CLR_{(A,S,)T}$ - property if there exist a sequence x_n such that

2010 *Mathematics Subject Classification*. Primary 54H25; Secondary 47H10
Keywords. metric space, fixed point, implicit relation, coincidence range property
Received: 10 January 2019; Accepted: 05 May 2019
Communicated by Vladimir Rakočević
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$$\lim_{n \longrightarrow \infty} Ax_n = \lim_{n \longrightarrow \infty} Sx_n = u \epsilon S(X) \cap T(X)$$

Remark 1.1. In all definitions 1.2 – 1.4 there exists some convergent sequences in X. We introduce a new type of common range property without limits of sequences.

Definition 1.5. (A, S) and T satisfy $CRP_{(A,S)T}$ - coincidence range property with respect to T , if there exists $u\epsilon C(A,S)$, with $z = Au\epsilon T(X)$.

Example 1.1. Let $X = (1, \infty)$ with the usual metric, and $Ax = (x^2 + 1)/2, Sx = (x + 1)/2$ and $Tx = x$, then $T(X) = [1, \infty)$, and $Sx = Ax$ implies x = 1. As a consequence, $A1 = S1 = z = 1\epsilon T(X) = [1, \infty)$.

2. Implicit relations

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition, by an implicit function in [9] and [10] and other papers. In 2008, Ali and Imdad [2] had introduced a new class of implicit relations. We will introduce a new class of implicit relations, similarly with [2].

Definition 2.1. Let F_C be a family of functions $F(t_1, , t_6) : R^6_+ \rightarrow R$ satisfying : (F1): $F(t, 0, 0, t, t, 0) > 0$, for all $t > 0$, (F2): $F(t, t, 0, 0, t, t) > 0$, for all $t > 0$. The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings satisfying $CRP_{(A,S)T}$ - property and an implicit relation.

Example 2.1. $F(t_1, ..., t_6) = t_1 - k.max\{t_2, ..., t_6\}$, where $k\epsilon[0, 1)$.

Example 2.2. $F(t_1, ..., t_6) = k.max\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\}$, where $k\epsilon[0, 1)$.

Example 2.3. $F(t_1, ..., t_6) = t_1 - k.max\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\}$, where $k\epsilon[0, 1)$.

Example 2.4. $F(t_1, ..., t_6) = t_1 - a.t_2 - b.max\{t_3, t_4\} - c.max\{t_5, t_6\}$, where $a, b, c \geq 0$ and $a + b + c < 1$.

Example 2.5. $F(t_1, ..., t_6) = t_1 - \alpha.max\{t_2, t_3, t_4\} - (1 - \alpha)(a.t_5 + b.t_6)$, where $\alpha\epsilon(0, 1)$, $a, b \geq 0$ and $a + b < 1$.

Example 2.6. $F(t_1, ..., t_6) = t_1 - a.t_2 - \frac{b.t_5+t_6}{1+t_3+t_4}$, where $a, b \geq 0$ and $a + 2b < 1$.

Example 2.7. $F(t_1, ..., t_6) = t_1^2 - t_1(a.t_2 + b.t_3 + c.t_4) - d.t_5.t_6$, where $a, b, c, d \geq 0$ and $a + b + c + d < 1$.

Example 2.8. $F(t_1, ..., t_6) = t_1 - max\{c.t_2, c.t_3, c.t_4, a.t_5 + b.t_6\}$, where $a, b, c \geq 0$ and $max\{c, a + b\} < 1$.

The purpose of this paper is to prove a general fixed point theorem for two pair of mappings satisfiying $CRP_{(A,S)T}$ - properties without the use of limits of mappings.

3. Main result:

Lemma 3.1 [1]. Let f, g be two weakly compatible mappings of a non empty set X. If f and g have a unique point w of coincidence where $w = fx = gx$, for that $x\epsilon X$, then w is the unique common fixed point of f and g.

Theorem 3.2 Let A, B, S, T be self mappings of a metric space such that: (3.1)$F(d(Ax, By)), d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ax, Ty)) \leq 0$ for all $x, y\epsilon X$ and some $F\epsilon F_C$.

If (A, S) and T satisfy $CRP_{(A,S)T}$ property then $C(B, T) \neq \Phi$. Moreover, if (A, S) and (B, T) are two pairs of weakly compatible mappings, then $A, B, S,$ and T have a unique common fixed point.

Proof: Since (A, S) and T satisfy $CRP_{(A,S)T}$ -property, there exist $v\epsilon X$ such that $z = Av = Sv$ with $z\epsilon T(X)$. Hence, there exists $u\epsilon X$ such that $z = T(u)$.

By 3.1. for $x = v$ and $y = u$ we obtain: $F(d(Av, Bu), d(Sv, Tu), d(Sv, Av) , d(Tu, Bu), d(Sv, Bu), d(Av, Tu)) \leq 0$, $F(d(z, Bu), 0, 0, d(z, Bu), d(z, Bu), 0) \leq 0$, A contradiction with (F1) if $d(z, Bu) > 0$, hence $d(z, Bu) = 0$. Which implies that $z = Bu = Tu$. And $C_{(B,T)} \neq \Phi$. Therefore $z = Av = Sv = Tu = Bu$. Therefore, z is a common point of coincidence of (A, S) and (B, T).

We prove that z is the unique point of coincidence for A and S. Suppose that $t = Aw = Bw$ for some $w\epsilon X$. By 3.1 we obtain for $x = w$ and $y = u$ that $F(d(Av, Bu), d(Sw, Tu), d(Sw, Aw) , d(Tu, Bu), d(Sw, Bu), d(Aw, Tu)) \leq 0$, $F(d(t, z), d(t, z), 0, 0, d(z, t), d(z, t)) \leq 0$. A contradiction of (F2) if $d(z, t) > 0$. Which implies $d(z, t) = 0$, i.e. $z = t$. And z is the unique point of coincidence of A and S. Similarly z is the unique point of coincidence, moreover, if (A, S) and (B, T) are weakly compatible, by Lemma 3.1, z is the unique common fixed point of A, B, S, T.

Remark 3.3: For the proof of this theorem we have to do the followings steps:

Step 1. Solve the equation $Sx = Ax$ on X and establish $C(A,S) = \{z|x\epsilon X$ and $Sx = Ax, z = Ax\}$. If $C(A,S) = \Phi$ the theorem is not applicable.

Step 2. If $C(A,S) \neq \Phi$ we have to select z from $C(A,S)$ such that exists an $x\epsilon X$ such that $T(x) = z$. As a consequence, A, S, T satisfy the $CRP_{(A,S)T}$ property.

Step 3. Verify if the pairs (A,S) and (B,T) are weakly compatible. I.e: solve the $Az = Sz, z\epsilon C(A,S)$ and similarly, for (B,T) : solve the $Bq = Tq$, $q\epsilon C(B,T)$. If one of those pairs are not weakly compatible, the theorem can not be applied. Stop.

Step 4. If the relation 3.1 is satisfied then, by Theorem 3.1, A, S, B, T have a unique fixed point: z.

Example 3.4 Let $x = [0,1]$ be a metric space with d, the usual metric and $Ax = 0, Sx = \frac{x}{x+2}, Bx = \frac{x}{3}, Tx = x$. If $Ax = Sx$ then x=0 and $C(A,S) = \{0\}$. Then, $z = 0, z\epsilon T(X) = X$. Hence,$(A,S)$ and T satisfy $CRP_{(A,S)T}$-property.

Moreover, $AS0 = SA0 = 0$, and $BT0 = TB0 = 0$, hence (A,S) and (B,T) are weakly compatible. Otherhand, $d(Ax,By) = \frac{y}{3}, d(Ty,By) = \frac{2y}{3}$, which implies, $d(Ax,By) \leq k.d(Ty,By)$, where $k\epsilon[\frac{1}{2},1)$. Then $d(Ax,By) \leq k.max\{d(Sx,Ty),d(Sx,Ax)\},d(Ty,By),d(Sx,By),d(Ax,Ty)$, with $k\epsilon[\frac{1}{2},1])$.

By Theorem 3.2, and Example 2.1, A,B,S and T have a unique common fixed point $z = 0$.

Remark 3.4 By Theorem 3.2 and example 2.2-2.8 we can obtain new particular results.

References

[1] M.Abbas and B.E.Rhoades, Common fixed points for non commuting mapping without continuity in generalized metric spaces, Applied Math. Comput. 215 (2009), 260–265.

[2] J.Ali and M.Imdad, An implicit function implies several contractive conditions, Sarajevo J. Math, 17,4 (2008), 269–285.

[3] M.Imdad, M.Panth and S.Chauhan, Fixed point theorems in Menger spaces satisfying $CLR_{(A,S)}$-property and application, J.Nonlinear Anal. Optim. 312,2 (2012), 225–236.

[4] M.Imdad, S.Chauhan and Z.Kadelburg, Fixed point theorem for mapping with common limited range property satisfying (Ψ,Φ)-weak contractive condition, Math. Sci.,7,16 (2013).

[5] M.Imdad and S.Chauhan, Employing common limit range property to prove unified metrical common fixed point theorems, Int.J.Anal.(2012) article ID 763261.

[6] M.Imdad, S.Sharma and S.Chauhan, Unifying a multitude of metrical fixed point theorem in symetric spaces, Filomat, 28,6,(2014), 1113-1132.

[7] B.Jungck, Common fixed point for noncontinuous nonself maps on a nonnumeric space, Far. East J, Maath, 4,1(1996), 198–218.

[8] E.Karapinar, D.K.Patel,M.Imdad and D.Gopal, Some nonunique common fixed point theorems in symetric spaces through $CLR_{S,T}$-property Int.J.Math.Math.Sci. 2013,Article ID 793965.

[9] V.Popa, Fixed point theorems for implicit contractive mappings, Stud. Cerc.St.Ser.Mat.Univ. Bacau, 7(1997),157-163.

[10] V.Popa, Some fixed point theorems for compatible mappings satisfying an implicit relation, Demonstratio Math. 33(1999), 157-162.

[11] V.Popa, Fixed point theorem for two pairs of mappings satisfying a new type of common limit range property. Filomat, 31:11(2017), 3181-3192.

[12] V.Popa, W.Sintunavarat and P.Kumarn, Common fixed points for a pair of weak comaptible mapping in fuzzy metric spaces, J.Appl.Math. 2011, Article ID 637958.