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Perception of BCC-Algebras Under the Bishops Principled-Philosophical Orientation: BCC-Algebra with Apartness

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Abstract. In this paper it is given a short introduction in a reconsideration about BCC-algebras under the light of Bishop's principled-philosophical orientation. At the first, it is introduced the concept of BCC-algebras into this orientation. Additionally, the consequences of the selected axioms in the determination of BCC-algebras with apartness are analyzed. Also, some substructures in the BCC-algebras with apartness that have no counterparts in the classical case and which appear as products of the chosen logical environment such as co-ideals are analyzed. At the end, two different results that can be viewed as the isomorphism theorem in for BCC-algebras are exposed.

1. Introduction

1.1. The classical case

In 1966, Y. Imai and K. Iséki in [10] introduced the concept of BCK-algebra. BCK-algebras form a quasivariety of algebras. K. Iski posed the interesting problem whether the class of BCK-algebras is a variety or not. In connection with this problem Y. Komori introduced in [11] a notion of BCC-algebra. The algebras we will consider are those based on a set X containing a constant '0' and an internal binary operation '$\cdot$'. The description of BCC-algebras determined on set with an apartness relation is taken from [6].

Definition 1.1 ([6]). *By a BCC-algebra is a non-empty set X together with a binary internal operation '$\cdot$' and a distinguished element '0' such that the following axioms are satisfied:*
(BCC1) $(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$;
(BCC2) $(\forall x \in X)(x \cdot x = 0)$;
(BCC3) $(\forall x \in X)(x \cdot 0 = 0)$;
(BCC4) $(\forall x \in X)(0 \cdot x = x)$; *and*
(BCC5) $(\forall x, y \in X)((x \cdot y = 0 \wedge y \cdot x = 0) \Longrightarrow x = y)$.

Remark 1.2. *Let us observe that their form is coherent with the interpretation in which $\cdot$ stay for $\Longrightarrow$, 0 for $\top$ and $=$ for the semantical equivalence usually denoted by $\equiv$.*

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The following three substructures play an important role in describing the properties of these algebras.

Definition 1.3. *Let $(X, \cdot, 0)$ be a BCC-algebra and let S be a subset of X. For a subset S, it says that it is:*
(a) *a* sub-algebra *of X if*
(1) $0 \in S$ *and*
(2) $(\forall x, y \in X)((x \in S \ \wedge \ y \in S) \implies x \cdot y \in S)$;
(b) ([8]) *an* ideal *in X if*
(1) $0 \in S$ *and*
(3) $(\forall x, y, z \in X)(((x \cdot y) \cdot z \in S \ \wedge \ y \in S) \implies x \cdot z \in S)$;
(c) ([20]) *a* proper filter *of X if*
(4) $\neg(0 \in S)$ *and*
(5) $(\forall x, y, z \in X)((\neg((x \cdot y) \cdot z \in S) \ \wedge \ x \cdot z \in S) \implies y \in S)$.

Proposition 1.4. *Every ideal of a BCC-algebra is a sub-algebra.*

Definition 1.5 ([7]). *On any BCC-algebra $(X, \cdot, 0)$ one can define the so-called natural order by putting*
(6) $(\forall x, y \in X)(x \leqslant y \iff x \cdot y = 0)$.

Remark 1.6. *Let us observe that the definition of natural order is coherent with the interpretation in which $\leqslant$ stay for the semantic consequence $\models$.*

Natural BCC-order has the following properties, which can be proved by direct verification by relying on (BCC1)
(7) $(\forall x, y \in X)(x \leqslant y \cdot x)$,
(8) $(\forall x, y, z \in X)(y \cdot z \leqslant (x \cdot y) \cdot (x \cdot z))$,
(9) $(\forall x, y, z \in X)(x \leqslant y \implies (y \cdot z \leqslant x \cdot z \ \wedge \ z \cdot x \leqslant z \cdot y))$.

The following three propositions describe the main features of the ideals of a BCC-algebra.

Proposition 1.7 ([8, 20]). *If S is an ideal of a BCC-algebra X, then the following formulas are valid:*
(10) $(\forall x, y \in X)((x \cdot y \in S \ \wedge \ y \in S) \implies x \in S)$;
(11) $(\forall y, z \in X)(y \in S \implies y \cdot z \in S)$ *and*
(12) $(\forall x, y \in S)((x \leqslant y \ \wedge \ y \in S) \implies x \in S)$.

Proposition 1.8 ([20], Theorem 3.2). *Let S be an ideal of a BCC-algebra X. The the relation $\prec$ over X, defined by*
(13) $(\forall x, y \in X)(x \prec y \iff x \cdot y \in S)$,
is a quasi-order on X, moreover the following formulas are valid:
(14) $(\forall x, y, z \in X)(x \prec y \implies x \cdot z \prec y \cdot z)$ *and*
(15) $(\forall x, y, z \in X)(x \prec y \implies z \cdot y \prec z \cdot x)$.

A proper filter in a BCC-algebra X has the following properties.

Proposition 1.9 ([20]). *Let X be a BCC-algebra and S a proper filter of X. Then*
(16) $(\forall x, y \in X)(\neg((x \cdot y \in S) \ \wedge \ x \in S) \implies y \in S)$,
(17) $(\forall x, y \in X)(x \cdot y \in S \implies y \in S)$
(18) $(\forall x, y \in X)((x \leqslant y \ \wedge \ x \in S) \implies y \in S)$.

In a comprehensive consideration of the properties of this kind of logical algebra, among others, the following researchers took part: Y. Komori [11], W. A. Dudek [6, 7, 9], X. Zhang [8] and D. A. Romano [20].

1.2. The Bishop's constructive case

In the following analysis, we will change the logical environment: We assume that the logical background of the incoming research is the Intuitionistic Logic **IL** instead of the Classic Logic **CL**. The tools allowed in the upcoming research are determined by the **IL** and the principled-philosophical orientation of Bishop's constructive mathematics **Bish**.

The reader interested in Bishop's constructive mathematics can refers to [1, 2, 12], while the role of **IL** in **Bish** can be found in [24]. Fundamentals of constructive algebra can be found in [12] and in the dissertations [13, 23]. Specific structures in the **Bish** frame are investigated in [3–5, 16] (semigroups) and in [14, 15] (rings, fields and modules), while a comprehensive overview about general algebraic structures can be found in [22].

The logical principle of "exclusion of the third - principle tertium non datur" (so-called "TND principle")

(TND) $F \vee \neg F$ for any formula F

is not an axiom in **IL**. In addition, the "double negation" principle is not a valid formula in **IL**. Therefore, it is possible that there is a difference relation whose negation is not the equality. We arrive at the essential part of this analysis: mathematical objects and mathematical structures should be developed in an environment that implies Intuitionistic Logic with equality relations and diversity relations. We assume that the carrier of an algebraic structure is supplied with a relation of equality '$=$' and a relation of diversity '$\neq$'. The diversity relation has the properties:

$(\forall x)\neg(x \neq x)$ (consistency) and

$(\forall x, y)(x \neq y \implies y \neq x)$ (symmetry) .

Of course, it is assumed that the relation of diversity is extensional with respect to the equality in the following sense

$= \circ \neq \subseteq \neq$ and $\neq \circ = \subseteq \neq$,

where '$\circ$' is the canonical notation for composition of relations. A diversity relation $\neq$ it is said to be an *apartness* if it meets the following additional condition

$(\forall x, y, z)(x \neq z \implies (x \neq y \vee y \neq z))$ (co-transitivity).

Our intention in this report is to describe the internal structures of BCC-algebras, assuming that the carriers of these algebraic structures are relational systems $(X, =, \neq)$. This commitment implies that any function f and any relation R on the selected set $(X, =, \neq)$ is strongly extensional in the following sense

$(\forall x_1, ..., x_n, x'_1, ..., x'_n \in X)((x_1, ..., x_n) \neq (x'_1, ..., x'_n) \iff (x_1 \neq x'_1 \vee ... \vee x_n \neq x'_n))$;

$(\forall x_1, ..., x_n, x'_1, ..., x'_n \in X)(f(x_1, ..., x_n) \neq f(x'_1, ..., x'_n) \implies (x_1 ..., x_n) \neq (x'_1 ... x'_m))$; and

$(\forall x_1, ..., x_n, x'_1, ..., x'_n \in X)((x_1, ..., x_n) \in R \implies ((x_1, ..., x_n) \neq (x'_1, ..., x'_n) \vee (x'_1, ..., x'_n) \in R))$.

For example, a subset S of a set X is strongly extensional if the following holds

$(\forall x, y \in X)(x \in S \implies (y \neq x \vee y \in S))$.

The first step in this intent is the introduction of the concept of BCC-algebras with apartness.

Definition 1.10. *A* BCC-algebra with apartness *is a non-empty set* $(X, =, \neq)$ *together with a binary internal operation* '$\cdot$' *and an element* '0' *such that the following axioms are satisfied:*

(BCC1) $(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$*;*

(BCC2) $(\forall x \in X)(x \cdot x = 0)$*;*

(BCC3) $(\forall x \in X)(x \cdot 0 = 0)$*;*

(BCC4) $(\forall x \in X)(0 \cdot x = x)$*; and*

(BCC5$^{\neq}$) $(\forall x, y \in X)(x \neq y \implies (x \cdot y \neq 0 \vee y \cdot x \neq 0))$.

Remark 1.11. *The differences between BCC-algebras in the classical case, defined by Definition 1.1, and and BCC-algebras with apartness, determined in the previous definition, determined in the previous definitions, that are immediately perceived are:*

- carrier $(X, =, \neq)$ *instead of carrier* $(X, =)$ *and*

- the axiom (BCC5$^{\neq}$) instead of the axiom (BCC5).

The important difference which is implicitly assumed between these determinations of BCC-algebras is the strongly extensionality of the internal binary operation in X in the following sense

$(\forall x, y, z, u \in X)(x \cdot y \neq z \cdot u \implies (x \neq z \vee y \neq u))$

The last formula is equivalent to

(BCC0$^{\neq}$) $(\forall x, y, z \in X)((x \cdot z \neq y \cdot z \vee z \cdot x \neq z \cdot y) \implies x \neq y)$.

In the next proposition we show that if the apartness is a *tight* relation, i.e. if

$(\forall x, y \in X)(\neg(x \neq y) \implies x = y)$,

then (BCC5$^{\neq}$) implies (BCC5).

Proposition 1.12. *Let $((X, =, \neq), \cdot, 0)$ be a BCC-algebra with a tight apartness, then (BCC5$^{\neq}$) implies (BCC5).*

Proof. Let $x, y \in X$ be arbitrary elements such that $x \cdot y = 0$ and $y \cdot x = 0$. If there were $x \neq y$, we would have $\neg(x \cdot y = 0) \wedge \neg(y \cdot x = 0)$, which is in contradiction with the hypothesis. Therefore, it must be $\neg(x \neq y)$. From here follows $x = y$ because the relation $\neq$ is a tight. □

1.3. Our intention

Since, in general, the apartness relation is not a tight on X, we conclude that the algebraic structures $((X, =, \neq), \cdot, 0)$ and $(X, =, \cdot, 0)$ differ in the formulas in which appears the axiom (BCC5$^{\neq}$) instead of the axiom (BCC5), or they are the result of deduction using this axiom. If we also add that the **IL** system is disconnected from the **CL** system for at least one basic axiom, the principle of "tertium not datur", the reader should acquire a minimum initial impression that the analysis of the BCC-algebras in different logical systems differs. Thus, the constructive aspect **Bish**, which includes the **IL** system and the Bishop's principled-philosophical orientation, is not only a different way of observing and analyzing this algebraic structure, but also observing and analyzing a significantly different algebraic structure than is the case in classical logical background.

In this article we will deal with the insight of the internal structure of the BCC-algebras with apartness. The specificity of the logical background in which the substructures of a BCC-algebra with apartness are observed and analyzed, provides to us some special substructures that do not have a counterpart. Should there be an academic interest in exploring such algebraic structures with apartness relation? - is a completely natural question that is posed in itself.

The answer to this question should always be affirmative, except in some extreme cases. For the academic community, it should be interesting (except in extreme cases) to find out how from some given suppositions deduce consequences in the framework of a chosen logical system in an acceptable way. On the other hand, there are many reasons for studying algebraic structures with diversity relation. For example,

- this is a way of proving that the axioms of the formal system are well chosen, since the addition of diversity relation could produce some inconsistencies throughout the theory; and

- allows to observe the presence of substructures that are almost invisible in the classical logical background.

2. Specificity of BCC-algebras with apartness

2.1. Concept of co-subalgebra

In what follows, we need the notion of disconnection. An element x of a BCC-algebra X it is said yo be disconnected from a subset T of the algebra X if it is valid

$(\forall t \in T)(x \neq t)$.

This relationship between the element x and the subset T in a BCC-algebra X will be denoted by $x \lhd T$. If S and T are subsets of a BCC-algebra with apartness X, we say that S is disconnected from T, in symbols $S \bowtie T$, if the formula

$(\exists s \in S)(s \lhd T) \vee (\exists t \in T)(t \lhd S)$.

is valid. It is easily verified that '$\bowtie$' is a diversity relation in the family $\mathfrak{P}(X)$ of all subsets of the set X. Though, in general, the $\bowtie$ is not an apartness relation.

Definition 2.1. *Let $((X, =, \neq), \cdot, 0)$ be a BCC-algebra with apartness and let T be a strongly extensional subset of X. A set T is a co-subalgebra if*
(1a) $0 \lhd T$ *and*
(2a) $(\forall x, y \in X)(x \cdot y \in T \implies (x \in T \ \vee \ y \in T))$.

Remark 2.2. *Speaking by language of classical algebras, the set T is a co-subalgebra of a BCC-algebra with apartness X if and only if it is a consistent subset of X.*

Our first proposition in this section will focus more on the concept of co-subalgebras.

Proposition 2.3. *Let S be a co-subalgebra of a BCC-algebra with apartness X. Then the set $S^{\lhd} = \{x \in X : x \lhd S\}$ is a sub-algebra of X.*

Proof. If S is a co-subalgebra of a BCC-algebra X with apartness, then $0 \in S^{\lhd}$ by definition. Let $x, y, s \in X$ be arbitrary elements such that $x \lhd S$, $y \lhd S$ and $s \in S$. Thus $s \in S$ implies $s \neq x \cdot y$ or $x \cdot y \in S$ because S is a strongly extensional subset of X. In the second case it would be $x \in S$ or $y \in S$. It is impossible. So, it has to be $x \cdot y \neq s \in S$ and $x \cdot y \in S^{\lhd}$. Therefore, the set $S^{\lhd}$ is a subalgebra of X. □

It is obvious that the subset $X_0 = \{x \in X : x \neq 0\}$ of a BCC-algebra X is a co-subalgebra of X. We accept that the empty set $\emptyset$ is also a co-subalgebra of X. Thus, the family $\mathfrak{C}(X)$ of all co-subalgebras of a BCC-algebra X is not empty.

Theorem 2.4. *The family $\mathfrak{C}(X)$ of all co-subalgebras of a BCC-algebra X with apartness is a complete lattice.*

Proof. Let $\{T_i\}_{i \in I}$ be a family of co-subalgebras of a BCC-algebra X with apartness. Then the union $\bigcup_{i \in I} T_i$ is a co-subalgebra of X too. First, the union $\bigcup_{i \in I} T_i$ is a strongly extensional subset of X. Indeed, let $x \in \bigcup_{i \in I} T_i$ and $y \in X$. Thus, there exists an index $i \in I$ such that $x \in T_i$. Then, $y \neq x \ \vee \ y \in T_i \subseteq \bigcup_{i \in I} T_i$ since T_i is a strongly extensional subset of X. Second, it is obvious that the condition $0 \lhd \bigcup_{i \in I} T_i$ is fulfilled. Let $x, y \in X$ be elements such that $x \cdot y \in \bigcup_{i \in I} T_i$. Then there exists $i \in I$ such that $x \cdot y \in T_i$. Thus, $x \in T_i \subseteq \bigcup_{i \in I} T_i$ or $y \in T_i \subseteq \bigcup_{i \in I} T_i$ since T_i is a co-subalgebra of X. So, the union $\bigcup_{i \in I} T_i$ is a co-subalgebra of X also.

Let $\mathfrak{A}$ be the family of all co-subalgebras of the BCC-algebra X with apartness contained in the intersection $\bigcap_{i \in I} T_i$. Then the union $\cup\mathfrak{A}$ is the maximal co-subalgebra of X contained in $\bigcap_{\in I} T_i$.

If we put $\sqcup_{i \in I} T_i = \bigcup_{i \in I} T_i$ and $\sqcap_{i \in I} T_i = \cup\mathfrak{A}$, then $(\mathfrak{C}(X), \sqcup, \sqcap)$ is a complete lattice. □

Corollary 2.5. *Let X be a BCC-algebra with apartness and let T be a subset of X. Then there exists the maximal co-subalgebra of X contained in T.*

Proof. The claim follows directly from the second part of the proof of the previous theorem. □

About algebraic structures determined on sets with apartness, the interested reader can look at articles [3–5, 14, 15, 22].

2.2. Concept of co-order

Let $X = ((X, =, \neq), \cdot, 0)$ be a BCC-algebra with apartness. In the following definition, we introduce the concept of co-order relation. This notion and its generalization "co-quasiorder relation" have been analyzed in many of our earlier published articles as, for example, in [16, 17]. Descriptions of some ordered algebraic structures under co-(quasi)order relation, a reader can be find in [22]. A relation $\nless$ is a *co-quasiorder relation* on X if it is consistent and co-transitive, i.e., if the formula

$(\forall x, y \in X)(x \nless y \implies x \neq y)$ (consistency) and
$(\forall x, y, z \in X)(x \nless z \implies (x \nless y \ \vee \ y \nless z))$ (co-transitivity).

are valid. A binary relation $\nleqslant$ is a *co-order* relation on X if in addition it is also linear, i.e. if the formula

$(\forall x, y \in X)(x \neq y \implies (x \nleqslant y \ \vee \ y \nleqslant x))$ (linearity).

holds. If there is a co-quasiorder (or a co-order) relation on a set (or an algebraic structure), then we say that the set (or the algebraic structure) is ordered under this relation. If X is a set with a binary operation (so-called *groupoid*) then the following implications must be satisfied:

$(\forall x, y, z \in X)(x \cdot z \not< y \cdot z \implies x \not< y)$ and

$(\forall x, y, z \in X)(z \cdot x \not< z \cdot y \implies x \not< y)$.

In this case we are talking about the compatibility of the binary operation with the co-quasiorder relation. In the language of classical algebra, the internal binary operation in the groupoid X is left and right cancellative with respect to the co-quasiorder (so-order) relation.

Definition 2.6. *Let $X = ((X, =, \neq), \cdot, 0)$ be a BCC-algebra with apartness. Then we define a relation '$\nleqslant$' in X by*

(6a) $(\forall x, y \in X)(x \nleqslant y \iff x \cdot y \neq 0)$.

In the following proposition we give some of basic properties of relation $\leqslant$.

Proposition 2.7. *The following formulas*

(19) $(\forall x \in X)\neg(x \nleqslant x)$

(20) $(\forall x, y \in X)(x \nleqslant y \implies x \neq y)$

(21) $(\forall x, y \in X)(x \neq y \implies (x \nleqslant y \vee y \nleqslant x))$

(22) $(\forall x, y, z \in X)(x \nleqslant z \implies (x \nleqslant y \vee y \nleqslant z))$

(23) $(\forall x, y, z \in X)(x \cdot z \nleqslant y \cdot z \implies y \nleqslant x)$

(24) $(\forall x, y, z \in X)(z \cdot x \nleqslant z \cdot y \implies x \nleqslant y)$

are valid formulas in any BCC-algebra with apartness.

Proof. (19) is a direct consequence of (BCC2).

(20) Let $x, y \in X$ such that $x \nleqslant y$. Then $x \cdot y \neq 0$. Thus $x \cdot y \neq 0 = y \cdot y$ by (BCC2). From here it follows $x \neq y$ in accordance with (BCC0$^{\neq}$).

(21) is a direct consequence of (BCC5$^{\neq}$).

(22) Let $x, y, z \in X$ be arbitrary elements such that $x \nleqslant z$. Then $x \cdot z \neq 0$. Thus $0 \cdot (x \cdot z) \neq 0 = (y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z))$ by (BCC4) and (BCC1). From here it follows $0 \neq y \cdot z$ or $0 \cdot (x \cdot z) \neq (x \cdot y) \cdot (x \cdot z)$ in accordance with (BCC0$^{\neq}$) and (BCC4). From here it follows $0 \neq x \cdot y$ in accordance with (BCC0$^{\neq}$). Finally, we have $y \nleqslant z \vee x \nleqslant z$. Therefore, the property of co-transitivity is proved.

(23) Let $x, y, z \in X$ be arbitrary element such that $x \cdot z \nleqslant y \cdot z$. Then $(x \cdot y) \cdot (y \cdot z) \neq 0$. Thus $(x \cdot z) \cdot (y \cdot z) \neq 0 = (x \cdot z) \cdot ((y \cdot x) \cdot (y \cdot z))$ by (BCC1) where we put $x = y$ and $y = x$. From here it follows $0 \cdot (y \cdot z) \neq (y \cdot x) \cdot (y \cdot z)$ according to (BCC0$^{\neq}$) and (BCC4). From here it follows $0 \neq y \cdot x$ according to (BCC0$^{\neq}$). So, $y \nleqslant x$. In this case we are talking about anti-compatibility.

(24) Let $x, y, z \in X$ be arbitrary elements such that $z \cdot x \nleqslant z \cdot y$. Then $(z \cdot x) \cdot (z \cdot y) \neq 0$. Thus $0 \cdot ((z \cdot x) \cdot (z \cdot y)) \neq 0 = (x \cdot y) \cdot ((z \cdot x) \cdot (z \cdot y))$ by (BCC4) and (BCC1) where we put $y = x$, $z = y$ and $x = z$. From here it follows $0 \neq x \cdot y$ in accordance with (BCC0$^{\neq}$). So, $x \nleqslant y$. □

Corollary 2.8. *The relation $\nleqslant$, defined above, is a co-order relation on a BCC-algebra X, moreover it is left compatible and right anti-compatible with the internal operation in the algebra X.*

Lemma 2.9. *Let X be a BCC-algebra with apartness. Then the formulas*

$(\forall x, y \in X)\neg(x \leqslant y \wedge x \nleqslant y)$;

$(\forall x, y, z \in X)((x \nleqslant z \wedge y \leqslant z) \implies x \nleqslant y)$ *and*

$(\forall x, y, z \in X)((x \leqslant y \wedge x \nleqslant z) \implies y \nleqslant z)$ *are valid.*

In the next proposition, we describe the left and right classes of this relation.

Proposition 2.10. *Let $((X, =, \neq), \cdot, 0)$ be a BCC-algebra with apartness and let a and b be elements of X. Then $L(a) = \{y \in X : a \nleqslant y\}$ is the left class and $R(b) = \{x \in X : x \nleqslant b\}$ is the right class of the co-order relation $\nleqslant$ generated by a and b respectively. Then $L(a)$ and $R(b)$ are strongly extensional subsets of X such that $a \lhd L(a)$ and $b \lhd R(b)$, moreover the following formulas*

$(\forall u, v \in X)(u \in L(a) \implies (v \in L(a) \vee v \nleqslant u))$ *and*

$(\forall u, v \in X)(u \in R(b) \implies (u \nleqslant b \vee v \in R(b)))$.

are valid

Proof. Let $t \in L(a)$ be an arbitrary element. Then $a \nleqslant t$. Thus $a \neq t$ and therefore $a \lhd L(a)$.

Let $u \in L(a)$ and $v \in X$. Then $a \nleqslant u$ and $v \in X$. Thus $a \nleqslant v$ or $v \nleqslant u$ by co-transitivity of $\nleqslant$. So, we have $v \in L(a) \vee v \nleqslant u$. Since the $\nleqslant$ is a consistent relation, we also have $u \in L(a) \vee v \neq u$. So, the set $L(a)$ is a strongly extensional subset of X.

For the set $R(b)$ the proof is analogous to the previous one. □

2.3. Concept of co-ideal

In this subsection, we will deal with the concept of co-ideals of a BCC-algebra with apartness which is a counterpart of the concepts of ideal in such algebras. The idea of a co-ideals in an algebraic structure with apartness was first exposed by W. Ruitenberg in his dissertation [23]. Further elaboration of this idea was contributed by this author in his dissertation [13] and in his texts devoted to commutative rings with apartness (see, for example [14, 15]). This idea can be found also in the Chapter 8 (Algebra) of [24]. The concepts of co-ideals and co-filters in sets with apartness (in semigroups with apartness) ordered under a co-quasiorder relation have been developed by this author (see, for example: [19, 22]).

Definition 2.11. *Let $((X, =, \neq), \cdot, 0)$ be a BCC-algebra with apartness. For a subset K in X, it is said that it is a co-ideal of X if the following formulas are valid*
(1a) $0 \lhd K$ *and*
(3a) $(\forall x, y, z \in X)(x \cdot z \in K \implies (x \cdot (y \cdot z) \in K \vee y \in K))$.

It is obvious that the set X_0 is a co-ideal in X. Indeed, if $x \cdot z \neq 0$, then $x \cdot z \neq x \cdot (y \cdot z) \neq$ or $x \cdot (y \cdot z) \neq 0$. The first option gives $0 \neq y$. So, we have $x \cdot (y \cdot z) \neq 0 \vee y \neq 0$. We also accept that the empty set $\emptyset$ is a co-ideal of X. So, the family $\mathfrak{K}(X)$ of all co-ideals of X is not empty.

In the following proposition, two important features of this substructure of a BCC-algebra with apartness are proved.

Proposition 2.12. *Let K be a co-ideal of a BCC-algebra with apartness. Then*
(10a) $(\forall x, y \in X)(y \in K \implies (x \cdot y \in K \vee x \in K))$ *and*
(11a) $(\forall x, y \in X)(x \cdot y \in K \implies y \in K)$.

Proof. If we put x = 0, y = x and z = y in (3a), we get (10a) in accordance with (BCC4).

If we put $y = x$ and $z = y$ in (3a), we obtain (11a) according to (BCC2) and (BCC3). □

Corollary 2.13. *Any co-ideal of a BCC-algebra X with apartness is a strongly extensional subset of X and it also holds*
(12a) $(\forall x, y \in X)(y \in K \implies (x \nleqslant y \vee x \in K))$

Proof. Let $x, y \in X$ be arbitrary elements such that $y \in K$. Then $x \cdot y \in K \vee x \in K$ by (10a). Thus $x \cdot y \neq 0$ or $x \in K$ since $0 \lhd K$. So, we have $x \nleqslant y \vee x \in K$. From here it follows $x \neq y \vee x \in K$ because $\nleqslant$ is a consistent relation. □

Corollary 2.14. *Each co-ideal in a BCC-algebra X with apartness is a co-subalgebra in X.*

Proof. The claim directly follows from (11a). □

Theorem 2.15. *The family $\mathfrak{K}(X)$ of all co-ideals of a BCC-algebra with apartness X is a complete lattice and $\mathfrak{K}(X) \subseteq \mathfrak{C}(X)$ holds.*

Proof. Let $\{K_i\}_{i \in I}$ be a family of co-ideals of a BCC-algebra with apartness X. It is clear that the condition $0 \lhd \bigcup_{i \in I} K_i$ is fulfilled because $0 \lhd K_i$ for every $i \in I$. Let $x, y, z \in X$ such that $x \cdot z \in \bigcup_{i \in I} K_i$. Then there exist an index $i \in I$ such that $x \cdot z \in K_i$. Thus, $x \cdot (y \cdot z) \in K_i \vee y \in K_i$, because K_i is a co-ideal of X. So, we have $x \cdot (y \cdot z) \in \bigcup_{i \in I} K_i$ or $y \in \bigcup_{i \in I} K_i$. Therefore, the subset $\bigcup_{i \in I} K_i$ is a co-ideal in X.

Let $\mathfrak{T}$ be the family of all co-ideals of X included in the intersection $\bigcap_{i \in I} K_i$. Then, the union $\cup\mathfrak{T}$ is the maximal co-ideal of X included in $\bigcap_{i \in I} K_i$.

If we put $\sqcup_{i \in I} K_i = \bigcup_{i \in I} K_i$ and $\sqcap_{i \in I} K_i = \cup\mathfrak{T}$, then $(\mathfrak{K}(X), \sqcup, \sqcap)$ is a complete lattice. □

Corollary 2.16. *If Y is a subset of a BCC-algebra X with apartness, then there is a maximal co-ideal of X contained in Y.*

Theorem 2.17. *Let K be a co-ideal of a BCC-algebra with apartness X. Then the relation $' \nprec '$ on X defined by*

(13a) $(\forall x, y \in X)(x \nprec y \iff x \cdot y \in K)$

is a co-quasiorder on X such that

(25a) $(\forall xy, z \in X)(z \cdot x \nprec z \cdot y \implies x \nprec y)$ *and*

(26a) $(\forall xy, z \in X)(y \cdot z \nprec x \cdot z \implies x \nprec y)$.

Proof. Let $x, y \in X$ be arbitrary elements such that $x \nprec y$. Then $x \cdot y \in K$. Thus $x \cdot y \neq 0$ because $0 \lhd K$. So, $x \nleqslant y$ and $x \neq y$ because the relation $\nleqslant$ is a consistent relation on X. Therefore, the relation $\nprec$ is a consistent relation on X included in $\nleqslant$.

Let $x, y, z \in X$ be arbitrary elements such that $x \nprec z$. Then $x \cdot z \in K$. Thus $(x \cdot y) \cdot (x \cdot z) \in K \vee x \cdot y \in K$ by (10a). Further on, from $(x \cdot y) \cdot (x \cdot z) \in K$ it follows $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) \in K$ or $y \cdot z \in K$. The first option is impossible because (BCC1). So, we have $x \nprec y \vee y \nprec z$. Therefore, the relation $\nprec$ is a co-transitive relation on X.

Let $x, y, z \in X$ be such that $z \cdot x \nprec z \cdot y$. Then $(z \cdot x) \cdot (z \cdot y) \in K$. Thus $(x \cdot y)((z \cdot x) \cdot (z \cdot y)) \in K \vee x \cdot y \in K$ by (10a). Since the first option is impossible due to (BCC1), where we put $x = z$, $y = x$ and $z = y$, we have $x \nprec y$. So, the relation $\nprec$ is a left cancellative relation with respect to the internal operation in X.

Let $x, y, z \in X$ be arbitrary elements such that $y \cdot z \nprec x \cdot z$. Then $(y \cdot z) \cdot (x \cdot z) \in K$. Thus $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) \in K$ or $x \cdot y \in K$ by (3a). Since the first option is impossible because $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0 \lhd K$ by (BCC1), it must be $x \cdot y \in K$. Therefore, $x \nprec y$. So, the relation $\nprec$ is right anti-cancellative with respect to the internal operation in X. □

2.4. Concept of co-congruence

The idea of co-congruence on an algebraic system was first introduced in the author's dissertation [13]. Something more about this term, determined on the algebraic structure of commutative rings with apartness, can be found in [14]. The commutative rings and modules above them with apartness one can found in [14, 15]. The semigroups with apartness are investigated in articles [3–5, 16, 22]. Let $(X, =, \neq)$ be a set with apartness. A relation q on X it called a *co-equality* relation on X if it is consistent, symmetric and co-transitive. Let $((X, =, \neq), \cdot)$ be a groupoid and let q be a co-equality relation on X. Then q is a *co-congruence* on X if the following

(27a) $(\forall x, y, u, v \in X)((x \cdot u, y \cdot v) \in q \implies ((x, y) \in q \vee (u, v) \in q))$

is valid. The previous formula is equivalent to the following two formulas

(28a) $(\forall x, y, u \in X)((x \cdot u, y \cdot u) \in q) \implies (x, y) \in q)$ and

(29a) $(\forall x, y, u \in X)((u \cdot x, u \cdot y) \in q \implies (x, y) \in q)$

If we put $v = u$ in (27a), we get (28a), because q is a consistent relation. Similarly, if we put $x = u$, $y = u$, $u = x$ and $v = y$ in (27a), we obtain (29a), since q is a consistent relation.

Conversely, suppose that formulas (28a) and (29a) are valid. From $(xu, yv) \in q$ follows $(x \cdot u, y \cdot u) \in q$ or $(y \cdot u, y \cdot v) \in q$ by co-transitivity of q. Thus $(x, y) \in q$ or $(u, v) \in q$ by (28a) and (29a). So, the formula (27a) is proved.

Definition 2.18. *Let $((X, =, \neq), \cdot, 0)$ be a BCC-algebra with apartness and let q be a co-equality relation on set X. Then q is a co-congruence on BCC-algebra with apartness if the formula*

$(\forall x, y, u, v \in X)((x \cdot u, y \cdot v) \in q \implies ((x, y) \in q \vee (u, v) \in q))$

is valid.

Lemma 2.19. *Any co-congruence q on a BCC-algebra with apartness X is a strongly extensional relation on X.*

Proof. Let $x, y, u, v \in X$ be arbitrary elements such that $(x, y) \in q$. Then $(x, u) \in q$ or $(u, v) \in q$ or $(v, y) \in q$ by co-transitivity of q. Thus $x \neq u$ or $(u, v) \in q$ or $v \neq y$ by consistency of q. So, we have $(x, y) \neq (u, v) \vee (u, v) \in q$. □

Proposition 2.20. *If q is a co-congruence on a BCC-algebra with apartness X, then the relation $q^{\triangleleft}$ is a congruence on X.*

Proof. It is known [21] that q is an equivalence on X. Let $x, y, u, v, t, s \in X$ be arbitrary elements such that $(x, y) \triangleleft q$, $(u, v) \triangleleft q$ and $(s, t) \in q$. Then $(s, t) \neq (x \cdot u, y \cdot v)$ or $(x \cdot u, y \cdot v) \in q$. If there were $(x \cdot u, y \cdot v) \in q$, we would have $(x, y) \in q \vee (u, v) \in q$ by (27a), which is impossible by the hypothesis. Therefore, it must be $(x \cdot u, y \cdot v) \neq (s, t) \in q$. So, $(x \cdot u, y \cdot v) \in q^{\triangleleft}$. Finally, $q^{\triangleleft}$ is a congruence on X. □

Proposition 2.21. *If q is a co-congruence on a BCC-algebra with apartness X, then the set $C_0 = \{x \in X : (0, x) \in q\}$ is a co-ideal of X.*

Proof. Let $x \in X$ be an arbitrary elements such that $x \in C_0$. Then $(0, x) \in q$ and $x \neq 0$ because q is a consistent relation.

Let $x, y \in X$ be such that $y \in C_0$. Then $(0, y) \in q$. Thus $(0, x{\cdot}y) \in q \vee (x{\cdot}y, 0{\cdot}y) \in q$, i.e. $(0, x{\cdot}y) \in q \vee (x, 0) \in q$. From here, it follows $x \cdot y \in C_0$ or $x \cdot (y \cdot z) \in C_0$. Therefore, C_0 is a co-ideal of X. □

Theorem 2.22. *The family $\mathfrak{Q}(X)$ of all co-congruences on a BCC-algebra with apartness X is a complete lattice.*

Proof. Let $\{q_i\}_{i\in I}$ be a family of co-congruences on X. It is clear that $\bigcup_{i\in I} q_i$ is a consistent and symmetric relation since each components q_i is a consistent and symmetric relation. It remains to show that the union is a co-transitive relation. Let $x, y, z \in X$ be arbitrary elements such that $(x, z) \in \bigcup_{i\in I} q_i$. Then there exists an index $i \in I$ such that $(x, z) \in q_i$. Thus $(x, y) \in q_i \subseteq \bigcup_{i\in I} q_i$ or $(y, z) \in q_i \subseteq \bigcup_{i\in I} q_i$. So, the relation $\bigcup_{i\in I} q_i$ is a co-transitive relation on X. Let $x, y, u, v \in X$ such that $(x \cdot u, y \cdot v) \in \bigcup_{i\in I} q_i$. Then there exists an index $i \in I$ such that $(x \cdot u, y \cdot v) \in q_i$. Thus $(x, y) \in q_i \subseteq \bigcup_{i\in I} q_i$ or $(u, v) \in q_i \subseteq \bigcup_{i\in I} q_i$. Therefore, the relation $\bigcup_{i\in I} q_i$ is a co-congruence on a BCC-algebra X with apartness.

Let $\mathfrak{T}$ be the family of all co-congruences included in the intersection $\bigcap_{i\in I} q_i$. Then the relation $\cup\mathfrak{T}$ is the maximal co-congruence included in the intersection $\bigcap_{i\in I} q_i$.

If we put $\sqcup_{i\in I} q_i = \bigcup_{i\in I} q_i$ and $\sqcap_{i\in I} q_i = \cup\mathfrak{T}$, then $(\mathfrak{Q}(X), \sqcup, \sqcap)$ is a complete lattice. □

Proposition 2.23. *Let K be a co-ideal of a BCC-algebra with apartness X. Then the relation $q = \not\prec \cup \not\prec^{-1}$ is a co-congruence on X and $C_0 = K$ holds.*

Proof. The relation $\not\prec$, defined by $x \not\prec y \iff x \cdot y \in K$, is a left cancellative and right anti-cancellative co-quasiorder relation on X, according to Theorem 2.17. By direct checking, it can be verified that the relation $q = \not\prec \cup \not\prec^{-1}$ is a left-cancellative and right-cancellative co-equality on X. So, the relation q is a co-congruence on X.

Let $x \in C_0$ be an arbitrary element. Then $(0, x) \in q = \not\prec \cup \not\prec^{-1}$. Thus $0 = x \cdot 0 \in K$ or $x = 0 \cdot x \in K$. So, $x \in K$, because the option $0 \in K$ is impossible by definition of the co-ideal K. Therefore, $C_0 \subseteq K$. Conversely, let $x \in K$ be an arbitrary element. Then $0 \cdot x = x \in K$. Thus $0 \not\prec x$ and $(0, x) \in \not\prec \cup \not\prec^{-1} = q$. So, $x \in C_0$. Therefore, $K \subseteq C_0$. Finally, $K = C_0$. □

Let e be a congruence and let q be a co-congruence q on a BCC-algebra with apartness X. (e, q) is an *associate pair* if

$$e \circ q \subseteq q \text{ and } q \circ e \subseteq q$$

holds. This type of relationship between a congruence e and a co-congruence q on a set X allows us to construct a factor set $X/(e, q) = \{[x] : x \in X\}$ in the following way:

$$(\forall x, y \in X)([x] = [y] \iff (x, y) \in e \text{ and } [x] \neq [y] \iff (x, y) \in q).$$

Let q be a co-congruence on a BCC-algebra X. Since the relation $q^{\triangleleft}$ and q are associate, we can construct the factor-set $X/(q^{\triangleleft}, q) = \{aq^{\triangleleft} : x \in X\}$ with

$$(\forall x, y \in X)(xq^{\triangleleft} = yq^{\triangleleft} \iff (x, y) \triangleleft q \text{ and } xq^{\triangleleft} \neq yq^{\triangleleft} \iff (x, y) \in q).$$

Also, we can construct the family $[X : q] = \{xq : x \in X\}$ with

$$(\forall x, y \in X)(xq = yq \iff (x, y) \lhd q \text{ and } xq \neq yq \iff (x, y) \in q).$$

Our intention is to construct quotient BCC-algebras using the sets $X/(e, q)$, $X/(q^{\lhd}, q)$ and $[X : q]$ as the carriers of BCC-algebraic structures. We will need the following lemma for this purpose.

Lemma 2.24. *Let (e, q) be a associate pair of a congruence and a co-congruence on a BCC-algebra with apartness X. Then the function '$\cdot$', defined by*

(30) $(\forall x, y \in X)([x] \cdot [y] = [x \cdot y])$

is well-defined an internal operation on $X/(e, q)$.

Proof. Since it is obvious that this defined operation '$\cdot$' is a total function, we only have to prove that it is a strongly extensional function. Assume $[x \cdot y] \neq [x' \cdot y']$. Then $(x \cdot x', y \cdot y') \in q$. Thus $(x, x') \in q \vee (y, y') \in q$ by (27a). So, we have $[x] \neq [x'] \vee [y] \neq [y']$. □

The following two lemmas can be proved analogously to the previous one.

Lemma 2.25. *Let q be a co-congruence on a BCC-algebra with apartness X. Then the function '$\cdot$', defined by*

(30′) $(\forall x, y \in X)(xq^{\lhd} \cdot yq^{\lhd} = (x \cdot y)q^{\lhd})$

is well-defined an internal operation in $X/(q^{\lhd}, q)$.

Lemma 2.26. *Let q be a co-congruence on a BCC-algebra with apartness X. Then the function '$\cdot$', defined by*

(30″) $(\forall x, y \in X)(xq \cdot yq = (x \cdot y)q)$

is well-defined an internal operation in $[X : q]$.

As it is easily seen, $X/(e, q)$ satisfies all axioms of a BCC-algebra except (BCC5$^{\neq}$). In the following theorem, we give one of the main results of this paper. We show that the structures $X/(e, q)$, $X/(q^{\lhd}, q)$ and $[X : q]$ are BCC-algebras with apartness, where q is defined by $(x, y) \in q \iff (x \cdot y \in K \vee y \cdot x \in K)$.

Theorem 2.27. *Let (e, q) be an associate pair of a congruence and a co-congruence on a BCC-algebra with apartness X, where the relation q is determined by a co-ideal K in X. Then $X/(e, q)$, $X/(q^{\lhd}, q)$ and $[X : q]$ are BCC-algebras with apartness.*

Proof. Axioms (BCC1) - (BCC4) concern equality and then their validity is guaranteed by the general result for a quotient of a BCC-algebra. It is sufficiently to check the validity of the axiom (BCC5$^{\neq}$). Let $x, y \in X$ be arbitrary elements such that $[x] \neq [y]$ ($xq^{\lhd} \neq yq^{\lhd}$, $xq \neq yq$, respectively). Then $(x, y) \in q = \nleqslant \cup \nleqslant^{-1}$, i.e. $x \nleqslant y \vee y \nleqslant x$. Thus $x \cdot y \in K$ or $y \cdot x \in K$. If we write this in the form $0 \cdot (x \cdot y) \in K \vee 0 \cdot (y \cdot x) \in K$, according to (BCC4), we have $(0, x \cdot y) \in q$ or $(0, y \cdot x) \in q$. □

2.5. Concept of strongly extensional homomorphism

Let $(X, =, \neq)$ and $(Y, =, \neq)$ be sets with apartness. A mapping $f : X \longrightarrow Y$ is
- injective if $(\forall x, x' \in X)(f(x) = f(x') \implies x = x')$ holds;
- an embedding if $(\forall x, x' \in X)(x \neq x' \implies f(x) \neq f(x'))$ holds.

Strongly extensional mappings (shortly: se-mappings) between ordered relational systems under co-quasiorder (co-order) relations have already been in the focus of this author (see, for example [17, 18, 22]).

Definition 2.28. *Let $((X, =, \neq), \cdot, 0)$ and $((Y, =, \neq), \cdot, 0)$ be BCC-algebras with apartness. A strongly extensional mapping $f : X \longrightarrow Y$ is a* strongly extensional homomorphism *(shortly: se-homomorphism) between BCC-algebras with apartness if*

(31) $f(0) = 0$ *and*

(32) $(\forall x, x' \in X)(f(x \cdot x') = f(x) \cdot f(x'))$.

A se-epimorphism is a surjective se-homomorphism, a se-monomorphism is an injective se-homomorphism, a se-isomorphism is a surjective and injective se-homomorphism that is also an embedding.

Let us note that a se-monomorphism does not have to be in general an embedding.

Remark 2.29. *The condition (31) is a direct consequence of requirement (32) and axiom (BCC2).*

By direct checking without any difficulty, one can prove the claims of the following lemmas.

Lemma 2.30. *Let $f : X \longrightarrow Y$ be a se-homomorphism between BCC-algebras with apartness. Then the relation $q = \{(x, x') \in X \times X : f(x) \neq f(x')\}$ is a co-congruence on X.*

Lemma 2.31. *If $f : X \longrightarrow Y$ is a se-homomorphism between BCC-algebras with apartness, then the mappings $\pi : X \longrightarrow X/(q^{\lhd}, q)$ and $\theta : X \longrightarrow [X : q]$, defined by $\pi(x) = xq^{\lhd}$ and $\theta(x) = xq$ (for any $x \in X$), respectively, are se-epimorphisms.*

The following theorem gives one of the important results in this article.

Theorem 2.32. *Let $f : X \longrightarrow Y$ be a se-homomorphism between BCC-algebras. Then there exist a unique se-epimorphism $\pi : X \ni x \longmapsto xq^{\lhd} \in X/(q^{\lhd}, q)$ and a unique embedding se-monomorphisms $g : X/(q^{\lhd}, q) \longrightarrow X$ such that $f = g \circ \pi$.*

Proof. 1. The mapping π. Let $x, x' \in X$ be arbitrary elements such that $\pi(x) \neq \pi(x')$. Then $zq^{\lhd} \neq x'q^{\lhd}$, i.e. $(x, x') \in q$. Thus $f(x) \neq f(x')$. From here, it follows immediately $x \neq x'$ because f is a se-mapping. So, the mapping π is a se-epimorphism.

2. The mapping g. Let $g : X/(q^{\lhd}, q) \longrightarrow Y$ be define by $g(xq^{\lhd}) = f(x) \quad (x \in X)$. Suppose $g(xq^{\lhd}) \neq g(x'q^{\lhd})$. Then $f(x) \neq f(x')$. Thus $x \neq x'$. So, the mapping g is a se-mapping. Suppose $xq^{\lhd} \neq x'q^{\lhd}$. Then $(x, x') \in q$, i.e. $f(x) \neq f(x')$. Thus $g(xq^{\lhd}) \neq g(x'q)$. Therefore, the mapping g is an embedding. Suppose $g(xq^{\lhd}) = g(x'q^{\lhd})$ for some $x, x' \in X$. Then $f(x) = f(x')$, i.e. $(x, x') \lhd q$. Thus $xq^{\lhd} = x'q^{\lhd}$. Therefore, g is an injective and then it is an embedding se-monomorphism.

3. $f = g \circ \pi$. Let $x \in X$ be an arbitrary element. Then $f(x) = g(xq^{\lhd}) = g(\pi(x)) = (g \circ \pi)(x)$. □

The previous theorem can be seen as the first isomorphism theorem between BCC-algebras with apartness. The following theorem can be seen as the first isomorphism theorem between BCC-algebras with apartness, also. The formulation of this theorem differs significantly from that of the previous one. This form does not have its counterpart in the classical case since it is the specificity of the chosen principled-philosophical orientation.

Theorem 2.33. *Let $f : X \longrightarrow Y$ be a se-homomorphism between BCC-algebras. Then there exist the se-epimorphism $\theta : X \ni x \longmapsto xq \in [X : q]$, the embedding se-monomorphisms $h : [X : q] \longrightarrow Y$ and the se-isomorphism $\varphi : X/(q^{\lhd}, q) \ni xq^{\lhd} \longmapsto zq \in [X : q]$ such that $f = h \circ \theta$ and $g = h \circ \varphi$.*

Proof. The existence of the mapping θ. Let $x, x' \in X$ be arbitrary elements such that $\theta(x) \neq \theta(x')$. Then $zq \neq x'q$, i.e. $(x, x') \in q$. Thus $f(x) \neq f(x')$. From here, follows immediately $x \neq x'$ because f is a strongly extensional mapping. So, the mapping θ is a se-epimorphism.

The existence of mapping h. Similarly, let $h : [X : q] \longrightarrow Y$ be define by $h(xq) = f(x)$ $(x \in X)$. Suppose $h(xq) \neq g(x'q)$ for some $xq, x'q \in [X : q]$. Then $f(x) \neq f(x')$, i.e. $(x, x') \in q$. Thus $xq \neq x'q$. So, the mapping h is se-mapping. Let $xq \neq x'q$ for some $xq, xq \in [X : q]$. Then $(x, x') \in q$, i.e. $f(x) \neq f(x')$. Thus $h(xq) \neq h(x'q)$. So, h is an embedding. Let $h(xq) = h(x'q)$ for some $xq, x'q \in [X : q]$. Then $fx) = f(x')$, i.e. $(x, x') \lhd q$. Thus, $xq = x'q$. So h is injective and then it is an embedding se-monomorphism.

The equality $f = h \circ \theta$. Let $x \in X$ be an arbitrary element. Then $f(x) = h(xq) = h(\theta(x)) = (h \circ \theta)(x)$.

The se-isomorphism φ. If we define $\varphi : X/(q^{\lhd}, q) \longrightarrow [X : q]$ by $\varphi(xq^{\lhd}) = xq$ for any $xq^{\lhd} \in X/(q^{\lhd}, q)$, by direct verification it can be proved that φ is a se-isomorphism and that $g = h \circ \varphi$. □

So, apart from the fact that the partition $X/(q^{\lhd}, q)$, determined by the equality $q^{\lhd}$ and the co-equality q, is seen as the carrier of the algebraic construction of the factor BCC-algebra $((X/(q^{\lhd}, q), =, \neq), \cdot, 0)$ we also consider the family $[X : q] = \{xq : x \in X\}$ of all classes of co-equality relation q generated by elements of X, the so-called *co-partition* on X, as the carrier for the algebraic construction of another BCC-algebra which is very tight connected with the aforementioned factor-algebra $X/(q^{\triangleleft}, q)$. In the previous two theorems, the co-congruence q plays the main role and enables us to construct new BCC-algebras with apartness and ensures the existence of se-homomorphisms g, h and φ that connect newly constructed BCC-algebras.

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