Filomat 33:19 (2019), 6315–6334
https://doi.org/10.2298/FIL1919315M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# An Extension of Darbo's Theorem via Measure of Non-Compactness with its Application in the Solvability of a System of Integral Equations

**Behnam Matani[a], Jamal Rezaei Roshan[a], Nawab Hussain[b]**

[a]*Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran*
[b]*Department of Mathematics, King Abdulaziz University, P.O.Box 80203, Jeddah 21589, Saudi Arabia*

**Abstract.** In this work, we present a new extension of Darbo's theorem for two different classes of altering distance functions via measure of non-compactness. Using two-variable contractions we obtain the well-known results in this literature (see [22]). We also use these results to discuss the existence of solutions for a system of integral equations. Finally, we provide an example to confirm the results obtained.

## 1. Introduction and Preliminaries

The measure of non-compactness is one of the most important and useful concepts in functional analysis. This subject which was initiated by the fundamental article of Kuratowski in [17] and has provided powerful tools for obtaining the solutions of a large variety of integral equations and systems of integral equations. In fixed point theory one of the most important results is due G. Darbo [13]. So far, many scholars have provided generalizations of Darbo's theorem and have been helped in solving the integral equations (for example see [1–18, 20–23]). In this paper, we present a new extension of Darbo's theorem for two different classes of altering distance functions via measure of non-compactness. Using two-variable contractions we obtain the well-known results in this literature. We also use these results to discuss the existence of solutions for a system of nonlinear integral equations and give a concrete example.

From now until the end of this work, let $E$ be a Banach space. Let us denote the set of real numbers with $\mathbb{R}$. Consider $\mathbb{R}_+ = [0, +\infty)$. We will denote by $\overline{B_r}$ the closed ball centered at $\theta$ with radius $r$. Considering $\mathcal{X} \subset E, \mathcal{X} \neq \emptyset$, assume that $\overline{\mathcal{X}}$ is the closure of the set $\mathcal{X}$ and $co\mathcal{X}$ denotes the closed convex hull of $\mathcal{X}$. Also we symbolize by $\mathcal{M}_E$ the family of all non-empty and bounded sets and by $\mathcal{N}_E$ subfamily consisting of all relatively compact sets.

**Definition 1.1.** *([11]) A function $\mu : \mathcal{M}_E \to \mathbb{R}_+$ is called a measure of non-compactness in $E$ if it satisfies the following hypothesis:*

$(BM1)$ The family $\ker \mu = \{\mathcal{X} \in \mathcal{M}_E : \mu(\mathcal{X}) = 0\} \neq \emptyset$ and $\ker \mu \subset \mathcal{N}_E$;

$(BM2)$ $\mathcal{X} \subset \mathcal{Y} \Rightarrow \mu(\mathcal{X}) \leq \mu(\mathcal{Y})$;

2010 *Mathematics Subject Classification*. Primary 47H08; Secondary 45G15, 47H10
*Keywords*. Measure of non-compactness, Systems of nonlinear integral equations, fixed point theorems
Received: 13 May 2019; Accepted: 03 November 2019
Communicated by Snežana Živković-Zlatanović
*Email addresses:* savadkouh1352@gmail.com (Behnam Matani), Jmlroshan@gmail.com, Jml.roshan@qaemiau.ac.ir (Jamal Rezaei Roshan), nhusain@kau.edu.sa (Nawab Hussain)

$(BM3)$ $\mu\left(\overline{\mathcal{X}}\right) = \mu\left(co\mathcal{X}\right) = \mu\left(\mathcal{X}\right);$

$(BM4)$ $\mu\left(\lambda\mathcal{X} + (1-\lambda)\mathcal{Y}\right) \leq \lambda\mu\left(\mathcal{X}\right) + (1-\lambda)\mu\left(\mathcal{Y}\right)$ for $\lambda \in [0,1]$;

$(BM5)$ If $(\mathcal{X}_k)$ is a sequence of closed sets from $\mathcal{M}_E$ such that $\mathcal{X}_{k+1} \subset \mathcal{X}_k$ for $k = 1,2,...,$ and $\lim_{k\to\infty}\mu\left(\mathcal{X}_k\right) = 0$, then the set $\mathcal{X}_\infty = \cap_{k=1}^{\infty}\mathcal{X}_k \neq \emptyset$.

The subfamily $\ker\mu$ defined in $(BM1)$ represents Kernel of $\mu$ and since $\mu\left(X_\infty\right) = \mu\left(\cap_{k=1}^{\infty}\mathcal{X}_k\right) \leq \mu\left(\mathcal{X}_k\right)$, we see that $\mu\left(\mathcal{X}_\infty\right) = 0$. Therefore $X_\infty \in \ker\mu$.

**Definition 1.2.** *We say that $l : [0,+\infty)^3 \to [0,+\infty)$ is a lower semi-continuous function, if for any arbitrary sequences $\{a_k\}$ and $\{b_k\}$ and $\{c_k\}$ of $[0,+\infty)$,*

$$l\left(\liminf_{k\to\infty} a_k, \liminf_{k\to\infty} b_k, \liminf_{k\to\infty} c_k\right) \leq \liminf_{k\to\infty} l\left(a_k, b_k, c_k\right).$$

*For example, $l_1(p,q,r) = \ln(p+q+r+1)$ and $l_2(p,q,r) = \max\{p,q,r\}$ are lower semicontinuous.*

**Theorem 1.3.** *([6]) Assume that $\mu_1,\mu_2,...,\mu_k$ are measures of non-compactness in Banach spaces $E_1,E_2,...,E_k$ respectively. Also, suppose that the function $G : [0,+\infty)^k \to [0,+\infty)$ is convex and $G(p_1,p_2,...,p_k) = 0 \Leftrightarrow p_i = 0, (i = 1,2,3,...,k)$. Then*

$$\widetilde{\mu}\left(\mathcal{X}\right) = G\left(\mu_1\left(\mathcal{X}_1\right), \mu_2\left(\mathcal{X}_2\right), ..., \mu_k\left(\mathcal{X}_k\right)\right),$$

*defines a measure of non-compactness in $E_1 \times E_2 \times ... \times E_k$ where $\mathcal{X}_i$ denotes the natural projections of $\mathcal{X}$ into $E_i$, for $i = 1,2,3,..k$.*

**Example 1.4.** *([6]) Consider $G(p,q,r) = p+q+r$ for every $(p,q,r) \in [0,+\infty)^3$, then $G$ has all conditions of Theorem 1.3. So, $\widetilde{\mu}\left(\mathcal{X}\right) = \mu\left(\mathcal{X}_1\right) + \mu\left(\mathcal{X}_2\right) + \mu\left(\mathcal{X}_3\right)$ for each $\mathcal{X} \subseteq E \times E \times E$ is the measure of non-compactness in $E \times E \times E$.*

**Theorem 1.5.** *(Schauder's fixed point theorem [3]) Assume that $C$ be a convex and closed subset of $E$. Then every compact, continuous map $T : C \to C$ has at least one fixed point.*

**Theorem 1.6.** *(Darbo's fixed point theorem [13]) Assume that $\Omega$ be a non-empty, bounded, closed and convex subset of $E$. Consider the constant $\lambda \in [0,1)$. Also, suppose that $T : \Omega \to \Omega$ is a continuous operator such that $\mu\left(T\left(\mathcal{X}\right)\right) \leq \lambda\mu\left(\mathcal{X}\right)$ for each $\mathcal{X} \subset \Omega$ with $\mathcal{X} \neq \emptyset$. Then $T$ has a fixed point in $\Omega$.*

Now, we introduce three different classes of functions that we need in the next section.

**Definition 1.7.** *Let $\Theta$ be the class of all functions $\theta : [0,+\infty)^3 \to [0,+\infty)$ satisfying the following hypothesis:*

$(A1)$ $\theta(p_1+p_2, q_1+q_2, r_1+r_2) \leq \theta(p_1,q_1,r_1) + \theta(p_2,q_2,r_2)$ for every $p_1,p_2,q_1,q_2,r_1,r_2 \in \mathbb{R}_+$,

$(A2)$ $\theta(p,q,r) = 0 \Leftrightarrow p = q = r = 0$, for every $p,q,r \in \mathbb{R}_+$,

$(A3)$ $\theta$ is lower semicontinuous.

For example, $\theta_1(p,q,r) = \ln(p+q+r+1)$ and $\theta_2(p,q,r) = \max\{p,q,r\}$ satisfy the above three properties.

**Definition 1.8.** *Let $\Phi$ be the class of all functions $\phi : [0,+\infty)^3 \to [0,+\infty)$ satisfying the following hypothesis:*

$(B1)$ $\phi$ is continuous and nondecreasing,

$(B2)$ $\phi(h,h,h) < h$ for every $h > 0$,

$(B3)$ $\frac{1}{3}\left(\phi(p_1,q_1,r_1) + \phi(p_2,q_2,r_2) + \phi(p_3,q_3,r_3)\right) \leq \phi\left(\frac{p_1+p_2+p_3}{3}, \frac{q_1+q_2+q_3}{3}, \frac{r_1+r_2+r_3}{3}\right)$ for every

$p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 \in \mathbb{R}_+$.

For example, $\phi_1(p,q,r) = \lambda_1 p + \lambda_2 q + \lambda_3 r$, where $\lambda_1, \lambda_2, \lambda_3 \in [0,1)$ with $\lambda_1 + \lambda_2 + \lambda_3 < 1$, and $\phi_2(p,q,r) = \ln\left(1 + \frac{p+q+r}{3}\right)$ satisfy the above three properties.

**Definition 1.9.** *Let $\Psi$ be the class of all functions $\psi : [0,+\infty)^2 \to [0,+\infty)$ sayisfying the following hypothesis:*

(C1) $\psi$ is continuous,

(C2) $\psi(h,h) \geq h$ for every $h > 0$.

For example, $\psi_1(p,q) = p+q$ and $\psi_2(p,q) = \sqrt{p^2+q^2}$ and $\psi_3(p,q) = e^{\sqrt{p^2+q^2}} - 1$, in which $p, q \in \mathbb{R}_+$ satisfy the above two properties.

Let $BC(\mathbb{R}_+)$ be the Banach space consisting of all defined, bounded and continuous functions on $\mathbb{R}_+$ equipped with the standard supremum norm

$$\|x\| = \sup\{|x(\tau)| : \tau \geq 0\}.$$

Fix $\mathcal{X} \subset BC(\mathbb{R}_+), \mathcal{X} \neq \emptyset$ and $L > 0$ and $\tau \in \mathbb{R}_+$. For $x \in \mathcal{X}$ and $\epsilon \geq 0$

$$\begin{aligned}
\omega^L(x,\epsilon) &= \sup\{|x(\tau) - x(\upsilon)| : \tau, \upsilon \in [0,L], |\tau - \upsilon| \leq \epsilon\}, \\
\omega^L(\mathcal{X},\epsilon) &= \sup\left\{\omega^L(x,\epsilon) : x \in \mathcal{X}\right\}, \\
\omega_0^L(\mathcal{X}) &= \lim_{\epsilon\to 0} \omega^L(\mathcal{X},\epsilon), \\
\omega_0(\mathcal{X}) &= \lim_{L\to\infty} \omega_0^L(\mathcal{X}), \\
\mathcal{X}(\tau) &= \{x(\tau) : x \in \mathcal{X}\},
\end{aligned}$$

and

$$\mu(\mathcal{X}) = \omega_0(\mathcal{X}) + \lim_{\tau\to\infty} \sup diam\mathcal{X}(\tau),$$

where

$$diam\mathcal{X}(\tau) = \sup\left\{\left|x(\tau) - y(\tau)\right| : x, y \in \mathcal{X}\right\}.$$

As mentioned in [11], $\mu(\mathcal{X})$ is the measure of non-compactness in $BC(\mathbb{R}_+)$.

## 2. Main results

Throughout the main results section, let us assume that $\Omega$ is a non-empty, bounded, closed, and convex subset of $E$. Also, assume $\mu$ is an arbitrary measure of non-compactness in $E$.

**Theorem 2.1.** *Assume that $\widetilde{\mu}$ be a measure of non-compactness as in Example 1.4 and $\psi \in \Psi, \theta \in \Theta$. Also, suppose $G : \Omega\times\Omega\times\Omega \longrightarrow \Omega\times\Omega\times\Omega$ is a continuous operator satisfying:*

$$\psi\left(\widetilde{\mu}(G(\mathcal{X})), \widetilde{\mu}(G(\mathcal{X}))\right) \leq \psi\left(\widetilde{\mu}(\mathcal{X}), \widetilde{\mu}(\mathcal{X})\right) - \theta\left(\widetilde{\mu}(\mathcal{X}), \widetilde{\mu}(\mathcal{X}), \widetilde{\mu}(\mathcal{X})\right), \tag{1}$$

for each $\mathcal{X} \subset \Omega\times\Omega\times\Omega$ with $\mathcal{X} \neq \emptyset$. Then $G$ has at least one fixed point in $\Omega\times\Omega\times\Omega$.

*Proof.* We define a sequence $\{\Omega_k\times\Omega_k\times\Omega_k\}_{k=1}^{\infty}$ inductively such that

$$\Omega_0\times\Omega_0\times\Omega_0 = \Omega\times\Omega\times\Omega, \Omega_k\times\Omega_k\times\Omega_k = coG\left(\Omega_{k-1}\times\Omega_{k-1}\times\Omega_{k-1}\right),$$

for $k = 1, 2, \cdots$. By given conditions, we get

$$\begin{aligned}
G(\Omega_0\times\Omega_0\times\Omega_0) &= G(\Omega\times\Omega\times\Omega)\subseteq\Omega\times\Omega\times\Omega=\Omega_0\times\Omega_0\times\Omega_0,\\
\Omega_1\times\Omega_1\times\Omega_1 &= coG(\Omega_0\times\Omega_0\times\Omega_0)\subseteq\Omega\times\Omega\times\Omega=\Omega_0\times\Omega_0\times\Omega_0,\\
&\;.\\
&\;.\\
&\;.\\
...\Omega_{k+1}\times\Omega_{k+1}\times\Omega_{k+1} &\subseteq \Omega_k\times\Omega_k\times\Omega_k\subseteq ...\subseteq\Omega_1\times\Omega_1\times\Omega_1\subseteq\Omega_0\times\Omega_0\times\Omega_0.
\end{aligned}$$

Next, if for an integer $K \geq 0$ we have $\widetilde{\mu}(\Omega_K\times\Omega_K\times\Omega_K) = 0$, then $\Omega_K\times\Omega_K\times\Omega_K$ is relatively compact. Hence, the proof is completed by using Theorem 1.5. Therefore we suppose that $\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k) > 0$ for every $k \geq 0$. Also with given assumptions, we obtain

$$\begin{aligned}
&\psi(\widetilde{\mu}(\Omega_{k-1}\times\Omega_{k-1}\times\Omega_{k-1}),\widetilde{\mu}(\Omega_{k-1}\times\Omega_{k-1}\times\Omega_{k-1}))\\
\leq\;&\psi(\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k),\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k))\\
&-\theta(\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k),\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k),\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k)). \qquad (2)
\end{aligned}$$

Since the sequence $\{\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k)\}_{k=1}^{\infty}$ is a nonincreasing and positive sequence, therefore, there is an $\alpha \geq 0$ such that $\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k)\to\alpha$, as $k\to\infty$. Moreover, we have

$$\begin{aligned}
\psi(\alpha,\alpha) &= \lim_{k\to\infty}\sup\psi(\widetilde{\mu}(\Omega_{k+1}\times\Omega_{k+1}\times\Omega_{k+1}),\widetilde{\mu}(\Omega_{k+1}\times\Omega_{k+1}\times\Omega_{k+1}))\\
&\leq \lim_{k\to\infty}\sup\psi(\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k),\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k))\\
&\quad-\lim_{k\to\infty}\inf\theta\begin{pmatrix}\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k),\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k),\\ \widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k)\end{pmatrix}\\
&\leq \lim_{k\to\infty}\sup\psi(\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k),\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k))\\
&\quad-\theta\begin{pmatrix}\lim_{k\to\infty}\inf\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k),\\ \lim_{k\to\infty}\inf\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k),\\ \lim_{k\to\infty}\inf\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k)\end{pmatrix}\\
&= \psi(\alpha,\alpha)-\theta(\alpha,\alpha,\alpha).
\end{aligned}$$

So, $\theta(\alpha,\alpha,\alpha) = 0$, and hence $\alpha = 0$. So, we conclude that $\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k)\to 0$, as $k\to\infty$. Now, since $\Omega_{k+1}\times\Omega_{k+1}\times\Omega_{k+1}\subseteq\Omega_k\times\Omega_k\times\Omega_k$, then from $(BM5)$, we conclude that $\Omega_\infty\times\Omega_\infty\times\Omega_\infty=\cap_{k=1}^{\infty}\Omega_k\times\Omega_k\times\Omega_k$ is a non-empty, convex,closed set, invariant under $G$ and $\Omega_\infty\times\Omega_\infty\times\Omega_\infty\in\ker\widetilde{\mu}$. So from Theorem 1.5 we deduce that $G$ has a fixed point in $\Omega_\infty\times\Omega_\infty\times\Omega_\infty$. Since $\Omega_\infty\times\Omega_\infty\times\Omega_\infty\subset\Omega\times\Omega\times\Omega$, then the proof is completed. □

**Theorem 2.2.** *Suppose $\psi\in\Psi$ is nondecreasing with $\psi(p_1+p_2,q_1+q_2)\leq\psi(p_1,q_1)+\psi(p_2,q_2)$ for every $p_1,p_2,q_1,q_2\in\mathbb{R}_+$ and $\theta\in\Theta$. Also assume that $G_i:\Omega\times\Omega\times\Omega\longrightarrow\Omega$ $(i=1,2,3)$ are continuous operators satisfying:*

$$\begin{aligned}
\psi(\mu(G_1(X_1\times X_2\times X_3)),\mu(G_1(X_1\times X_2\times X_3))) &\leq \frac{1}{3}\psi\begin{pmatrix}\mu(X_1)+\mu(X_2)+\mu(X_3),\\ \mu(X_1)+\mu(X_2)+\mu(X_3)\end{pmatrix}\\
&\quad-\theta(\mu(X_1),\mu(X_2),\mu(X_3)),\\
\psi(\mu(G_2(X_1\times X_2\times X_3)),\mu(G_2(X_1\times X_2\times X_3))) &\leq \frac{1}{3}\psi\begin{pmatrix}\mu(X_1)+\mu(X_2)+\mu(X_3),\\ \mu(X_1)+\mu(X_2)+\mu(X_3)\end{pmatrix}\\
&\quad-\theta(\mu(X_2),\mu(X_3),\mu(X_1)),\\
\psi(\mu(G_3(X_1\times X_2\times X_3)),\mu(G_3(X_1\times X_2\times X_3))) &\leq \frac{1}{3}\psi\begin{pmatrix}\mu(X_1)+\mu(X_2)+\mu(X_3),\\ \mu(X_1)+\mu(X_2)+\mu(X_3)\end{pmatrix}\\
&\quad-\theta(\mu(X_3),\mu(X_1),\mu(X_2)), \qquad (3)
\end{aligned}$$

for each $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \subset \Omega$. Then there exist $\tau^*, \upsilon^*, \rho^* \in \Omega$ such that

$$\begin{cases} G_1(\tau^*, \upsilon^*, \rho^*) & = \tau^* \\ G_2(\tau^*, \upsilon^*, \rho^*) & = \upsilon^* \\ G_3(\tau^*, \upsilon^*, \rho^*) & = \rho^* \end{cases}. \tag{4}$$

*Proof.* Consider $\widetilde{\mu}$ as defined in Example 1.4. We define $\widetilde{G}$ on $\Omega \times \Omega \times \Omega$ as following:gg

$$\widetilde{G}(\tau, \upsilon, \rho) = (G_1(\tau, \upsilon, \rho), G_2(\tau, \upsilon, \rho), G_3(\tau, \upsilon, \rho)).$$

Clearly, $\widetilde{G}$ is continuous on $\Omega \times \Omega \times \Omega$ by its definition. We will show that $\widetilde{G}$ satisfies all the hypothesis of Theorem 2.1. For this purpose, let $\mathcal{X} \subset \Omega \times \Omega \times \Omega, \mathcal{X} \neq \emptyset$. Then, by axiom $(BM2)$ of Definition 1.1 and relation (3) we obtain

$$\begin{aligned}
\psi\left(\widetilde{\mu}\left(\widetilde{G}(\mathcal{X})\right), \widetilde{\mu}\left(\widetilde{G}(\mathcal{X})\right)\right) &\leq \psi\left(\begin{array}{c} \widetilde{\mu}\left(\begin{array}{c} G_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3) \times G_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3) \\ \times G_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3) \end{array}\right), \\ \widetilde{\mu}\left(\begin{array}{c} G_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3) \times G_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3) \\ \times G_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3) \end{array}\right) \end{array}\right) \\
&= \psi\left(\begin{array}{c} \mu(G_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) + \mu(G_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) \\ +\mu(G_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)), \\ \mu(G_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) + \mu(G_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) \\ +\mu(G_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) \end{array}\right) \\
&\leq \psi(\mu(G_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)), \mu(G_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3))) \\
&\quad +\psi(\mu(G_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)), \mu(G_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3))) \\
&\quad \psi(\mu(G_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)), \mu(G_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3))) \\
&\leq \frac{1}{3}\psi(\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3), \mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3)) \\
&\quad -\theta(\mu(\mathcal{X}_1), \mu(\mathcal{X}_2), \mu(\mathcal{X}_3)) \\
&\quad +\frac{1}{3}\psi(\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3), \mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3)) \\
&\quad -\theta(\mu(\mathcal{X}_2), \mu(\mathcal{X}_3), \mu(\mathcal{X}_3)) \\
&\quad +\frac{1}{3}\psi(\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3), \mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3)) \\
&\quad -\theta(\mu(\mathcal{X}_3), \mu(\mathcal{X}_1), \mu(\mathcal{X}_2)) \\
&= \psi(\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3), \mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3)) \\
&\quad -\left(\begin{array}{c} \theta(\mu(\mathcal{X}_1), \mu(\mathcal{X}_2), \mu(\mathcal{X}_3)) + \theta(\mu(\mathcal{X}_2), \mu(\mathcal{X}_3), \mu(\mathcal{X}_1)) \\ +\theta(\mu(\mathcal{X}_3), \mu(\mathcal{X}_1), \mu(\mathcal{X}_2)) \end{array}\right) \\
&\leq \psi(\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3), \mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3)) \\
&\quad -\theta\left(\begin{array}{c} \mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3), \mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3), \\ \mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3) \end{array}\right) \\
&= \psi(\widetilde{\mu}(\mathcal{X}), \widetilde{\mu}(\mathcal{X})) - \theta(\widetilde{\mu}(\mathcal{X}), \widetilde{\mu}(\mathcal{X}), \widetilde{\mu}(\mathcal{X})).
\end{aligned}$$

So, from Theorem 2.1 we deduce that $\widetilde{G}$ has a fixed point, that is, there exist $\tau^*, \upsilon^*, \rho^* \in \Omega$ such that

$$(\tau^*, \upsilon^*, \rho^*) = \widetilde{G}(\tau^*, \upsilon^*, \rho^*) = (G_1(\tau^*, \upsilon^*, \rho^*), G_2(\tau^*, \upsilon^*, \rho^*), G_3(\tau^*, \upsilon^*, \rho^*)),$$

which means (4) is satisfied. □

**Corollary 2.3.** *Suppose $\lambda_1, \lambda_2, \lambda_3$ are nonnegative constants with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Also assume that $G_i : \Omega \times \Omega \times \Omega \longrightarrow \Omega$ $(i = 1, 2, 3)$ are continuous operators satisfying:*

$$\mu\left(G_i\left(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3\right)\right) \leq \frac{\lambda_1}{3}\mu\left(\mathcal{X}_1\right) + \frac{\lambda_2}{3}\mu\left(\mathcal{X}_2\right) + \frac{\lambda_3}{3}\mu\left(\mathcal{X}_3\right),$$

*for each $X_1, X_2, X_3 \subseteq \Omega$. Then there exist $\tau^*, \upsilon^*, \rho^* \in \Omega$ such that*

$$\begin{cases} G_1\left(\tau^*, \upsilon^*, \rho^*\right) & = \tau^* \\ G_2\left(\tau^*, \upsilon^*, \rho^*\right) & = \upsilon^* \\ G_3\left(\tau^*, \upsilon^*, \rho^*\right) & = \rho^* \end{cases}.$$

*Proof.* Considering $\psi(p, q) = p + q$ and $\theta(p, q, r) = \frac{2}{3}\left[(1 - \lambda_1)p + (1 - \lambda_2)q + (1 - \lambda_3)r\right]$ in Theorem 2.2 the result is desirable. □

**Corollary 2.4.** *Consider the constant $\lambda$ with $0 \leq \lambda < 1$. Also assume that $G_i : \Omega \times \Omega \times \Omega \longrightarrow \Omega$ $(i = 1, 2, 3)$ are continuous operators satisfying:*

$$\mu\left(G_i\left(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3\right)\right) \leq \lambda \max\left\{\mu\left(\mathcal{X}_1\right), \mu\left(\mathcal{X}_2\right), \mu\left(\mathcal{X}_3\right)\right\},$$

for each $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \subseteq \Omega$. Then there exist $\tau^*, \upsilon^*, \rho^* \in \Omega$ such that

$$\begin{cases} G_1\left(\tau^*, \upsilon^*, \rho^*\right) & = \tau^* \\ G_2\left(\tau^*, \upsilon^*, \rho^*\right) & = \upsilon^* \\ G_3\left(\tau^*, \upsilon^*, \rho^*\right) & = \rho^* \end{cases}.$$

*Proof.* Considering $\psi(p, q) = p + q$ and $\theta(p, q, r) = 2(1 - \lambda)\max\{p, q, r\}$ in Theorem 2.2 the result is desirable. □

**Corollary 2.5.** *Suppose $G_i : \Omega \times \Omega \times \Omega \longrightarrow \Omega$ $(i = 1, 2, 3)$ are continuous operators satisfying:*

$$\mu\left(G_i\left(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3\right)\right) \leq \frac{\mu\left(\mathcal{X}_1\right) + \mu\left(\mathcal{X}_2\right) + \mu\left(\mathcal{X}_3\right)}{3} - \ln\left(\mu\left(\mathcal{X}_1\right) + \mu\left(\mathcal{X}_2\right) + \mu\left(\mathcal{X}_3\right) + 1\right),$$

for each $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \subseteq \Omega$. Then there exist $\tau^*, \upsilon^*, \rho^* \in \Omega$ such that

$$\begin{cases} G_1\left(\tau^*, \upsilon^*, \rho^*\right) & = \tau^* \\ G_2\left(\tau^*, \upsilon^*, \rho^*\right) & = \upsilon^* \\ G_3\left(\tau^*, \upsilon^*, \rho^*\right) & = \rho^* \end{cases}.$$

*Proof.* Considering $\psi(p, q) = p + q$ and $\theta(p, q, r) = 2\ln(p + q + r + 1)$ in Theorem 2.2 the result is desirable. □

**Corollary 2.6.** *Consider the constant $\lambda$ with $0 \leq \lambda \leq 1$. Also assume that $G_i : \Omega \times \Omega \times \Omega \longrightarrow \Omega$ $(i = 1, 2, 3)$ are continuous operators satisfying:*

$$\mu\left(G_i\left(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3\right)\right) \leq \left(1 - \lambda^2\right)\left(\frac{\mu\left(\mathcal{X}_1\right) + \mu\left(\mathcal{X}_2\right) + \mu\left(\mathcal{X}_3\right)}{9}\right),$$

for each $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \subseteq \Omega$. Then there exist $\tau^*, \upsilon^*, \rho^* \in \Omega$ such that

$$\begin{cases} G_1\left(\tau^*, \upsilon^*, \rho^*\right) & = \tau^* \\ G_2\left(\tau^*, \upsilon^*, \rho^*\right) & = \upsilon^* \\ G_3\left(\tau^*, \upsilon^*, \rho^*\right) & = \rho^* \end{cases}.$$

*Proof.* Considering $\psi(p,q) = \sqrt{p+q}$ and $\theta(p,q,r) = \frac{\lambda}{3}\sqrt{p+q+r}$ in Theorem 2.2 the result is desirable. □

**Theorem 2.7.** *Assume that $\widetilde{\mu}$ is a measure of non-compactness as in Example 1.4 and $\phi \in \Phi, \psi \in \Psi$. Also suppose $F : \Omega\times\Omega\times\Omega \longrightarrow \Omega\times\Omega\times\Omega$ be a continuous operator satisfying:*

$$\psi\left(\widetilde{\mu}(F(\mathcal{X})), \widetilde{\mu}(F(\mathcal{X}))\right) \le \phi\left(\widetilde{\mu}(\mathcal{X}), \widetilde{\mu}(\mathcal{X}), \widetilde{\mu}(\mathcal{X})\right), \tag{5}$$

for each $X \subset \Omega\times\Omega\times\Omega$ with $\mathcal{X} \neq \emptyset$. Then $F$ has at least one fixed point in $\Omega\times\Omega\times\Omega$.

*Proof.* We define a sequence $\{\Omega_k\times\Omega_k\times\Omega_k\}_{k=1}^{\infty}$ inductively such that

$$\Omega_0\times\Omega_0\times\Omega_0 = \Omega\times\Omega\times\Omega, \Omega_k\times\Omega_k\times\Omega_k = coF\left(\Omega_{k-1}\times\Omega_{k-1}\times\Omega_{k-1}\right),$$

for $k = 1, 2, \cdots$. By given conditions, we obtain

$$\begin{array}{rcl} F(\Omega_0\times\Omega_0\times\Omega_0) &=& F(\Omega\times\Omega\times\Omega) \subseteq \Omega\times\Omega\times\Omega = \Omega_0\times\Omega_0\times\Omega_0, \\ \Omega_1\times\Omega_1\times\Omega_1 &=& coF(\Omega_0\times\Omega_0\times\Omega_0) \subseteq \Omega\times\Omega\times\Omega = \Omega_0\times\Omega_0\times\Omega_0, \\ && . \\ && . \\ && . \\ \cdots\Omega_{k+1}\times\Omega_{k+1}\times\Omega_{k+1} &\subseteq& \Omega_k\times\Omega_k\times\Omega_k \subseteq \cdots \subseteq \Omega_1\times\Omega_1\times\Omega_1 \subseteq \Omega_0\times\Omega_0\times\Omega_0. \end{array}$$

If for an integer $K \ge 0$ we have $\widetilde{\mu}(\Omega_K\times\Omega_K\times\Omega_K) = 0$, then $\Omega_K\times\Omega_K\times\Omega_K$ is relatively compact. Hence, the proof is completed by using Theorem 1.5. Therefore, we suppose that $\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k) > 0$ for each $k \ge 0$. Now, by given conditions, we get

$$\begin{array}{rl} & \psi\left(\widetilde{\mu}(\Omega_{k+1}\times\Omega_{k+1}\times\Omega_{k+1}), \widetilde{\mu}(\Omega_{k+1}\times\Omega_{k+1}\times\Omega_{k+1})\right) \\ = & \psi\left(\widetilde{\mu}(coF(\Omega_k\times\Omega_k\times\Omega_k)), \widetilde{\mu}(coF(\Omega_k\times\Omega_k\times\Omega_k))\right) \\ = & \psi\left(\widetilde{\mu}(F(\Omega_k\times\Omega_k\times\Omega_k)), \widetilde{\mu}(F(\Omega_k\times\Omega_k\times\Omega_k))\right) \\ \le & \phi\left(\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k), \widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k), \widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k)\right). \end{array} \tag{6}$$

Moreover, the sequence $\{\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k)\}_{k=1}^{\infty}$ is a nonincreasing and positive sequence of real numbers, therefore, there is an $\alpha \ge 0$, $\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k) \to \alpha$, as $k \to \infty$. We show that $\alpha = 0$. If we assume that $\alpha > 0$, then from (6), we get

$$\begin{array}{rcl} \psi(\alpha,\alpha) &=& \psi\left(\lim\limits_{k\to\infty}\widetilde{\mu}(\Omega_{k+1}\times\Omega_{k+1}\times\Omega_{k+1}), \lim\limits_{k\to\infty}\widetilde{\mu}(\Omega_{k+1}\times\Omega_{k+1}\times\Omega_{k+1})\right) \\ &\le& \phi\left(\lim\limits_{k\to\infty}\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k), \lim\limits_{k\to\infty}\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k), \lim\limits_{k\to\infty}\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k)\right) \\ &=& \phi(\alpha,\alpha,\alpha) < \alpha. \end{array}$$

Which is contradiction. Therefore, we conclude that $\widetilde{\mu}(\Omega_k\times\Omega_k\times\Omega_k) \to 0$, as $k \to \infty$. Now, since $\Omega_{k+1}\times\Omega_{k+1}\times\Omega_{k+1} \subseteq \Omega_k\times\Omega_k\times\Omega_k$, then from (BM5), we conclude that $\Omega_\infty\times\Omega_\infty\times\Omega_\infty = \cap_{k=1}^{\infty}\Omega_k\times\Omega_k\times\Omega_k$ is a non-empty, covex, closed set, invatiant under $F$ and $\Omega_\infty\times\Omega_\infty\times\Omega_\infty \in \ker\widetilde{\mu}$. So, from Theorem 1.5 we conclude that $F$ has a fixed point in $\Omega_\infty\times\Omega_\infty\times\Omega_\infty$. Since $\Omega_\infty\times\Omega_\infty\times\Omega_\infty \subset \Omega\times\Omega\times\Omega$, then the proof is completed. □

**Theorem 2.8.** *Suppose $\psi \in \Psi$ is nondecreasing with $\psi(p_1+p_2, q_1+q_2) \le \psi(p_1,q_1)+\psi(p_2,q_2)$ for every $p_1, p_2, q_1, q_2 \in \mathbb{R}_+$ and $\phi \in \Phi$. Also assume that $F_i : \Omega\times\Omega\times\Omega \longrightarrow \Omega$ $(i = 1, 2, 3)$ are continuous operators satisfying:*

$$\begin{array}{rcl} \psi\left(\mu(F_1(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3)), \mu(F_1(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3))\right) &\le& \phi\left(\mu(\mathcal{X}_1), \mu(\mathcal{X}_2), \mu(\mathcal{X}_3)\right), \\ \psi\left(\mu(F_2(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3)), \mu(F_2(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3))\right) &\le& \phi\left(\mu(\mathcal{X}_2), \mu(\mathcal{X}_3), \mu(\mathcal{X}_1)\right), \\ \psi\left(\mu(F_3(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3)), \mu(F_3(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3))\right) &\le& \phi\left(\mu(\mathcal{X}_3), \mu(\mathcal{X}_1), \mu(\mathcal{X}_2)\right), \end{array} \tag{7}$$

for each $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \subseteq \Omega$. Then there exist $\tau^*, \upsilon^*, \rho^* \in \Omega$ such that

$$\begin{cases} F_1(\tau^*, \upsilon^*, \rho^*) & = \tau^* \\ F_2(\tau^*, \upsilon^*, \rho^*) & = \upsilon^* \\ F_3(\tau^*, \upsilon^*, \rho^*) & = \rho^* \end{cases}. \tag{8}$$

*Proof.* Consider $\widetilde{\mu}$ as defined in Example 1.4. We define $\widetilde{F}$ on $\Omega \times \Omega \times \Omega$ as following:

$$\widetilde{F}(\tau, \upsilon, \rho) = (F_1(\tau, \upsilon, \rho), F_2(\tau, \upsilon, \rho), F_3(\tau, \upsilon, \rho)).$$

Clearly, $\widetilde{F}$ is continuous on $\Omega \times \Omega \times \Omega$ by its definition. We will show that $\widetilde{F}$ satisfies all the hypothesis of Theorem 2.7. For this purpose, let $\mathcal{X} \subset \Omega \times \Omega \times \Omega, \mathcal{X} \neq \emptyset$. Then, by $(BM2)$ and (7) we obtain

$$\begin{aligned}
\psi\left(\widetilde{\mu}\left(\widetilde{F}(\mathcal{X})\right), \widetilde{\mu}\left(\widetilde{F}(\mathcal{X})\right)\right) &\leq \psi\begin{pmatrix} \widetilde{\mu}\begin{pmatrix} F_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3) \times F_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3) \\ \times F_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3) \end{pmatrix}, \\ \widetilde{\mu}\begin{pmatrix} F_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3) \times F_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3) \\ \times F_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3) \end{pmatrix} \end{pmatrix} \\
&= \psi\begin{pmatrix} \mu(F_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) + \mu(F_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) \\ +\mu(F_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)), \\ \mu(F_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) + \mu(F_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) \\ +\mu(F_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) \end{pmatrix} \\
&\leq \psi(\mu(F_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)), \mu(F_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3))) \\
&\quad +\psi(\mu(F_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)), \mu(F_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3))) \\
&\quad +\psi(\mu(F_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)), \mu(F_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3))) \\
&\leq \phi(\mu(\mathcal{X}_1), \mu(\mathcal{X}_2), \mu(\mathcal{X}_3)) + \phi(\mu(\mathcal{X}_2), \mu(\mathcal{X}_3), \mu(\mathcal{X}_1)) \\
&\quad +\phi(\mu(\mathcal{X}_3), \mu(\mathcal{X}_1), \mu(\mathcal{X}_2)) \\
&\leq 3\phi\begin{pmatrix} \frac{\mu(\mathcal{X}_1)+\mu(\mathcal{X}_2)+\mu(\mathcal{X}_3)}{3}, \frac{\mu(\mathcal{X}_1)+\mu(\mathcal{X}_2)+\mu(\mathcal{X}_3)}{3}, \\ \frac{\mu(\mathcal{X}_1)+\mu(\mathcal{X}_2)+\mu(\mathcal{X}_3)}{3} \end{pmatrix}.
\end{aligned} \tag{9}$$

Now from (9) and taking $\widehat{\mu} = \frac{1}{3}\widetilde{\mu}$, we obtain

$$\psi\left(\widehat{\mu}\left(\widetilde{F}(\mathcal{X})\right), \widehat{\mu}\left(\widetilde{F}(\mathcal{X})\right)\right) \leq \phi(\widehat{\mu}(\mathcal{X}), \widehat{\mu}(\mathcal{X}), \widehat{\mu}(\mathcal{X})). \tag{10}$$

Hence, by Theorem 2.7 $\widetilde{F}$ has a fixed point, i.e., there exist $\tau^*, \upsilon^*, \rho^* \in \Omega$ such that

$$(\tau^*, \upsilon^*, \rho^*) = \widetilde{F}(\tau^*, \upsilon^*, \rho^*) = (F_1(\tau^*, \upsilon^*, \rho^*), F_2(\tau^*, \upsilon^*, \rho^*), F_3(\tau^*, \upsilon^*, \rho^*)),$$

which means (8) is satisfied. □

**Corollary 2.9.** *Assume that $F_i : \Omega \times \Omega \times \Omega \longrightarrow \Omega$ $(i = 1, 2, 3)$ are continuous operators satisfying:*

$$\begin{aligned}
\mu(F_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) &\leq \phi(\mu(\mathcal{X}_1), \mu(\mathcal{X}_2), \mu(\mathcal{X}_3)), \\
\mu(F_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) &\leq \phi(\mu(\mathcal{X}_2), \mu(\mathcal{X}_3), \mu(\mathcal{X}_1)), \\
\mu(F_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) &\leq \phi(\mu(\mathcal{X}_3), \mu(\mathcal{X}_1), \mu(\mathcal{X}_2)),
\end{aligned}$$

for each $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \subseteq \Omega$, where $\phi \in \Phi$. Then there exist $\tau^*, \upsilon^*, \rho^* \in \Omega$ such that

$$\begin{cases} F_1(\tau^*, \upsilon^*, \rho^*) & = \tau^* \\ F_2(\tau^*, \upsilon^*, \rho^*) & = \upsilon^* \\ F_3(\tau^*, \upsilon^*, \rho^*) & = \rho^* \end{cases}.$$

*Proof.* Considering $\psi(p,q) = \dfrac{p+q}{2}$ in Theorem 2.8 the result is desirable. $\square$

**Corollary 2.10.** *Suppose $\lambda_1, \lambda_2, \lambda_3$ are nonnegative constants with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Also assume that $F_i : \Omega \times \Omega \times \Omega \longrightarrow \Omega$ $(i = 1, 2, 3)$ are continuous operators satisfying:*

$$\begin{array}{rcl} \mu(F_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) & \leq & \lambda_1\mu(\mathcal{X}_1) + \lambda_2\mu(\mathcal{X}_2) + \lambda_3\mu(\mathcal{X}_3), \\ \mu(F_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) & \leq & \lambda_1\mu(\mathcal{X}_2) + \lambda_2\mu(\mathcal{X}_3) + \lambda_3\mu(\mathcal{X}_1), \\ \mu(F_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) & \leq & \lambda_1\mu(\mathcal{X}_3) + \lambda_2\mu(\mathcal{X}_1) + \lambda_3\mu(\mathcal{X}_2), \end{array}$$

*for each $X_1, X_2, X_3 \subseteq \Omega$. Then there exist $\tau^*, \upsilon^*, \rho^* \in \Omega$ such that*

$$\begin{cases} F_1(\tau^*, \upsilon^*, \rho^*) & = \tau^* \\ F_2(\tau^*, \upsilon^*, \rho^*) & = \upsilon^* \\ F_3(\tau^*, \upsilon^*, \rho^*) & = \rho^* \end{cases}.$$

*Proof.* Considering $\psi(p,q) = p+q$ and $\phi(p,q,r) = 2\lambda_1 p + 2\lambda_2 q + 2\lambda_3 r$ in Theorem 2.8 the result is desirable. $\square$

**Corollary 2.11.** *Assume that $F_i : \Omega \times \Omega \times \Omega \longrightarrow \Omega$ $(i = 1, 2, 3)$ are continuous operators satisfying:*

$$\mu(F_i(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) \leq \frac{1}{\sqrt{2}} \ln\left(1 + \frac{\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2) + \mu(\mathcal{X}_3)}{3}\right),$$

*for each $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \subseteq \Omega$. Then there exist $\tau^*, \upsilon^*, \rho^* \in \Omega$ such that*

$$\begin{cases} F_1(\tau^*, \upsilon^*, \rho^*) & = \tau^* \\ F_2(\tau^*, \upsilon^*, \rho^*) & = \upsilon^* \\ F_3(\tau^*, \upsilon^*, \rho^*) & = \rho^* \end{cases}.$$

*Proof.* Considering $\psi(p,q) = \sqrt{p^2 + q^2}$ and $\phi(p,q,r) = \ln\left(1 + \frac{p+q+r}{3}\right)$ in Theorem 2.8 the result is desirable. $\square$

**Corollary 2.12.** *Suppose $\lambda_1, \lambda_2, \lambda_3$ are nonegative constants with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Also assume that $F_i : \Omega \times \Omega \times \Omega \longrightarrow \Omega$ $(i = 1, 2, 3)$ are continuous operators satisfying:*

$$\begin{array}{rcl} \mu(F_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) + \ln(1 + \mu(F_1(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3))) & \leq & \lambda_1\mu(\mathcal{X}_1) + \lambda_2\mu(\mathcal{X}_2) + \lambda_3\mu(\mathcal{X}_3), \\ \mu(F_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) + \ln(1 + \mu(F_2(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3))) & \leq & \lambda_1\mu(\mathcal{X}_2) + \lambda_2\mu(\mathcal{X}_3) + \lambda_3\mu(\mathcal{X}_1), \\ \mu(F_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)) + \ln(1 + \mu(F_3(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3))) & \leq & \lambda_1\mu(\mathcal{X}_3) + \lambda_2\mu(\mathcal{X}_1) + \lambda_3\mu(\mathcal{X}_2), \end{array}$$

*for each $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \subseteq \Omega$. Then there exist $\tau^*, \upsilon^*, \rho^* \in \Omega$ such that*

$$\begin{cases} F_1(\tau^*, \upsilon^*, \rho^*) & = \tau^* \\ F_2(\tau^*, \upsilon^*, \rho^*) & = \upsilon^* \\ F_3(\tau^*, \upsilon^*, \rho^*) & = \rho^* \end{cases}.$$

*Proof.* Considering $\psi(p,q) = \frac{p+q}{2} + \ln\left(1 + \frac{p+q}{2}\right)$ and $\phi(p,q,r) = \lambda_1 p + \lambda_2 q + \lambda_3 r$ in Theorem 2.8 the result is desirable. $\square$

## 3. Application and Example

Consider the following system of integral equations:

$$\begin{cases} x(\tau) = A_1(\tau) + h_1(\tau, x(\varepsilon_1(\tau)), y(\varepsilon_1(\tau)), z(\varepsilon_1(\tau))) \\ +f_1\left(\begin{array}{c} \tau, x(\varepsilon_1(\tau)), y(\varepsilon_1(\tau)), z(\varepsilon_1(\tau)), \\ \theta_1\left(\int_0^{\beta_1(\tau)} g_1(\tau, v, x(\sigma_1(v)), y(\sigma_1(v)), z(\sigma_1(v)))\, dv\right) \end{array}\right), \\ y(\tau) = A_2(\tau) + h_2(\tau, x(\varepsilon_2(\tau)), y(\varepsilon_2(\tau)), z(\varepsilon_2(\tau))) \\ +f_2\left(\begin{array}{c} \tau, x(\varepsilon_2(\tau), y(\varepsilon_2(\tau)), z(\varepsilon_2(\tau))), \\ \theta_2\left(\int_0^{\beta_2(\tau)} g_2(\tau, v, x(\sigma_2(v)), y(\sigma_2(v)), z(\sigma_2(v)))\, dv\right) \end{array}\right), \\ z(\tau) = A_3(\tau) + h_3(\tau, x(\varepsilon_3(\tau)), y(\varepsilon_3(\tau)), z(\varepsilon_3(\tau))) \\ +f_3\left(\begin{array}{c} \tau, x(\varepsilon_3(\tau)), y(\varepsilon_3(\tau)), z(\varepsilon_3(\tau)), \\ \theta_3\left(\int_0^{\beta_3(\tau)} g_3(\tau, v, x(\sigma_3(v)), y(\sigma_3(v)), z(\sigma_3(v)))\, dv\right) \end{array}\right) \end{cases} . \tag{11}$$

**Theorem 3.1.** *Let*

(I) $A_i : \mathbb{R}_+ \to \mathbb{R}, i = 1, 2, 3$ are continuous and bounded functions with

$$M_i = \sup\{|A_i(\tau)| : \tau \in \mathbb{R}_+\}.$$

(II) $\varepsilon_i, \sigma_i, \beta_i : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions and $\varepsilon_i(\tau) \to \infty$ as $\tau \to \infty$, for $i = 1, 2, 3$,

(III) $\theta_i : \mathbb{R}_+ \to \mathbb{R}$ with $\theta_i(0) = 0$ are continuous functions and consider the positive constants $\alpha_i, \delta_i$ with

$$|\theta_i(\tau_1) - \theta_i(\tau_2)| \le \delta_i |\tau_1 - \tau_2|^{\alpha_i}, \tag{12}$$

for every $\tau_1, \tau_2 \in \mathbb{R}_+, i = 1, 2, 3$,

(IV) $|f_i(\tau, 0, 0, 0, 0)|$ and $|h_i(\tau, 0, 0, 0)|, (i = 1, 2, 3)$ are bounded on $\mathbb{R}_+$, that is,

$$M_i' = \sup\{|f_i(\tau, 0, 0, 0, 0)| : \tau \in \mathbb{R}_+\} < \infty,$$
$$M_i'' = \sup\{|h_i(\tau, 0, 0, 0)| : \tau \in \mathbb{R}_+\} < \infty.$$

(V) $f_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\phi_i \in \Phi$ and $\varphi_i : \mathbb{R}_+ \to \mathbb{R}$ are nondecreasing continuous functions with $\varphi_i(0) = 0$, for $i = 1, 2, 3$,

$$|h_i(\tau, x, y, z) - h_i(\tau, u, v, w)| \le \frac{1}{2}\phi_i(|x - u|, |y - v|, |z - w|),$$
$$|f_i(\tau, x, y, z, m) - f_i(\tau, u, v, w, n)| \le \frac{1}{2}\phi_i(|x - u|, |y - v|, |z - w|) + \varphi_i(|m - n|), \tag{13}$$

for every $\tau \ge 0, x, y, z, m, n, u, v, w \in \mathbb{R}$,

(VI) $g_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (i = 1, 2, 3)$ are continuous and also we have

$$\lim_{\tau \to \infty} \int_0^{\beta_i(\tau)} \left| \begin{array}{c} g_i(\tau, v, x(\sigma_i(v)), y(\sigma_i(v)), z(\sigma_i(v))) \\ -g_i(\tau, v, u(\sigma_i(v)), v(\sigma_i(v)), w(\sigma_i(v))) \end{array} \right| dv$$
$$= 0, \tag{14}$$

and,

$$M_i''' = \sup\left\{ \begin{array}{c} \left|\int_0^{\beta_i(\tau)} g_i\left(\tau, \upsilon, x\left(\sigma_i\left(\upsilon\right)\right), y\left(\sigma_i\left(\upsilon\right)\right), z\left(\sigma_i\left(\upsilon\right)\right)\right) d\upsilon\right|^{\alpha_i} : \tau \in \mathbb{R}_+, \\ x, y, z \in BC\left(\mathbb{R}_+\right) \end{array} \right\}, \tag{15}$$

$(VII)$ The following inequality for a $\rho > 0$ is valid.

$$M_i + \phi_i\left(\kappa, \kappa, \kappa\right) + M_i' + M_i'' + \varphi_i\left(\delta_i M_i'''\right) < \kappa, (i = 1, 2, 3). \tag{16}$$

Then the system (11) has at least one solution in $BC\left(\mathbb{R}_+\right) \times BC\left(\mathbb{R}_+\right) \times BC\left(\mathbb{R}_+\right)$.

*Proof.* Consider the operators $T_i : BC\left(\mathbb{R}_+\right) \times BC\left(\mathbb{R}_+\right) \times BC\left(\mathbb{R}_+\right) \to BC\left(\mathbb{R}_+\right), (i = 1, 2, 3)$ by the formula:

$$\begin{aligned} T_i\left(x, y, z\right)\left(\tau\right) &= A_i\left(\tau\right) + h_i\left(\tau, x\left(\varepsilon_i\left(\tau\right)\right), y\left(\varepsilon_i\left(\tau\right)\right), z\left(\varepsilon_i\left(\tau\right)\right)\right) \\ &+ f_i\left( \begin{array}{c} \tau, x\left(\varepsilon_i\left(\tau\right)\right), y\left(\varepsilon_i\left(\tau\right)\right), z\left(\varepsilon_i\left(\tau\right)\right), \\ \theta_i\left(\int_0^{\beta_i(\tau)} g_i\left(\tau, \upsilon, x\left(\sigma_i\left(\upsilon\right)\right), y\left(\sigma_i\left(\upsilon\right)\right), z\left(\sigma_i\left(\upsilon\right)\right)\right) d\upsilon\right) \end{array} \right). \end{aligned} \tag{17}$$

Since $A_i, h_i$ and $f_i$ , $(i = 1, 2, 3)$ are continuous, then $T_i$ , $(i = 1, 2, 3)$ are continuous. Also with given assumptions, we get

$$\begin{aligned} \left|T_i\left(x, y, z\right)\left(\tau\right)\right| &\leq \left|A_i\left(\tau\right)\right| + \left|h_i\left(\tau, x\left(\varepsilon_i\left(\tau\right)\right), y\left(\varepsilon_i\left(\tau\right)\right), z\left(\varepsilon_i\left(\tau\right)\right)\right) - h_i\left(\tau, 0, 0, 0\right)\right| \\ &+ \left|h_i\left(\tau, 0, 0, 0\right)\right| \\ &+ \left( \begin{array}{c} \left| \begin{array}{c} f_i\left( \begin{array}{c} \tau, x\left(\varepsilon_i\left(\tau\right)\right), y\left(\varepsilon_i\left(\tau\right)\right), z\left(\varepsilon_i\left(\tau\right)\right), \\ \theta_i\left(\int_0^{\beta_i(\tau)} g_i\left( \begin{array}{c} \tau, \upsilon, \\ x\left(\sigma_i\left(\upsilon\right)\right), y\left(\sigma_i\left(\upsilon\right)\right), z\left(\sigma_i\left(\upsilon\right)\right) \end{array} \right) d\upsilon\right) \end{array} \right) \\ -f_i\left(\tau, 0, 0, 0, 0\right) \end{array} \right| \\ +\left|f_i\left(\tau, 0, 0, 0, 0\right)\right| \end{array} \right) \\ &\leq M_i + M_i' + M_i'' + \frac{1}{2}\phi_i\left(\left|x\left(\varepsilon_i\left(\tau\right)\right)\right|, \left|y\left(\varepsilon_i\left(\tau\right)\right)\right|, \left|z\left(\varepsilon_i\left(\tau\right)\right)\right|\right) \\ &+ \frac{1}{2}\phi_i\left(\left|x\left(\varepsilon_i\left(\tau\right)\right)\right|, \left|y\left(\varepsilon_i\left(\tau\right)\right)\right|, \left|z\left(\varepsilon_i\left(\tau\right)\right)\right|\right) \\ &+\varphi_i\left(\left|\theta_i\left(\int_0^{\beta_i(\tau)} g_i\left(\tau, \upsilon, x\left(\sigma_i\left(\upsilon\right)\right), y\left(\sigma_i\left(\upsilon\right)\right), z\left(\sigma_i\left(\upsilon\right)\right)\right) d\upsilon\right)\right|\right) \\ &\leq M_i + M_i' + M_i'' + \phi_i\left(\left|x\left(\varepsilon_i\left(\tau\right)\right)\right|, \left|y\left(\varepsilon_i\left(\tau\right)\right)\right|, \left|z\left(\varepsilon_i\left(\tau\right)\right)\right|\right) \\ &+ \varphi_i\left(\left|\theta_i\left(\int_0^{\beta_i(\tau)} g_i\left(\tau, \upsilon, x\left(\sigma_i\left(\upsilon\right)\right), y\left(\sigma_i\left(\upsilon\right)\right), z\left(\sigma_i\left(\upsilon\right)\right)\right) d\upsilon\right)\right|\right) \\ &\leq M_i + M_i' + M_i'' + \phi_i\left(\left\|x\right\|, \left\|y\right\|, \left\|z\right\|\right) \\ &+ \varphi_i\left(\delta_i\left|\int_0^{\beta_i(\tau)} g_i\left(\tau, \upsilon, x\left(\sigma_i\left(\upsilon\right)\right), y\left(\sigma_i\left(\upsilon\right)\right), z\left(\sigma_i\left(\upsilon\right)\right)\right) d\upsilon\right|^{\alpha_i}\right) \\ &\leq M_i + M_i' + M_i'' + \phi_i\left(\left\|x\right\|, \left\|y\right\|, \left\|z\right\|\right) + \varphi_i\left(\delta_i M_i'''\right), \end{aligned} \tag{18}$$

that shows $T_i, (i = 1, 2, 3)$, are well defined. Also, condition $(VII)$ and relation (18) imply that $T_i\left(\bar{B}_\rho \times \bar{B}_\rho \times \bar{B}_\rho\right) \subseteq \bar{B}_\rho$.

Now, we show that $T_i, i = 1, 2, 3$, are continuous on $\bar{B}_\rho \times \bar{B}_\rho \times \bar{B}_\rho$. Fix arbitrarily $\varepsilon > 0$. Consider

$(x,y,z),(u,v,w) \in \overline{B_\rho} \times \overline{B_\rho} \times \overline{B_\rho}$ with $\left\|(x,y,z)-(u,v,w)\right\| < \frac{\epsilon}{2}$. we obtain

$$\begin{aligned}
\left|T_i(x,y,z)(\tau) - T_i(u,v,w)(\tau)\right| &\le \left| \begin{array}{c} h_i(\tau, x(\varepsilon_i(\tau)), y(\varepsilon_i(\tau)), z(\varepsilon_i(\tau))) \\ -h_i(\tau, u(\varepsilon_i(\tau)), v(\varepsilon_i(\tau)), w(\varepsilon_i(\tau))) \end{array} \right| \\
&+ \left| \begin{array}{c} f_i\left( \begin{array}{c} \tau, x(\varepsilon_i(\tau)), y(\varepsilon_i(\tau)), z(\varepsilon_i(\tau)), \\ \theta_i\left(\int_0^{\beta_i(\tau)} g_i(\tau, v, x(\sigma_i(v)), y(\sigma_i(v)), z(\sigma_i(v)))\, dv\right) \end{array} \right) \\ -f_i\left( \begin{array}{c} \tau, x(\varepsilon_i(\tau)), y(\varepsilon_i(\tau)), z(\varepsilon_i(\tau)), \\ \theta_i\left(\int_0^{\beta_i(\tau)} g_i(\tau, v, u(\sigma_i(v)), v(\sigma_i(v)), w(\sigma_i(v)))\, dv\right) \end{array} \right) \end{array} \right| \\
&\le \frac{1}{2}\phi_i\left( \begin{array}{c} \left|x(\varepsilon_i(\tau)) - u(\varepsilon_i(\tau))\right|, \\ \left|y(\varepsilon_i(\tau)) - v(\varepsilon_i(\tau))\right|, \\ \left|z(\varepsilon_i(\tau)) - w(\varepsilon_i(\tau))\right| \end{array} \right) \\
&+ \frac{1}{2}\phi_i\left( \begin{array}{c} \left|x(\varepsilon_i(\tau)) - u(\varepsilon_i(\tau))\right|, \\ \left|y(\varepsilon_i(\tau)) - v(\varepsilon_i(\tau))\right|, \\ \left|z(\varepsilon_i(\tau)) - w(\varepsilon_i(\tau))\right| \end{array} \right) \\
&+ \varphi_i\left( \left| \begin{array}{c} \theta_i\left(\int_0^{\beta_i(\tau)} g_i(\tau, v, x(\sigma_i(v)), y(\sigma_i(v)), z(\sigma_i(v)))\, dv\right) \\ -\theta_i\left(\int_0^{\beta_i(\tau)} g_i(\tau, v, u(\sigma_i(v)), v(\sigma_i(v)), w(\sigma_i(v)))\, dv\right) \end{array} \right| \right) \\
&\le \frac{1}{2}\phi_i\left(\|x-u\|, \left\|y-v\right\|, \|z-w\|\right) \\
&+ \frac{1}{2}\phi_i\left(\|x-u\|, \left\|y-v\right\|, \|z-w\|\right) \\
&+ \varphi_i\left(\delta_i \left| \int_0^{\beta_i(\tau)} \left( \begin{array}{c} g_i(\tau, v, x(\sigma_i(v)), y(\sigma_i(v)), z(\sigma_i(v))) \\ -g_i(\tau, v, u(\sigma_i(v)), v(\sigma_i(v)), w(\sigma_i(v))) \end{array} \right) dv \right|^{\alpha_i} \right) \\
&\le \phi_i\left(\|x-u\|, \left\|y-v\right\|, \|z-w\|\right) \\
&+ \varphi_i\left(\delta_i \left| \int_0^{\beta_i(\tau)} \left( \begin{array}{c} g_i(\tau, v, x(\sigma_i(v)), y(\sigma_i(v)), z(\sigma_i(v))) \\ -g_i(\tau, v, u(\sigma_i(v)), v(\sigma_i(v)), w(\sigma_i(v))) \end{array} \right) dv \right|^{\alpha_i} \right). \qquad (19)
\end{aligned}$$

Furthermore, from relation (14), we have

$$\begin{aligned}
&\varphi_i\left(\delta_i \left| \int_0^{\beta_i(\tau)} \left( \begin{array}{c} g_i(\tau, v, x(\sigma_i(v)), y(\sigma_i(v)), z(\sigma_i(v))) \\ -g_i(\tau, v, u(\sigma_i(v)), v(\sigma_i(v)), w(\sigma_i(v))) \end{array} \right) dv \right|^{\alpha_i} \right) \\
&\le \frac{\epsilon}{2}, \qquad (20)
\end{aligned}$$

for every $x, y, z, u, v, w \in BC(\mathbb{R}_+)$.

If $\tau > L$, then from relations (19) and (20), we obtain

$$\begin{aligned}
\left|T_i(x,y,z)(\tau) - T_i(u,v,w)(\tau)\right| &\le \phi_i\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right) + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad (21)
\end{aligned}$$

If $\tau \in [0, L]$, then we get

$$\begin{aligned}
&\left|T_i(x,y,z)(\tau) - T_i(u,v,w)(\tau)\right| \\
&\le \phi_i\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right) + \varphi_i\left(\delta_i \left| \int_0^{\beta_i(\tau)} \left( \begin{array}{c} g_i(\tau, v, x(\sigma_i(v)), y(\sigma_i(v)), z(\sigma_i(v))) \\ -g_i(\tau, v, u(\sigma_i(v)), v(\sigma_i(v)), w(\sigma_i(v))) \end{array} \right) dv \right|^{\alpha_i} \right) \\
&< \frac{\epsilon}{2} + \varphi_i\left(\delta_i \left(\beta_i^L \omega(\epsilon)\right)^{\alpha_i}\right), \qquad (22)
\end{aligned}$$

where

$$\omega(\epsilon)=\sup\left\{\begin{array}{c}\left|g_i(\tau,v,x,y,z)-g_i(\tau,v,u,v,w)\right|:\tau\in[0,L],v\in\left[0,\beta_i^L\right],\\ x,y,z,u,v,w\in[-\rho,\rho],\left\|(x,y,z)-(u,v,w)\right\|<\frac{\epsilon}{2}\end{array}\right\},$$
$$\beta_i^L=\sup\{\beta_i(\tau):\tau\in[0,L]\}.$$

Using the continuity of $g_i, i=1,2,3$ on $[0,L]\times\left[0,\beta_i^L\right]\times[-\rho,\rho]\times[-\rho,\rho]\times[-\rho,\rho]$, we have $\omega(\epsilon)\to 0$, as $\epsilon\to 0$

and by continuity $\varphi_i, i=1,2,3$, we obtain

$$\varphi_i\left(\delta_i\left(\beta_i^L\omega(\epsilon)\right)^{\alpha_i}\right)\to 0,$$

as $\epsilon\to 0$. Therefore, from relations (21) and (22), we conclude that $T_i, i=1,2,3$, are continuous functions from $\bar{B}_\rho\times\bar{B}_\rho\times\bar{B}_\rho$ into $\overline{B}_\rho$. Next, we show that $T_i, i=1,2,3$, satisfies the conditions of Corollary 2.9. For this purpose, suppose $L,\epsilon\in\mathbb{R}_+,\tau_1,\tau_2\in[0,L]$ with $|\tau_1-\tau_2|\le\epsilon$ and $\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3$ are arbitrary non-empty subsets of $\overline{B}_\rho$.

Let $(x,y,z)\in\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3$. We can assume that $\beta_i(\tau_1)<\beta_i(\tau_2)$. Consequently,

$$\begin{aligned}
&T_i(x,y,z)(\tau_1)-T_i(x,y,z)(\tau_2)\\
\le\ & |A_i(\tau_1)-A_i(\tau_2)|+\left|\begin{array}{c}h_i(\tau_2,x(\varepsilon_i(\tau_2)),y(\varepsilon_i(\tau_2)),z(\varepsilon_i(\tau_2)))\\ -h_i(\tau_2,x(\varepsilon_i(\tau_1)),y(\varepsilon_i(\tau_1)),z(\varepsilon_i(\tau_1)))\end{array}\right|\\
&+\left|\begin{array}{l}f_i\left(\begin{array}{c}\tau_2,x(\varepsilon_i(\tau_2)),y(\varepsilon_i(\tau_2)),z(\varepsilon_i(\tau_2)),\\ \theta_i\left(\int_0^{\beta_i(\tau_2)}g_i\left(\begin{array}{c}\tau_2,v,x(\sigma_i(v)),y(\sigma_i(v)),\\ z(\sigma_i(v))\end{array}\right)dv\right)\end{array}\right)\\ -f_i\left(\begin{array}{c}\tau_2,x(\varepsilon_i(\tau_1)),y(\varepsilon_i(\tau_1)),z(\varepsilon_i(\tau_1)),\\ \theta_i\left(\int_0^{\beta_i(\tau_2)}g_i\left(\begin{array}{c}\tau_2,v,x(\sigma_i(v)),y(\sigma_i(v)),\\ z(\sigma_i(v))\end{array}\right)dv\right)\end{array}\right)\end{array}\right|\\
&+\left|\begin{array}{l}f_i\left(\begin{array}{c}\tau_2,x(\varepsilon_i(\tau_1)),y(\varepsilon_i(\tau_1)),z(\varepsilon_i(\tau_1)),\\ \theta_i\left(\int_0^{\beta_i(\tau_2)}g_i\left(\begin{array}{c}\tau_2,v,x(\sigma_i(v)),y(\sigma_i(v)),\\ z(\sigma_i(v))\end{array}\right)dv\right)\end{array}\right)\\ -f_i\left(\begin{array}{c}\tau_1,x(\varepsilon_i(\tau_1)),y(\varepsilon_i(\tau_1)),z(\varepsilon_i(\tau_1)),\\ \theta_i\left(\int_0^{\beta_i(\tau_2)}g_i\left(\begin{array}{c}\tau_2,v,x(\sigma_i(v)),y(\sigma_i(v)),\\ z(\sigma_i(v))\end{array}\right)dv\right)\end{array}\right)\end{array}\right|\\
&+\left|\begin{array}{l}f_i\left(\begin{array}{c}\tau_1,x(\varepsilon_i(\tau_1)),y(\varepsilon_i(\tau_1)),z(\varepsilon_i(\tau_1)),\\ \theta_i\left(\int_0^{\beta_i(\tau_2)}g_i\left(\begin{array}{c}\tau_2,v,x(\sigma_i(v)),y(\sigma_i(v)),\\ z(\sigma_i(v))\end{array}\right)dv\right)\end{array}\right)\\ -f_i\left(\begin{array}{c}\tau_1,x(\varepsilon_i(\tau_1)),y(\varepsilon_i(\tau_1)),z(\varepsilon_i(\tau_1)),\\ \theta_i\left(\int_0^{\beta_i(\tau_2)}g_i\left(\begin{array}{c}\tau_1,v,x(\sigma_i(v)),y(\sigma_i(v)),\\ z(\sigma_i(v))\end{array}\right)dv\right)\end{array}\right)\end{array}\right|\\
&+\left|\begin{array}{l}f_i\left(\begin{array}{c}\tau_1,x(\varepsilon_i(\tau_1)),y(\varepsilon_i(\tau_1)),z(\varepsilon_i(\tau_1)),\\ \theta_i\left(\int_0^{\beta_i(\tau_2)}g_i(\tau_1,v,x(\sigma_i(v)),y(\sigma_i(v)),z(\sigma_i(v)))dv\right)\end{array}\right)\\ -f_i\left(\begin{array}{c}\tau_1,x(\varepsilon_i(\tau_1)),y(\varepsilon_i(\tau_1)),z(\varepsilon_i(\tau_1)),\\ \theta_i\left(\int_0^{\beta_i(\tau_1)}g_i(\tau_1,v,x(\sigma_i(v)),y(\sigma_i(v)),z(\sigma_i(v)))dv\right)\end{array}\right)\end{array}\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \omega^L(A_i,\epsilon)+\omega^L_{\rho,H}(f_i,\epsilon)+\omega^L_\rho(h_i,\epsilon)\\
&\quad+\frac{1}{2}\phi_i\begin{pmatrix} |x(\varepsilon_i(\tau_2))-x(\varepsilon_i(\tau_1))|,\left|y(\varepsilon_i(\tau_2))-y(\varepsilon_i(\tau_1))\right|,\\ |z(\varepsilon_i(\tau_2))-z(\varepsilon_i(\tau_1))|\end{pmatrix}\\
&\quad+\frac{1}{2}\phi_i\begin{pmatrix} |x(\varepsilon_i(\tau_2))-x(\varepsilon_i(\tau_1))|,\left|y(\varepsilon_i(\tau_2))-y(\varepsilon_i(\tau_1))\right|,\\ |z(\varepsilon_i(\tau_2))-z(\varepsilon_i(\tau_1))|\end{pmatrix}\\
&\quad+\varphi_i\left(\left|\begin{array}{c}\theta_i\left(\int_0^{\beta_i(\tau_2)}g_i(\tau_2,\upsilon,x(\sigma_i(\upsilon)),y(\sigma_i(\upsilon)),z(\sigma_i(\upsilon)))\,d\upsilon\right)\\ -\theta_i\left(\int_0^{\beta_i(\tau_2)}g_i(\tau_1,\upsilon,x(\sigma_i(\upsilon)),y(\sigma_i(\upsilon)),z(\sigma_i(\upsilon)))\,d\upsilon\right)\end{array}\right|\right)\\
&\quad+\varphi_i\left(\left|\theta_i\left(\int_{\beta_i(\tau_1)}^{\beta_i(\tau_2)}g_i(\tau_1,\upsilon,x(\sigma_i(\upsilon)),y(\sigma_i(\upsilon)),z(\sigma_i(\upsilon)))\,d\upsilon\right)\right|\right)\\
&\leq \omega^L(A_i,\epsilon)+\omega^L_{\rho,H}(f_i,\epsilon)+\omega^L_\rho(h_i,\epsilon)\\
&\quad+\phi_i\left(\omega^L\left(x,\omega^L(\varepsilon_i,\epsilon)\right),\omega^L\left(y,\omega^L(\varepsilon_i,\epsilon)\right),\omega^L\left(z,\omega^L(\varepsilon_i,\epsilon)\right)\right)\\
&\quad+\varphi_i\left(\delta_i\left(\beta_i^L\omega^L_\rho(g_i,\epsilon)\right)^{\alpha_i}\right)+\varphi_i\left(\delta_i\left(H\omega^L(\beta_i,\epsilon)\right)^{\alpha_i}\right). \qquad (23)
\end{aligned}
$$

where

$$
\begin{aligned}
\omega^L(A_i,\epsilon)&=\sup\{|A_i(\tau_1)-A_i(\tau_2)|:\tau_1,\tau_2\in[0,L],|\tau_1-\tau_2|\leq\epsilon\},\\
\omega^L_\rho(h_i,\epsilon)&=\sup\left\{\begin{array}{c}\left|h_i(\tau_2,x,y,z)-h_i(\tau_1,x,y,z)\right|:\tau_1,\tau_2\in[0,L],\\ |\tau_1-\tau_2|\leq\epsilon,x,y,z\in[-\rho,\rho]\end{array}\right\},\\
\omega^L(\varepsilon_i,\epsilon)&=\sup\{|\varepsilon_i(\tau_1)-\varepsilon_i(\tau_2)|:\tau_1,\tau_2\in[0,L],|\tau_1-\tau_2|\leq\epsilon\},\\
\omega^L\left(x,\omega^L(\varepsilon_i,\epsilon)\right)&=\sup\left\{|x(\tau_1)-x(\tau_2)|:\tau_1,\tau_2\in[0,L],|\tau_1-\tau_2|\leq\omega^L(\varepsilon_i,\epsilon)\right\},\\
H&=\beta_i^L\sup\left\{\left|g_i(\tau,\upsilon,x,y,z)\right|:\tau\in[0,L],\upsilon\in\left[0,\beta_i^L\right],x,y,z\in[-\rho,\rho]\right\},\\
\omega^L_{\rho,H}(f_i,\epsilon)&=\sup\left\{\begin{array}{c}\left|f_i(\tau_2,x,y,z,p)-f_i(\tau_1,x,y,z,p)\right|:\tau_1,\tau_2\in[0,L],\\ |\tau_1-\tau_2|\leq\epsilon,x,y,z\in[-\rho,\rho],p\in[-\delta_iH^{\alpha_i},\delta_iH^{\alpha_i}]\end{array}\right\},\\
\omega^L_\rho(g_i,\epsilon)&=\sup\left\{\begin{array}{c}\left|g_i(\tau_1,\upsilon,x,y,z)-g_i(\tau_2,\upsilon,x,y,z)\right|:\tau_1,\tau_2\in[0,L],\\ |\tau_1-\tau_2|\leq\epsilon,\upsilon\in\left[0,\beta_i^L\right],x,y,z\in[-\rho,\rho]\end{array}\right\},\\
\omega^L(\beta_i,\epsilon)&=\sup\left\{\left|\beta_i(\tau_1)-\beta_i(\tau_2)\right|:\tau_1,\tau_2\in[0,L],|\tau_1-\tau_2|\leq\epsilon\right\}.
\end{aligned}
$$

Since $(x,y,z)$ was an arbitrary element of the set $\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3$ in relation (23), so we get

$$
\begin{aligned}
\omega^L(T_i(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3),\epsilon)&\leq\omega^L(A_i,\epsilon)+\omega^L_\rho(h_i,\epsilon)+\omega^L_{\rho,H}(f_i,\epsilon)\\
&\quad+\phi_i\left(\omega^L\left(\mathcal{X}_1,\omega^L(\varepsilon_i,\epsilon)\right),\omega^L\left(\mathcal{X}_2,\omega^L(\varepsilon_i,\epsilon)\right),\omega^L\left(\mathcal{X}_3,\omega^L(\varepsilon_i,\epsilon)\right)\right)\\
&\quad+\varphi_i\left(\delta_i\left(\beta_i^L\omega^L_\rho(g_i,\epsilon)\right)^{\alpha_i}\right)+\varphi_i\left(\delta_iH\omega^L(\beta_i,\epsilon)^{\alpha_i}\right). \qquad (24)
\end{aligned}
$$

Using continuity of $f_i,g_i,h_i$ on $[0,L]\times[-\rho,\rho]\times[-\rho,\rho]\times[-\rho,\rho]\times[-\delta_iH^{\alpha_i},\delta_iH^{\alpha_i}]$, $[0,L]\times\left[0,\beta_i^L\right]\times[-\rho,\rho]\times[-\rho,\rho]\times[-\rho,\rho]$, $[0,L]\times[-\rho,\rho]\times[-\rho,\rho]\times[-\rho,\rho]$, we have

$$
\begin{aligned}
\omega^L_{\rho,H}(f_i,\epsilon)&\to 0,\\
\omega^L_\rho(g_i,\epsilon)&\to 0,\\
\omega^L_\rho(h_i,\epsilon)&\to 0.
\end{aligned}
$$

Moreover, using continuity of $\varepsilon_i,\beta_i$ and $A_i$, we conclude that

$$\omega^L(\varepsilon_i,\epsilon)\to 0,\omega^L(\beta_i,\epsilon)\to 0,\omega^L(A_i,\epsilon)\to 0$$

as $\epsilon \to 0$, Therefore we obtain

$$\varphi_i\left(\delta_i\left(\beta_i^L\omega_\rho^L(g_i,\epsilon)\right)^{\alpha_i}\right)+\varphi_i\left(\delta_i\left(H\omega^L(\beta_i,\epsilon)\right)^{\alpha_i}\right)\to 0$$

as $\epsilon \to 0$. Now by letting $\epsilon \to 0$ in relation (24), we obtain

$$\omega_0^L\left(T_i\left(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3\right)\right)\leq\phi_i\left(\omega_0^L\left(\mathcal{X}_1\right),\omega_0^L\left(\mathcal{X}_2\right),\omega_0^L\left(\mathcal{X}_3\right)\right). \tag{25}$$

Also, by letting $L\to\infty$ in relation (25), we get

$$\omega_0\left(T_i\left(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3\right)\right)\leq\phi_i\left(\omega_0\left(\mathcal{X}_1\right),\omega_0\left(\mathcal{X}_2\right),\omega_0\left(\mathcal{X}_3\right)\right). \tag{26}$$

Furthermore, for every $(x,y,z),(u,v,w)\in\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3, t\in\mathbb{R}_+$, we get

$$\begin{aligned}
\left|\begin{array}{c}T_i(x,y,z)(\tau)-\\ T_i(u,v,w)(\tau)\end{array}\right| &\leq \left|\begin{array}{c}h_i\left(\tau,x\left(\varepsilon_i(\tau)\right),y\left(\varepsilon_i(\tau)\right),z\left(\varepsilon_i(\tau)\right)\right)\\ -h_i\left(\tau,u\left(\varepsilon_i(\tau)\right),y\left(\varepsilon_i(\tau)\right),z\left(\varepsilon_{i(\tau)}\right)\right)\end{array}\right|\\
&+\left|\begin{array}{c}f_i\left(\begin{array}{c}\tau,x\left(\varepsilon_i(\tau)\right),y\left(\varepsilon_i(\tau)\right),z\left(\varepsilon_i(\tau)\right),\\ \theta_i\left(\int_0^{\beta_i(\tau)}g_i\left(\tau,v,x\left(\sigma_i(v)\right),y\left(\sigma_i(v)\right),z\left(\sigma_i(v)\right)\right)dv\right)\end{array}\right)\\ -f_i\left(\begin{array}{c}\tau,u\left(\varepsilon_i(\tau)\right),v\left(\varepsilon_i(\tau)\right),w\left(\varepsilon_i(\tau)\right),\\ \theta_i\left(\int_0^{\beta_i(\tau)}g_i\left(\tau,v,u\left(\sigma_i(v)\right),v\left(\sigma_i(v)\right),w\left(\sigma_i(v)\right)\right)dv\right)\end{array}\right)\end{array}\right|\\
&\leq\frac{1}{2}\phi_i\left(\left|x\left(\varepsilon_i(\tau)\right)-u\left(\varepsilon_i(\tau)\right)\right|,\left|y\left(\varepsilon_i(\tau)\right)-v\left(\varepsilon_i(\tau)\right)\right|,\left|\begin{array}{c}z\left(\varepsilon_i(\tau)\right)\\ -w\left(\varepsilon_i(\tau)\right)\end{array}\right|\right)\\
&+\frac{1}{2}\phi_i\left(\left|x\left(\varepsilon_i(\tau)\right)-u\left(\varepsilon_i(\tau)\right)\right|,\left|y\left(\varepsilon_i(\tau)\right)-v\left(\varepsilon_i(\tau)\right)\right|,\left|\begin{array}{c}z\left(\varepsilon_i(\tau)\right)\\ -w\left(\varepsilon_i(\tau)\right)\end{array}\right|\right)\\
&+\varphi_i\left(\left|\begin{array}{c}\theta_i\left(\int_o^{\beta_i(\tau)}g_i\left(\tau,v,x\left(\sigma_i(v)\right),y\left(\sigma_i(v)\right),z\left(\sigma_i(v)\right)\right)dv\right)\\ -\theta_i\left(\int_0^{\beta_i(\tau)}g_i\left(\tau,v,u\left(\sigma_i(v)\right),v\left(\sigma_i(v)\right),w\left(\sigma_i(v)\right)\right)dv\right)\end{array}\right|\right)\\
&\leq\frac{1}{2}\phi_i\left(diam\mathcal{X}_i\left(\varepsilon_i(\tau)\right),diam\mathcal{X}_2\left(\varepsilon_i(\tau)\right),diam\mathcal{X}_3\left(\varepsilon_i(\tau)\right)\right)\\
&+\frac{1}{2}\phi_i\left(diam\mathcal{X}_i\left(\varepsilon_i(\tau)\right),diam\mathcal{X}_2\left(\varepsilon_i(\tau)\right),diam\mathcal{X}_3\left(\varepsilon_i(\tau)\right)\right)\\
&+\varphi_i\left(\delta_i\left|\int_0^{\beta_i(\tau)}\left(\begin{array}{c}g_i\left(\tau,v,x\left(\sigma_i(v)\right),y\left(\sigma_i(v)\right),z\left(\sigma_i(v)\right)\right)\\ -g_i\left(\tau,v,u\left(\sigma_i(v)\right),v\left(\sigma_i(v)\right),w\left(\sigma_i(v)\right)\right)\end{array}\right)dv\right|^{\alpha_i}\right)\\
&\leq\phi_i\left(diam\mathcal{X}_1\left(\varepsilon_i(\tau)\right),diam\mathcal{X}_2\left(\varepsilon_i(\tau)\right),diam\mathcal{X}_3\left(\varepsilon_i(\tau)\right)\right)\\
&+\varphi_i\left(\delta_i\left|\int_0^{\beta_i(\tau)}\left(\begin{array}{c}g_i\left(\tau,v,x\left(\sigma_i(v)\right),y\left(\sigma_i(v)\right),z\left(\sigma_i(v)\right)\right)\\ -g_i\left(\tau,v,u\left(\sigma_i(v)\right),v\left(\sigma_i(v)\right),w\left(\sigma_i(v)\right)\right)\end{array}\right)dv\right|^{\alpha_i}\right).
\end{aligned} \tag{27}$$

Because $(x,y,z)$ and $(u,v,w)$ and $\tau$, were chosen arbitrary in (27), we will have

$$\begin{aligned}
&diamT_i\left(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3\right)(\tau)\\
\leq\ &\phi_i\left(diam\mathcal{X}_1\left(\varepsilon_i(\tau)\right),diam\mathcal{X}_2\left(\varepsilon_i(\tau)\right),diam\mathcal{X}_3\left(\varepsilon_i(\tau)\right)\right)\\
&+\varphi_i\left(\delta_i\left|\int_0^{\beta_i(\tau)}\left(\begin{array}{c}g_i\left(\tau,v,x\left(\sigma_i(v)\right),y\left(\sigma_i(v)\right),z\left(\sigma_i(v)\right)\right)\\ -g_i\left(\tau,v,u\left(\sigma_i(v)\right),v\left(\sigma_i(v)\right),w\left(\sigma_i(v)\right)\right)\end{array}\right)dv\right|^{\alpha}\right).
\end{aligned} \tag{28}$$

By taking $\tau\to\infty$ in relation (28), then using (14) we obtain

$$\limsup_{\tau\to\infty} diamT_i\left(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3\right)(\tau)\leq\phi_i\left(\begin{array}{c}\limsup_{\tau\to\infty}diam\mathcal{X}_1\left(\varepsilon_i(\tau)\right),\\ \limsup_{\tau\to\infty}diam\mathcal{X}_2\left(\varepsilon_i(\tau)\right),\\ \limsup_{\tau\to\infty}diam\mathcal{X}_3\left(\varepsilon_i(\tau)\right)\end{array}\right). \tag{29}$$

From relation (26) together with relation (29), we obtain

$$\begin{aligned}
&\omega_0\left(T_i\left(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3\right)\right)+\limsup_{\tau\to\infty} diam T_i\left(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3\right)(\tau)\\
&\le \phi_i\left(\omega_o\left(\mathcal{X}_1\right),\omega_o\left(\mathcal{X}_2\right),\omega_o\left(\mathcal{X}_3\right)\right)\\
&+\phi_i\begin{pmatrix}\limsup_{\tau\to\infty} diam\mathcal{X}_1\left(\varepsilon_i(\tau)\right),\\ \limsup_{\tau\to\infty} diam\mathcal{X}_2\left(\varepsilon_i(\tau)\right),\\ \limsup_{\tau\to\infty} diam\mathcal{X}_3\left(\varepsilon_i(\tau)\right)\end{pmatrix}\\
&\le 3\phi_i\begin{pmatrix}\dfrac{\omega_0\left(\mathcal{X}_1\right)+\limsup_{\tau\to\infty} diam\mathcal{X}_1\left(\varepsilon_i(\tau)\right)}{3},\\ \dfrac{\omega_0\left(\mathcal{X}_2\right)+\limsup_{\tau\to\infty} diam\mathcal{X}_2\left(\varepsilon_i(\tau)\right)}{3},\\ \dfrac{\omega_0\left(\mathcal{X}_3\right)+\limsup_{\tau\to\infty} diam\mathcal{X}_3\left(\varepsilon_i(\tau)\right)}{3}\end{pmatrix}.
\end{aligned}\tag{30}$$

So, from relation (30), we conclude that

$$\frac{1}{3}\mu\left(T_i\left(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3\right)\right)\le\phi_i\left(\frac{\mu\left(\mathcal{X}_1\right)}{3},\frac{\mu\left(\mathcal{X}_2\right)}{3},\frac{\mu\left(\mathcal{X}_3\right)}{3}\right),$$

and by taking $\overline{\mu}=\frac{1}{3}\mu$, we get

$$\overline{\mu}\left(T_i\left(\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3\right)\right)\le\phi_i\left(\overline{\mu}\left(\mathcal{X}_1\right),\overline{\mu}\left(\mathcal{X}_2\right),\overline{\mu}\left(\mathcal{X}_3\right)\right),$$

Thus, by applying Corollary 2.9, the proof is complete. □

Finally, we present the following example and we investigate the conditions of Theorem 3.1 for existence of a solution.

**Example 3.2.** *Let us consider the following system of integral equations*

$$\left\{\begin{aligned}
x(\tau)&=\tfrac{1}{5}e^{-\tau^2}+\tfrac{1}{8(1+\tau^2)}\left(\cos x\left(\sqrt{\tau}\right)+\ln\left(1+\left|y\left(\sqrt{\tau}\right)\right|\right)+\sin z\left(\sqrt{\tau}\right)\right)+\tfrac{1}{7}e^{-\tau^2}\\
&\quad+\tfrac{1}{8(1+\tau^2)}\left(x\left(\sqrt{\tau}\right)+y\left(\sqrt{\tau}\right)+z\left(\sqrt{\tau}\right)\right)\\
&\quad+\arctan\left(\int_0^{\sqrt{\tau}}\left(\tfrac{v}{e^{\tau}}\right)\left(\tfrac{x(v^2)\left|\sin y(v^2)\right|\left|\cos z(v^2)\right|}{(1+x^2(v^2))\left(1+\sin^2 y(v^2)\right)(1+\cos^2 z(v^2))}\right)dv\right),\\
y(\tau)&=\tfrac{\tau^2}{5(1+\tau^2)}+\tfrac{\tau^2}{8\left(1+\tau^4\right)}\left(\cos x(\tau)+\ln\left(1+\left|y(\tau)\right|\right)+\sin z(\tau)\right)+\tfrac{1}{7}e^{-\tau^2}\\
&\quad+\tfrac{\tau^2}{8\left(1+\tau^4\right)}\left(x(\tau)+y(\tau)+z(\tau)\right)\\
&\quad+\sin\left(\int_0^{\tau}\left(\tfrac{v}{e^{\tau^2}}\right)\left(\tfrac{y^2(v)\left(1+\cos^2 x(v)\right)\left(1+\sin^2 z(v)\right)}{\left(1+y^2(v)\right)\left(1+\sin^2 x(v)\right)\left(1+\cos^2 z(v)\right)}\right)dv\right),\\
z(\tau)&=\tfrac{1}{5\sqrt{1+\tau^2}}+\tfrac{\tau^2}{8(1+\tau^3)}\left(\cos x(\tau)+\ln\left(1+\left|y(\tau)\right|\right)+\sin z(\tau)\right)+\tfrac{1}{7}e^{-\tau^2}\\
&\quad+\tfrac{\tau^2}{8(1+\tau^3)}\left(x(\tau)+y(\tau)+z(\tau)\right)\\
&\quad+\ln\left(1+\int_0^{\tau^2}\left(\tfrac{\sqrt{v}}{e^{\tau^3}}\right)\left(\tfrac{x^2\left|\cos y(v)\right|+y^2\left|\cos z(v)\right|+z^2\left|\cos x(v)\right|}{(1+x^2(v^2))\left(1+y^2(v^2)\right)(1+z^2(v^2))}\right)dv\right)
\end{aligned}\right. .\tag{31}$$

Here

$$\begin{aligned}
h_1(\tau,x,y,z) &= \frac{1}{8(1+\tau^2)}\left(\cos x+\ln\left(1+|y|\right)+\sin z\right),\\
h_2(\tau,x,y,z) &= \frac{\tau^2}{8(1+\tau^4)}\left(\cos x+\ln\left(1+|y|\right)+\sin z\right),\\
h_3(\tau,x,y,z) &= \frac{\tau^2}{8(1+\tau^3)}\left(\cos x+\ln\left(1+|y|\right)+\sin z\right),\\
f_1(\tau,x,y,z,m) &= \frac{1}{7}e^{-\tau^2}+\frac{1}{8(1+\tau^2)}(x+y+z)+\frac{m}{2},\\
f_2(\tau,x,y,z,m) &= \frac{1}{7}e^{-\tau^2}+\frac{\tau^2}{8(1+\tau^4)}(x+y+z)+\frac{m}{2},\\
f_3(\tau,x,y,z,m) &= \frac{1}{7}e^{-\tau^2}+\frac{\tau^2}{8(1+\tau^3)}(x+y+z)+\frac{m}{2},\\
g_1(\tau,v,x,y,z) &= \left(\frac{v}{e^{\tau}}\right)\frac{x\left|\sin y\right|\left|\cos z\right|}{(1+x^2)\left(1+\sin^2 y\right)(1+\cos^2 z)},\\
g_2(\tau,v,x,y,z) &= \left(\frac{v}{e^{\tau^2}}\right)\frac{y^2\left(1+\cos^2 x\right)\left(1+\sin^2 z\right)}{(1+y^2)\left(1+\sin^2 x\right)(1+\cos^2 z)},\\
g_3(\tau,v,x,y,z) &= \left(\frac{\sqrt{v}}{e^{\tau^3}}\right)\frac{x^2\left|\cos y\right|+y^2\left|\cos z\right|+z^2\left|\cos x\right|}{(1+x^2)(1+y^2)(1+z^2)},
\end{aligned}$$

and

$$\begin{aligned}
&A_1(\tau)=\frac{1}{5}e^{-\tau^2}, A_2(\tau)=\frac{\tau^2}{5(1+\tau^2)}, A_3(\tau)=\frac{1}{5\sqrt{1+\tau^2}}, \varepsilon_1(\tau)=\sqrt{\tau}, \varepsilon_2(\tau)=\tau,\\
&\varepsilon_3(\tau)=\tau, \sigma_1(\tau)=\tau^2, \sigma_2(\tau)=\tau, \sigma_3(\tau)=\tau, \beta_1(\tau)=\sqrt{\tau}, \beta_2(\tau)=\tau, \beta_3(\tau)=\tau^2,\\
&\theta_1(\tau)=\arctan\tau,\ \theta_2(\tau)=\sin\tau, \theta_3(\tau)=\ln(1+\tau),\\
&\phi_1(\tau,v,u)=\frac{1}{4}(\tau+v+u), \phi_2(\tau,v,u)=\frac{1}{4}(\tau+v+u), \phi_3(\tau,v,u)=\frac{1}{4}(\tau+v+u),\\
&\varphi_1(\tau)=\frac{\tau}{2}, \varphi_2(\tau)=\frac{\tau}{2}, \varphi_3(\tau)=\frac{\tau}{2}.
\end{aligned}$$

Clearly conditions $(I)$ and $(II)$ and $(III)$ are valid. Obviously we have, $M_i=\frac{1}{5}, \delta_i=1$ and $\alpha_i=1, i=1,2,3$.

Clearly, $\left|f_i(\tau,0,0,0,0)\right|=\frac{1}{7}e^{-\tau^2}, i=1,2,3,$ are bounded and $M_i^{'}=\frac{1}{7}$. Also $h_i(\tau,0,0,0), i=1,2,3,$ are bounded and $M_i^{''}=\frac{1}{8}$. Therefore, the condition $(IV)$ is valid.

Obviously, $f_i$ and $h_i$, $i=1,2,3$, are continuous. Let $\tau\in\mathbb{R}_+$, then we get

$$\begin{aligned}
\left|f_1(\tau,x,y,z,m)-f_1(\tau,u,v,w,n)\right| &= \left|\begin{array}{c}\frac{1}{8(1+\tau^2)}(x+y+z)+\frac{m}{2}\\ -\left(\frac{1}{8(1+\tau^2)}(u+v+w)+\frac{n}{2}\right)\end{array}\right|\\
&\le \frac{1}{8(1+\tau^2)}\left(|x-u|+\left|y-v\right|+|z-w|\right)+\frac{1}{2}|m-n|\\
&\le \frac{1}{8}\left(|x-u|+\left|y-v\right|+|z-w|\right)+\frac{1}{2}|m-n|\\
&= \frac{1}{2}\times\frac{1}{4}\left(|x-u|+\left|y-v\right|+|z-w|\right)+\frac{1}{2}|m-n|\\
&= \frac{1}{2}\phi_1\left(|x-u|,\left|y-v\right|,|z-w|\right)+\varphi_1(|m-n|). \qquad (32)
\end{aligned}$$

Similarly, we obtain the following two relations:

$$\left|f_2(\tau,x,y,z,m)-f_2(\tau,u,v,w,n)\right| \leq \frac{1}{2}\phi_2\left(|x-u|,\left|y-v\right|,|z-w|\right)+\varphi_2(|m-n|),$$
$$\left|f_3(\tau,x,y,z,m)-f_3(\tau,u,v,w,n)\right| \leq \frac{1}{2}\phi_3\left(|x-u|,\left|y-v\right|,|z-w|\right)+\varphi_3(|m-n|).$$

If $\tau\in\mathbb{R}_+$ and $x,y,z,u,v,w\in\mathbb{R}$ with $\left|y\right|\geq|v|$, then we get

$$\begin{aligned}
\left|h_1(\tau,x,y,z)-h_1(\tau,u,v,w)\right| &\leq \frac{1}{8(1+\tau^2)}|\cos x-\cos u| \\
&+\frac{1}{8(1+\tau^2)}\left|\ln\left(1+\left|y\right|\right)-\ln(1+|v|)\right| \\
&+\frac{1}{8(1+\tau^2)}|\sin z-\sin w| \\
&\leq \frac{1}{8}|x-u|+\frac{1}{8}\left|\ln\frac{\left(1+\left|y\right|\right)}{(1+|v|)}\right|+\frac{1}{8}|z-w| \\
&\leq \frac{1}{8}|x-u|+\frac{1}{8}\ln\left(1+\left|y-v\right|\right)+\frac{1}{8}|z-w| \\
&\leq \frac{1}{8}\left(|x-u|+\left|y-v\right|)+|z-w|\right) \\
&= \frac{1}{2}\times\frac{1}{4}\left(|x-u|+\left|y-v\right|+|z-w|\right) \\
&= \frac{1}{2}\phi_1\left(|x-u|,\left|y-v\right|,|z-w|\right).
\end{aligned}$$

Similarly, we obtain the following two relations:

$$\left|h_2(\tau,x,y,z)-h_2(\tau,u,v,w)\right| \leq \frac{1}{2}\phi_2\left(|x-u|,\left|y-v\right|,|z-w|\right),$$
$$\left|h_3(\tau,x,y,z)-h_3(\tau,u,v,w)\right| \leq \frac{1}{2}\phi_3\left(|x-u|,\left|y-v\right|,|z-w|\right).$$

Therefore, the condition $(V)$ is valid.
Clearly, $g_i, i=1,2,3$, are continuous. For every $\tau,\upsilon\in\mathbb{R}_+$ and $x,y,z\in\mathbb{R}$, by easy calculations we get

$$\left|g_1(\tau,\upsilon,x,y,z)-g_1(\tau,\upsilon,u,v,w)\right|\leq\tfrac{2\upsilon}{e^{\tau}},$$

$$\left|g_2(\tau,\upsilon,x,y,z)-g_2(\tau,\upsilon,u,v,w)\right|\leq\tfrac{8\upsilon}{e^{\tau^2}},$$

$$\left|g_3(\tau,\upsilon,x,y,z)-g_3(\tau,\upsilon,u,v,w)\right|\leq\tfrac{6\sqrt{\upsilon}}{e^{\tau^3}},$$

Hence,

$$\lim_{\tau\to\infty}\int_0^{\beta_1(\tau)}\left|\begin{array}{c} g_1(\tau,\upsilon,x(\sigma_1(\upsilon)),y(\sigma_1(\upsilon)),z(\sigma_1(\upsilon))) \\ -g_1(\tau,\upsilon,u(\sigma_1(\upsilon)),v(\sigma_1(\upsilon)),w(\sigma_1(\upsilon)))\end{array}\right|d\upsilon\leq\lim_{\tau\to\infty}\int_0^{\sqrt{\tau}}\tfrac{2\upsilon}{e^{\tau}}d\upsilon=\lim_{\tau\to\infty}\tfrac{\tau}{e^{\tau}}=0,$$

$$\lim_{\tau\to\infty}\int_0^{\beta_2(\tau)}\left|\begin{array}{c} g_2(\tau,\upsilon,x(\sigma_2(\upsilon)),y(\sigma_2(\upsilon)),z(\sigma_2(\upsilon))) \\ -g_2(\tau,\upsilon,u(\sigma_2(\upsilon)),v(\sigma_2(\upsilon)),w(\sigma_2(\upsilon)))\end{array}\right|d\upsilon\leq\lim_{\tau\to\infty}\int_0^{\tau}\tfrac{8\upsilon}{e^{\tau^2}}d\upsilon=\lim_{\tau\to\infty}\tfrac{4\tau^2}{e^{\tau^2}}=0,$$

$$\lim_{\tau\to\infty}\int_0^{\beta_3(\tau)}\left|\begin{array}{c} g_3(\tau,\upsilon,x(\sigma_3(\upsilon)),y(\sigma_3(\upsilon)),z(\sigma_3(\upsilon))) \\ -g_3(\tau,\upsilon,u(\sigma_3(\upsilon)),v(\sigma_3(\upsilon)),w(\sigma_3(\upsilon)))\end{array}\right|d\upsilon\leq\lim_{\tau\to\infty}\int_0^{\tau^2}\tfrac{6\sqrt{\upsilon}}{e^{\tau^3}}d\upsilon=\lim_{\tau\to\infty}\tfrac{4\tau^3}{e^{\tau^3}}=0,$$

Also, we get

$$\left|\int_0^{\beta_1(\tau)} g_1\left(\tau,\upsilon,x\left(\sigma_1\left(\upsilon\right)\right),y\left(\sigma_1\left(\upsilon\right)\right),z\left(\sigma_1\left(\upsilon\right)\right)\right)d\upsilon\right| \le \int_0^{\sqrt{\tau}} \frac{\upsilon}{e^{\tau}}d\upsilon = \frac{\tau}{2e^{\tau}},$$

$$\left|\int_0^{\beta_2(\tau)} g_2\left(\tau,\upsilon,x\left(\sigma_2\left(\upsilon\right)\right),y\left(\sigma_2\left(\upsilon\right)\right),z\left(\sigma_2\left(\upsilon\right)\right)\right)d\upsilon\right| \le \int_0^{\tau} \frac{4\upsilon}{e^{\tau^2}}d\upsilon = \frac{2\tau^2}{e^{\tau^2}},$$

$$\left|\int_0^{\beta_3(\tau)} g_3\left(\tau,\upsilon,x\left(\sigma_3\left(\upsilon\right)\right),y\left(\sigma_3\left(\upsilon\right)\right),z\left(\sigma_3\left(\upsilon\right)\right)\right)d\upsilon\right| \le \int_0^{\tau^2} \frac{3\sqrt{\upsilon}}{e^{\tau^3}}d\upsilon = \frac{2\tau^3}{e^{\tau^3}}.$$

Hence

$$\begin{aligned} M_1''' &= \sup\left\{\frac{\tau}{2e^{\tau}} : \mathbb{R}_+\right\} = \frac{1}{2e}, \\ M_2''' &= \sup\left\{\frac{\tau^2}{2e^{\tau^2}} : \mathbb{R}_+\right\} = \frac{2}{e}, \\ M_2''' &= \sup\left\{\frac{2\tau^3}{e^{\tau^3}} : \mathbb{R}_+\right\} = \frac{2}{e}. \end{aligned} \tag{33}$$

Therefore, the condition $(VI)$ is valid.

Now from (33) along with $M_i = \frac{1}{5}, M_i' = \frac{1}{7}, M_i'' = \frac{1}{8}$ and $\delta_i = 1, (i = 1,2,3)$ in (16), we get

$$\begin{aligned} \frac{1}{5}+\frac{1}{7}+\frac{1}{8}+\frac{1}{4e} &< \frac{\kappa}{4}, \\ \frac{1}{5}+\frac{1}{7}+\frac{1}{8}+\frac{1}{e} &< \frac{\kappa}{4}. \end{aligned}$$

Hence, the condition $(VII)$ is valid for each $\kappa > \frac{131}{70} + \frac{4}{e}$ .

Thus, all the assumptions from $(I) - (VII)$ are satisfied. Hence by Theorem 3.1 we conclude that the system (11) has a solution in $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

## References


[1] R. P. Agarwal, N. Hussain and M. A. Taoudi, Fixed Point Theorems in Ordered Banach Spaces and Applications to Nonlinear Integral Equations, Abstr. Appl. Anal., Volume 2012, Article ID 245872, 15 pages.

[2] A. Aghajani, R. Allahyari, M. Mursaleen, A generalization of Darbo's theorem with application to the solvability of systems of integral equations, J. Comput. Appl. Math., 260 (2014) 68-77.

[3] A. Aghajani, J. Banas, N. Sabzali, Some generalizations of Darbo's fixed point theorem with applications, Bull. Belg. Math. Soc. Simon Stevin., 20 (2) (2013) 345-358.

[4] A. Aghajani, N. Sabzali, Existence of coupled fixed points via measure of non-compactness and applications, J. Nonlinear Convex Anal., Vol.14, No.5 (2014) 941-952.

[5] A. Aghajani, A. Shole Haghighi, Existence of solutions for a class of functional integral equations of Volterra type in two variables via measure of non-compactness, IJST, 2014, 38A1 : 1-8.

[6] R. R. Akmerov, M. I. Kamenski, A. S. Potapov, A. E. Rodkina, B. N. Sadovskii, Measure of non-compactness and condensing operators, Birkhauser-Verlag, Basel, 1992.

[7] R. Allahyari, R. Arab, and A. Shole Haghighi, Existence of solutions for some classes of integro-differential equations via measure of non-compactness, Electron. J. Qual. Theo., No.41 (2015), 1-18.

[8] R. Arab, Some fixed point theorems in generalized Darbo's fixed point theorem and the existence of solutions for system of integral equations, J. Korean Math. Soc., 52 (1) (2015), 125-139.

[9] J. Banas, On measures of non-compactness in Banach spaces, Comment. Math. Univ. Carolin., 21 (1980) 131-143.

[10] J. Banas, Measures of non-compactness in the spaces of continuous tempered functions, Demonstration Math., 14 (1981) 127-133.

[11] J. Banas, K. Goebel, Measures of non-compactness in Banach space, Lecture Notes in Pure and Applied Mathematics, vol.60, Marcel Dekker, New York, 1980.

[12] V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mapping in partially ordered metric spaces, Nonlinear Anal., 65 (2006) 1379-1393.

[13] G. Darbo, Punti uniti in trasformazioni a codomio non compatto, Rend. Sem. Mat. Uni. Padova, 24 (1955) 84-92.

[14] J. G. Falset, K. Latrach, On Darbo-Sadovskii's fixed point theorems type for abstract measure of (weak) non-compactness, Bull. Belg. Math. Soc. Stevin 22 (2015) 761-779.

[15] A. Hajji, A generalization of Darbo's fixed point and common solutions of equations in Banach spaces, Fixed Point Theory Appl., 2013, 2013 : 62.
[16] V. Karakaya, M. Mursaleen, N. E. H. Bouzara, Measure of non-compactness in the study of solutions for a system of integral equations, arXiv : 1507.00185 v1 [math. FA] 1 jul 2015.
[17] K. Kuratowski, Sur les espaces complets, Fund. Math.,15 (1930) 301 − 309.
[18] M. Mursaleen, S. A. Mohiuddine, Applications of measure of non-compactness to the infinite system of differential equations in $l_p$ spaces, Nonlinear Anal., 75 (2012) 2111 − 2115.
[19] N. Hussain, V. Parvaneh, J.R. Roshan, Fixed point results for $G$-$\alpha$-contractive maps with application to boundary value problems, The Scientific World J., Vol. 2014, Article ID 585964.
[20] N. Hussain, A.R. Khan and Ravi P. Agarwal, Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces, J. Nonlinear Convex Anal., 11 (3), (2010), 475-489.
[21] V. Rakocevic, measure of noncompactness and some applications, Filomat (Nis) 12:2 (1998) , 87-120.
[22] J. R. Roshan, Existence of solutions for a class of system of functional integral equation via measure of non-compactness, J. Comput. Appl. Math., 313 (2017) 129-141.
[23] A. Samadi, M. B. Ghaemi, An extention of Darbo′s fixed point theorem and its applications to coupled fixed point and integral equations, Filomat 28 : 4 (2014) 879-886.