Generalized Inequalities of Warped Product Submanifolds of Nearly Kenmotsu f-Manifolds

Yavuz Selim Balkan[a], Ali H. Alkhaldi[b], Aliya Naaz Siddiqui[c], Akram Ali[b]

[a]Department of Mathematics, Faculty of Art and Sciences, Duzce University, 81620, Duzce Turkey
[b]Department of Mathematics, College of Science, King Khalid University, 61413 Abha, Saudi Arabia
[c]Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia, New Delhi- 110025, India

Abstract. In the present paper, we establish two general sharp inequalities for the squared norm of second fundamental form for mixed totally geodesic warped product pseudo-slant submanifolds of the form $M_\perp \times_f M_\theta$ and $M_\theta \times_f M_\perp$, in a nearly Kenmotsu f-manifold $\overline{M}$, which include the squared norm of the warping function and slant angle. Also, equality cases are verified. We proved that some previous results are trivial from our results. Moreover, we generalized the inequality theorems [3] and [26] from our derived results.

1. Introduction

The concept of warped product has been playing a crucial role in the theory of general relativity which provides the best mathematical model of the universe. The warped product model was successfully applied in general relativity and semi-Riemannian geometry in the direction to build basic cosmological models such as the Robertson-Walker spacetime, the Friedmann cosmological model and the standard static spacetime. The warped product, a natural generalization of Riemannian product, was introduced to construct the manifolds with negative curvature by Bishop and O'Neill in 1969 [8]. Bejancu [7] studied a special class *CR*-submanifolds of Kähler manifolds, and generalized class of holomorphic and totall real submanifolds. Later on, taking account of CR-submanifold, in [12] Chen initiated the concept of nontrvial *CR*-warped product submanifold in a Kähler manifold by considering the base is holomorphic and fiber is totally real submanifolds. He also established a general inequality for the second fundamental form in terms of warping functions for an arbitrary *CR*-warped product in an arbitrary Kähler manifold. After that many authors derived the geometric inequalities of warped product submanifolds in different ambient spaces ([1]-[3], [19], [20], [25]). Recently, Şahin [24] constructed a general inequality for warped product pseudo slant isometrically immersed in a Kähler manifold under the name of warped product hemi-slant submanifold.

On the other hand, Yano defined and studied the $(2n + s)$-dimensional globally framed metric f-manifold which is a natural generalization complex manifolds and contact manifolds [27]. In [17] Ishihara and Yano

investigate the integrability of the structures defined on these manifolds. Blair introduced three classes of globally framed metric f-manifold as K-manifolds, S-manifolds and C-manifolds [9]. It should be noted that class of Kenmotsu manifold is a remarkable class of almost contact manifolds with negative curvature -1 and realize as warped product which was defined in [18]. Moreover Falcitelli and Pastore defined almost Kenmotsu f-manifold which is a generalization of an almost Kenmotsu manifold [13]. Öztürk et. al generalized the almost C-manifolds and almost Kenmotsu f-manifold and they introduced almost α-cosymplectic f-manifolds [22]. Recently, Balkan studied a globally framed version of nearly Kenmotsu manifolds and obtained the fundamental properties of these type manifolds [5].

The main objective of this paper to consider nearly Kenmotsu f-manifolds and compute some geometric sharp inequalities of non-trivial warped product pseudo slant submanifolds. We prove the existence of the warped product pseudo slant submanifolds in nearly Kenmotsu f-manifolds. It well known that the warped product pseudo slant submanifolds are natural extensions of CR-warped product submanifold with some geometric condition.

2. Preliminaries

Let $\overline{M}$ be $(2n + s)$-dimensional manifold and φ is a non-null $(1, 1)$ tensor field on M. If φ satisfies

$$\varphi^3 + \varphi = 0, \tag{1}$$

then φ is called an f-structure and $\overline{M}$ is called an f-manifold [27]. If $rank\varphi = 2n$ or $rank\varphi = 2n + 1$, i.e., $s = 0$ or $s = 1$, then φ is called an almost complex structure or an almost contact structure, respectively [14]. On the other hand, $rank\varphi$ is always constant [23]. On an f-manifold $\overline{M}$, P_1 and P_2 operators are defined by

$$P_1 = -\varphi^2, \quad P_2 = \varphi^2 + I, \tag{2}$$

which satisfy

$$\begin{aligned} P_1 + P_2 &= I, & P_1^2 &= P_1, & P_2^2 &= P_2, \\ \varphi P_1 = P_1 \varphi &= \varphi, & P_2 \varphi &= \varphi P_2 = 0. \end{aligned} \tag{3}$$

These properties show that P_1 and P_2 are complement projection operators. There are D and $D^\perp$ distributions with respect to P_1 and P_2 operators, respectively [28]. Moreover $\dim(D) = 2m$ and $\dim(D^\perp) = s$. Let $\overline{M}$ be $(2m + s)$-dimensional f-manifold and φ is a $(1, 1)$ tensor field, ξ_i is vector field and η^i is 1-form for each $1 \le i \le s$ on M, respectively. If the following properties are satisfied

$$\eta^j(\xi_i) = \delta_i^j, \tag{4}$$

and

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \tag{5}$$

then $\left(\varphi, \xi_i, \eta^i\right)$ is called globally framed f-structure or simply framed f-structure and $\overline{M}$ is called globally framed f-manifold or simply framed f-manifold [21]. For a framed f-manifold $\overline{M}$, the following properties hold [21]:

$$\varphi \xi_i = 0, \tag{6}$$

$$\eta^i \circ \varphi = 0. \tag{7}$$

On a framed f-manifold $\overline{M}$ if there exists a Riemannian metric which satisfies

$$\eta^i(X) = g(X, \xi_i), \tag{8}$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X) \eta^i(Y),$$
(9)

for all vector fields X, Y on $\overline{M}$, then $\overline{M}$ is called a framed metric f-manifold [15]. On a framed metric f-manifold, the fundamental 2-form Φ is defined by

$$\Phi(X, Y) = g(X, \varphi Y),$$
(10)

for all vector fields X, Y on $\overline{M}$ [15]. For a framed metric f-manifold, if the following holds

$$N_\varphi + 2 \sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0,$$
(11)

then $\overline{M}$ is called normal framed metric f-manifold, where N_φ denotes the Nijenhuis torsion tensor of φ [16]. A globally framed metric f-manifold $\overline{M}$ is called Kenmotsu f-manifold if it satisfies

$$\left(\overline{\nabla}_X \varphi\right) Y = \sum_{k=1}^{s} \left\{ g(\varphi X, Y) \xi_k - \eta^k(Y) \varphi X \right\},$$
(12)

for all vector fields X, Y on $\overline{M}$ [22]. Furthermore, if a globally framed metric f-manifold $\overline{M}$ satisfies

$$\left(\overline{\nabla}_X \varphi\right) Y + \left(\overline{\nabla}_Y \varphi\right) X = - \sum_{k=1}^{s} \left\{ \eta^k(X) \varphi Y + \eta^k(Y) \varphi X \right\}$$
(13)

then it is called a nearly Kenmotsu f-manifold. It is easily seen that every Kenmotsu f-manifold is a nearly Kenmotsu f-manifold, but the converse is not true. When a nearly Kenmotsu f-manifold $\overline{M}$ is normal, it turns to a Kenmotsu f-manifold [5]. On a nearly Kenmotsu f-manifold $\overline{M}$, the following identity hold:

$$\overline{\nabla}_X \xi_i = -\varphi^2 X,$$
(14)

for any vector field X on $\overline{M}$ [5].

Remark 2.1. *From (13), it is clear that if $s = 0$ then $\overline{M}$ is become nearly Kaehler manifold [26]. If $s = 1$ then the manifold $\overline{M}$ is called nearly Kenmotsu manifold [3].*

Now we recall some basic facts of submanifold from [11]. Let M be a submanifold immersed in $\overline{M}$. We also denote by g the induced metric on M. Let TM be the Lie algebra of vector fields in M and $T^\perp M$ the set of all vector fields normal to M. Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and $\overline{M}$, respectively. Then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y)$$
(15)

and

$$\overline{\nabla}_X V = -A_V X + \nabla_X^\perp V$$
(16)

respectively, for any vector fields X, Y on $\overline{M}$ and any $V \in T^\perp M$. Here, $\nabla^\perp$ is normal connection in the normal bundle, h is second fundamental form of M and A_V is the Weingarten endomorphism associated with V. On the other hand, there is a relation between A_V and h such that

$$g(A_V X, Y) = g(h(X, Y), V).$$
(17)

The mean curvature vector H is defined by $H = \dfrac{1}{m} trace(h) = \dfrac{1}{m} \sum_{i=1}^{m} h(e_i, e_i)$, where m is the dimension of M and $\{e_1, \ldots, e_m\}$ is a local orthonormal frame of the tangent bundle TM of M. M is said to be minimal, totally geodesic and totally umbilical if H vanishes identically and $h = 0$,

$$h(X, Y) = g(X, Y) H, \tag{18}$$

respectively. Furthermore, the second fundamental form h satisfies

$$\left(\overline{\nabla}_X h\right)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{19}$$

3. Submanifolds of Globally Framed Metric f-manifolds

In this section, let us recall some basic properties of submanifolds of globally framed metric f-manifolds from [4].

Definition 3.1. *Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. For any vector field X on M, we can write*

$$\varphi X = TX + NX, \tag{20}$$

where TX and NX are respectively tangent and normal components of φX, respectively. Similarly, for each $V \in \Gamma(T^\perp M)$, we have

$$\varphi V = tV + nV. \tag{21}$$

Here, tV is tangent component and nV is normal component of φV.

Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. Then the following identities hold:

$$T^2 = -I + \sum_{k=1}^{s} \eta^k \otimes \xi_k - tN, \quad NT + nN = 0, \tag{22}$$

$$Tt + tn = 0, \quad Nt + n^2 = -I, \tag{23}$$

where I denotes the identity transformation. Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. Then, for any vector fields X, Y on M and $V \in \Gamma(T^\perp M)$, the following identities hold:

$$\left(\overline{\nabla}_X \varphi\right) Y = \overline{\nabla}_X \varphi Y - \varphi \overline{\nabla}_X Y \tag{24}$$

$$\left(\overline{\nabla}_X T\right) Y = \nabla_X TY - T \nabla_X Y, \tag{25}$$

$$\left(\overline{\nabla}_X N\right) Y = \nabla_X^\perp NY - N \nabla_X Y, \tag{26}$$

where ∇ is the Levi-Civita connection.

Definition 3.2. *Let M be a submanifold of a globally framed metric f-manifold $\overline{M}$ and let M be tangent to the structure vector fields ξ_k for each $1 \le k \le s$. For each nonzero vector X tangent to M at p, we denote by $0 \le \theta(X) \le \dfrac{\pi}{2}$, the angle between φX and $T_p M$, known as the Wirtinger angle of X. If the $\theta(X)$ is constant, that is, independent of the choice of $p \in M$ and $X \in T_p M - \{\xi_k\}$, for each $1 \le k \le s$, then M is said to be a slant submanifold and the constant angle θ is called slant angle of the slant submanifold*

Here, if $\theta = 0$, M is invariant submanifold and if $\theta = \dfrac{\pi}{2}$, then M is an anti-invariant submanifold. A slant submanifold is proper slant if it is neither invariant nor anti-invariant submanifold.

Definition 3.3. *Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. Then the TM (tangent bundle of M) can be decomposed as*

$$TM = \sum_{k=1}^{s} D_\theta \oplus \xi_k, \tag{27}$$

where for each $1 \le k \le s$ the ξ_k denotes the distributions spanned by the structure vector fields ξ_k and D_θ is complementary of distributions ξ_k in TM, known as the slant distribution on M.

Theorem 3.4. *Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. Then M is a slant submanifold if and only if there exists a constant $\mu \in [0, 1]$ such that*

$$T^2 = -\mu \left(I - \sum_{k=1}^{s} \eta^k \otimes \xi_k \right). \tag{28}$$

Moreover, if θ is the slant angle of M, then $\mu = \cos^2 \theta$.

Corollary 3.5. *Let M be a slant submanifold of a globally framed metric f-manifold $\overline{M}$ with slant angle θ. Then for any vector fields X, Y on M, we find*

$$g(TX, TY) = \cos^2 \theta \left\{ g(X, Y) - \sum_{k=1}^{s} \eta^k(X)\, \eta^k(Y) \right\} \tag{29}$$

and

$$g(NX, NY) = \sin^2 \theta \left\{ g(X, Y) - \sum_{k=1}^{s} \eta^k(X)\, \eta^k(Y) \right\}. \tag{30}$$

Definition 3.6. *Let M be a submanifold of a a globally framed metric f-manifold $\overline{M}$ We say that M is a pseudo-slant submanifold [10] if there exist two orthogonal distributions D_θ and $D^\perp$ such that*
    1) The TM tangent bundle of M admits the orthogonal direct decomposition $TM = D^\perp \oplus D_\theta$, where for each $1 \le k \le s$ $\xi_k \in \Gamma(D_\theta)$.
    2) The distribution $D^\perp$ is anti-invariant i. e., $\varphi(D^\perp) \subset (T^\perp M)$.
    3) The distribution D_θ is slant with angle $\theta \ne \dfrac{\pi}{2}$, that is, the angle between D_θ and $\varphi(D_\theta)$ is a constant.

A pseudo-slant submanifold of a globally framed metric f-manifold is called mixed totally geodesic if $h(X, Z) = 0$ for all $X \in \Gamma(D^\perp)$ and $Z \in \Gamma(D_\theta)$. Now let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space TM and e_r belongs to the orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of a normal bundle $T^\perp M$, then we define

$$h_{ij}^r = g\left(h\left(e_i, e_j\right), e_r\right) \quad \text{and} \quad \|h\|^2 = \sum_{i,\,j=1}^{n} g\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right). \tag{31}$$

On the other hand, for a differentiable function λ on M, we have

$$\|\nabla\lambda\|^2 = \sum_{i=1}^{n} (e_i(\lambda))^2, \tag{32}$$

where the gradient $grad\lambda$ is defined by $g(\nabla\lambda, X) = X\lambda$, for any vector field $X \in \Gamma(TM)$.

4. Warped Product Pseudo-Slant Submanifolds

In this section, we investigate some fundamental properties of warped product pseudo-slant submanifolds of a nearly Kenmotsu f-manifold. First, we recall the definition of warped product manifolds and provide the useful lemma from [8] which will be used in the proof of our main results.

Definition 4.1. *Let (M_1, g_{M_1}) and (M_2, g_{M_2}) be two Riemannian manifolds with Riemannian metrics g_{M_1} and g_{M_2}, respectively and f is a positive differentiable function on M_1. The warped product $M_1 \times_f M_2$ of M_1 and M_2 is the Riemannian manifold $(M_1 \times M_2, g)$, where*

$$g = g_{M_1} \times f^2 g_{M_2}.$$

More explicitly , if U is tangent to $M = M_1 \times_f M_2$ at (p, q), then

$$\|U\|^2 = \left\|d_{\pi_1} U\right\|^2 + f^2 \left\|d_{\pi_2} U\right\|^2$$

where for $i = 1, 2$, π_i are the canonical projections of $M_1 \times M_2$ on M_1 and M_2 respectively.

Lemma 4.2. *Let $M = M_1 \times_f M_2$ be a warped product manifold. Then we have*
    (i) $\nabla_X Y \in \Gamma(TM_1)$,
    (ii) $\nabla_Z X = \nabla_X Z = (X \ln f) Z$,
    (iii) $\nabla_Z W = \nabla_Z^\circ W - g(Z, W) \nabla \ln f$,
    for all X, Y on M_1 and Z, W on M_2.

where ∇ and ∇° denote the Levi-Civita connections on M_1 and M_2,
respectively. Moreover, $\nabla \ln f$, is the gradient of $\ln f$, which is defined by $g(\nabla \ln f, U) = U \ln f$. A warped product manifold $M = M_1 \times_f M_2$ is trivial if the warping function f is constant. If $M = M_1 \times_f M_2$ is a warped product manifold then M_1 is totally geodesic and M_2 is totally umbilical submanifold of M.

Motivated from the definition of pseudo-slant submanifolds, two types of warped product pseudo-slant submanifolds can be constructed. According to them, we have the following cases

$$\text{(i) } M_\theta \times_f M_\perp \quad \text{and} \quad \text{(ii) } M_\perp \times_f M_\theta, \tag{33}$$

where we consider the the structure vector fields $\xi'_k s$ are tangent to base manifold in both cases.

Let us consider the first case of warped product pseudo-slant submanifold of type $M_\theta \times_f M_\perp$ in a nearly Kenmotsu f−manifold.

Now, we prove some lemmas for the next section. We begin with the following.

Lemma 4.3. *Let $M = M_\theta \times_f M_\perp$ be a non-trivial warped product pseudo slant submanifold of a nearly Kenmotsu f-manifold $\overline{M}$. Then we have*

    (i) $g(h(Z, Z), NTX) = g(h(Z, TX), \varphi Z) + \left\{ s \sum_{k=1}^{s} \eta^k(X) - (X \ln f) \right\} \cos^2 \theta \|Z\|^2$,

    (ii) $g(h(Z, Z), NX) = g(h(Z, X), \varphi Z) - (TX \ln f) \|Z\|^2$,

    for any X on M_θ and Z on $M_\perp$, where the structure vector fields $\xi'_k s$ are tangent to M_θ.

Proof. By using (15) and (20), then we get

$$g(h(Z, Z), NTX) = g(\overline{\nabla}_Z Z, NTX) = g(\overline{\nabla}_Z Z, \varphi TX) - g(\overline{\nabla}_Z Z, T^2 X).$$

From (28), it follows

$$g\left(h\left(Z,\,Z\right),\,NTX\right) = -g\left(\varphi\overline{\nabla}_Z Z,\,TX\right) + \cos^2\theta\left\{g\left(\overline{\nabla}_Z Z,\,X\right) - \sum_{k=1}^{s}\eta^k\left(X\right)g\left(\overline{\nabla}_Z Z,\,\xi_k\right)\right\}.$$

By virtue of properties, we deduce

$$g\left(h\left(Z,\,Z\right),\,NTX\right) = g\left(\left(\overline{\nabla}_Z\varphi\right)Z,\,TX\right) - g\left(\overline{\nabla}_Z\varphi Z,\,TX\right) + \cos^2\theta g\left(\nabla_Z X,\,Z\right) + \cos^2\theta\sum_{k=1}^{s}\eta^k\left(X\right)g\left(\overline{\nabla}_Z\xi_k,\,Z\right).$$

Taking into account of $\sum_{k=1}^{s}\left(\xi_k\ln f\right) = s$ and using Lemma 4.2 (ii), we get (i). The second property of lemma can be easily gotten by interchanging X by TX in (i) of this lemma. This completes proof of lemma. $\square$

For the second case of warped product pseudo-slant submanifold of type $M_\perp \times_f M_\theta$ where structure vector fields ξ_s are tangent to base manifold. We derive some important lemmas which will be used in our main results.

Lemma 4.4. *Let* $M = M_\perp \times_f M_\theta$ *be a warped product pseudo slant submanifold of a nearly Kenmotsu f-manifold* $\overline{M}$ *such that the structure vector fields $\xi'_k s$ are tangent to $M_\perp$, for $1 \leq k \leq s$. Then we have*

$$g\left(h\left(X,\,TX\right),\,\varphi Z\right) = g\left(h\left(X,\,Z\right),\,NTX\right) + \frac{1}{3}\left\{\sum_{k=1}^{s}\eta^k\left(Z\right) - \left(Z\ln f\right)\right\}\cos^2\theta\left\|X\right\|^2,$$

for any vector field X on M_θ and Z on $M_\perp$.

Proof. Let $M = M_\perp \times_f M_\theta$ be a warped product pseudo slant submanifold of a nearly Kenmotsu f-manifold $\overline{M}$. By using (15), we find

$$g\left(h\left(X,\,TX\right),\,\varphi Z\right) = -g\left(\varphi\overline{\nabla}_X TX,\,Z\right).$$

By the virtue of the covariant derivative of φ, we get

$$g\left(h\left(X,\,TX\right),\,\varphi Z\right) = g\left(\left(\overline{\nabla}_X\varphi\right)TX,\,Z\right) - g\left(\overline{\nabla}_X\varphi TX,\,Z\right).$$

From (13) and Theorem 3.4, we obtain

$$g\left(h\left(X,\,TX\right),\,\varphi Z\right) = -g\left(\left(\overline{\nabla}_{TX}\varphi\right)X,\,Z\right) - \cos^2\theta g\left(\overline{\nabla}_X Z,\,X\right) + g\left(h\left(X,\,Z\right),\,NTX\right).$$

Hence by using Lemma 4.2 (ii) and the covariant derivative of φ, we derive

$$g\left(h\left(X,\,TX\right),\,\varphi Z\right) = -g\left(\overline{\nabla}_{TX}\varphi X,\,Z\right) - g\left(\overline{\nabla}_{TX}X,\,\varphi Z\right) + g\left(h\left(X,\,Z\right),\,NTX\right) - \cos^2\theta\left(Z\ln f\right)\left\|X\right\|^2.$$

Taking into account of (16) and (20), we deduce

$$2g\left(h\left(X,\,TX\right),\,\varphi Z\right) = g\left(\nabla_{TX}Z,\,TX\right) - g\left(\overline{\nabla}_{TX}NX,\,Z\right) + g\left(h\left(X,\,Z\right),\,NTX\right) - \cos^2\theta\left(Z\ln f\right)\left\|X\right\|^2.$$

Now, by using Lemma 4.2 (ii) and (29), it follows that

$$2g\left(h\left(X,\,TX\right),\,\varphi Z\right) = g\left(h\left(TX,\,Z\right),\,NX\right) + g\left(h\left(X,\,Z\right),\,NTX\right). \tag{34}$$

On the other hand, for any X on M_θ and Z on $M_\perp$, we conclude that

$$g\left(h\left(X,\,Z\right),\,NTX\right) = g\left(\overline{\nabla}_Z X,\,NTX\right).$$

Since ξ_k is tangent to $TM_\perp$ for each $1 \le k \le s$, (15) and (20) imply

$$g\left(h\left(X,\,Z\right),\,NTX\right) = -g\left(\varphi\overline{\nabla}_Z X,\,TX\right) + \cos^2 g\left(\nabla_Z X,\,X\right).$$

Again by using Lemma 4.2 (ii) and the covariant derivative of φ, it is obtained that

$$g\left(h\left(X,\,Z\right),\,NTX\right) = g\left(\left(\overline{\nabla}_Z\varphi\right)X,\,TX\right) - g\left(\overline{\nabla}_Z\varphi X,\,TX\right)$$
$$+ \cos^2\theta\left(Z\ln f\right)\|X\|^2.$$

By the virtue of (13) and (29), the last equation takes the form

$$g\left(h\left(X,\,Z\right),\,NTX\right) = -g\left(\left(\overline{\nabla}_X\varphi\right)Z,\,TX\right) - \sum_{k=1}^{s}\eta^k\left(Z\right)\cos^2\theta\|X\|^2 - g\left(\nabla_Z TX,\,TX\right) - g\left(\overline{\nabla}_Z NX,\,TX\right)$$
$$+ \cos^2\theta\left(X\ln f\right)\|Z\|^2.$$

By using Lemma 4.2 (ii), (16) and (29), it follows that

$$g\left(h\left(X,\,Z\right),\,NTX\right) = -g\left(\overline{\nabla}_X\varphi Z,\,TX\right) - g\left(\overline{\nabla}_X Z,\,\varphi TX\right) - \sum_{k=1}^{s}\eta^k\left(Z\right)\cos^2\theta\|X\|^2 + g\left(h\left(Z,\,TX\right),\,NX\right).$$

From Lemma 4.2 (ii) and (15), we have

$$g\left(h\left(X,\,TX\right),\,\varphi Z\right) = 2g\left(h\left(X,\,Z\right),\,NTX\right) - \left\{\left(Z\ln f\right) - \sum_{k=1}^{s}\eta^k\left(Z\right)\right\}\cos^2\theta\|X\|^2 - g\left(h\left(Z,\,TX\right),\,NX\right) \quad (35)$$

Hence from (34) and (35), we obtain

$$g\left(h\left(X,\,TX\right),\,\varphi Z\right) = g\left(h\left(X,\,Z\right),\,NTX\right) + \frac{1}{3}\left\{\sum_{k=1}^{s}\eta^k\left(Z\right) - \left(Z\ln f\right)\right\}\cos^2\theta\|X\|^2,$$

which gives us the desired result. $\quad\square$

As a consequence of this lemma, we can give the following corollary.

Corollary 4.5. *Let $M = M_\perp \times_f M_\theta$ be a totally geodesic warped product pseudo slant submanifold of a nearly Kenmotsu f-manifold $\overline{M}$. Then at least one of the following statements is true:*
(i) M is an anti-invariant submanifold,
(ii) M is an invariant submanifold or,
(iii) $\left(Z\ln f\right) = \sum_{k=1}^{s}\eta^k\left(Z\right).$

Lemma 4.6. *Let $M = M_\perp \times_f M_\theta$ be a warped product pseudo slant submanifold of a nearly Kenmotsu f-manifold $\overline{M}$. Then the followings hold.*
(i) $g\left(h\left(X,\,X\right),\,\varphi Z\right) = g\left(h\left(Z,\,X\right),\,NX\right),$
(ii) $g\left(h\left(TX,\,TX\right),\,\varphi Z\right) = g\left(h\left(Z,\,TX\right),\,NTX\right),$
for any X on M_θ and Z on $M_\perp$.

Proof. By using (15), we have

$$g\left(h\left(X,\,X\right),\,\varphi Z\right) = g\left(\overline{\nabla}_X X,\,\varphi Z\right) = -g\left(\varphi\overline{\nabla}_X X,\,Z\right).$$

From the properties, we obtain

$$g\left(h\left(X,\,X\right),\,\varphi Z\right) = g\left(\left(\overline{\nabla}_X\varphi\right)X,\,Z\right) - g\left(\overline{\nabla}_X\varphi X,\,Z\right).$$

By virtue of (9) and (13), then we derive

$$g\left(h\left(X,\,X\right),\,\varphi Z\right)=g\left(TX,\,\overline{\nabla}_{X}Z\right)-g\left(\overline{\nabla}_{X}NX,\,Z\right).$$

Taking into account of Lemma 4.2 (ii) and (15), it follows that

$$g\left(h\left(X,\,X\right),\,\varphi Z\right)=\left(Z\ln f\right)g\left(TX,\,X\right)+g\left(A_{NX}X,\,Z\right).$$

Since X and TX are orthogonal vector fields and in view of (17), we conclude that

$$g\left(h\left(X,\,X\right),\,\varphi Z\right)=g\left(h\left(X,\,Z\right),\,NX\right),\tag{36}$$

which gives us (i). By interchanging X by TX in (36), we have the last result of this lemma. $\quad\square$

5. Inequality for a Warped Product Pseudo Slant Submanifold of the form $M_{\perp}\times_{f}M_{\theta}$

In this section, we obtain a geometric inequality of warped product pseudo slant submanifold in terms of the second fundamental form such that ξ_{k} is tangent to the anti-invariant submanifold and the mixed totally geodesic submanifold for each $1\leq k\leq s$.

Now let $M=M_{\perp}\times_{f}M_{\theta}$ be m-dimensional warped product pseudo slant submanifold of $(2m+s)$-dimensional nearly Kenmotsu f-manifold $\overline{M}$ with M_{θ} of dimension $d_{1}=2\beta$ and $M_{\perp}$ of dimension $d_{2}=\alpha+s$, where M_{θ} and $M_{\perp}$ are the integral manifolds of D_{θ} and $D^{\perp}$, respectively. Then we consider $\{e_{1},\,\ldots e_{\alpha},\,e_{d_{2}=\alpha+1}=\xi_{1},\,\ldots,\,e_{d_{2}=\alpha+s}=\xi_{s}\}$ and $\{e_{\alpha+s+1}=e_{1}^{*},\,\ldots e_{\alpha+s+\beta+1}=e_{\beta}^{*},\,e_{\alpha+s+\beta+2}=e_{\beta+1}^{*}=\sec\theta Te_{1}^{*},\,\ldots,\,e_{\alpha+s+2\beta}=e_{2\beta}^{*}=\sec\theta Te_{2\beta}^{*}\}$ are orthonormal basis of $D^{\perp}$ and D_{θ}, respectively. Hence the orthonormal basis of the normal subbundles $\varphi D^{\perp}$, ND_{θ} and μ are $\{e_{q+1}=\overline{e}_{1}=\varphi e_{1},\,\ldots,\,e_{q+\alpha}=\overline{e}_{\alpha}=\varphi e_{\alpha}\}$, $\{e_{q+\alpha+1}=\overline{e}_{\alpha+1}=\widetilde{e}_{1}=\csc\theta Ne_{1}^{*},\,\ldots,\,e_{q+\alpha+\beta}=\overline{e}_{\alpha+\beta}=\widetilde{e}_{\beta}=\csc\theta Ne_{\beta}^{*},\,e_{q+\alpha+\beta+1}=\overline{e}_{\alpha+\beta+1}=\widetilde{e}_{\beta+1}=\csc\theta NTe_{1}^{*},\,\ldots e_{m+\alpha+2\beta}=\overline{e}_{\alpha+2\beta}=\widetilde{e}_{2\beta}=\csc\theta NTe_{\beta}^{*}\}\{e_{2m-1}=\overline{e}_{m},\,\ldots,\,e_{2n+s}=\overline{e}_{2(n-m+s)}\}$, respectively.

Theorem 5.1. *Let $M=M_{\perp}\times_{f}M_{\theta}$ be q-dimensional mixed totally geodesic warped product pseudo slant submanifold of a $(2m+s)$-dimensional nearly Kenmotsu f-manifold $\overline{M}$ such that $\xi_{k}\in\Gamma\left(TM_{\perp}\right)$, where $M_{\perp}$ is an anti-invariant submanifold of dimension d_{2}, and M_{θ} is a proper slant submanifold of dimension d_{1} of $\overline{M}$. Then*
(i) The squared norm of the second fundamental form of M is given by

$$\|h\|^{2}\geq\frac{2\beta}{9}\cos^{2}\theta\left\{\left\|\nabla^{\perp}\ln f\right\|^{2}-s^{2}\right\},\tag{37}$$

where $\nabla^{\perp}\ln f$ denotes the gradient of $\ln f$ with respect to M_{θ}.
(ii) If the equality case holds in (37), then $M_{\perp}$ is totally geodesic and M_{θ} is a totally umbilical submanifold into $\overline{M}$.

Proof. The squared norm of the second fundamental form is defined by

$$\|h\|^{2}=\|h\left(D_{\theta},\,D_{\theta}\right)\|^{2}+\left\|h\left(D^{\perp},\,D^{\perp}\right)\right\|^{2}+2\left\|h\left(D_{\theta},\,D^{\perp}\right)\right\|^{2}.$$

Since M is mixed totally geodesic we obtain

$$\|h\|^{2}=\|h\left(D_{\theta},\,D_{\theta}\right)\|^{2}+\left\|h\left(D^{\perp},\,D^{\perp}\right)\right\|^{2}.\tag{38}$$

From (31), we have

$$\|h\|^{2}\geq\sum_{l=q+1}^{2m+s}\sum_{i,\,j=1}^{2\beta}g\left(h\left(e_{i}^{*},\,e_{j}^{*}\right),\,e_{l}\right)^{2}.$$

By rewriting the last equation as in the components of $\varphi D^{\perp}$, ND_{θ} and v, then we get

$$\|h\|^2 \geq \sum_{l=1}^{\alpha} \sum_{i,j=1}^{2\beta} g\left(h\left(e_i^*, e_j^*\right), \bar{e}_l\right)^2 + \sum_{l=\alpha+1}^{2\beta+\alpha} \sum_{i,j=1}^{2\beta} g\left(h\left(e_i^*, e_j^*\right), \bar{e}_l\right)^2 + \sum_{l=q}^{2(m-q+s)} \sum_{i,j=1}^{2\beta} g\left(h\left(e_i^*, e_j^*\right), \bar{e}_l\right)^2, \tag{39}$$

which implies

$$\|h\|^2 \geq \sum_{l=1}^{\alpha} \sum_{i,j=1}^{2\beta} g\left(h\left(e_i^*, e_j^*\right), \bar{e}_l\right)^2.$$

By considering another adapted frame for D^{θ}, we derive

$$\|h\|^2 \geq \sum_{i=1}^{\alpha} \sum_{r,k=1}^{\beta} g\left(h\left(e_r^*, e_k^*\right), \bar{e}_i\right)^2 + \sec^2\theta \sum_{i=1}^{\alpha} \sum_{r,k=1}^{\beta} g\left(h\left(Te_r^*, e_k^*\right), \bar{e}_i\right)^2 + \sec^2\theta \sum_{i=1}^{\alpha} \sum_{r,k=1}^{\beta} g\left(h\left(e_r^*, Te_k^*\right), \bar{e}_i\right)^2$$

$$+ \sec^4\theta \sum_{i=1}^{p} \sum_{r,k=1}^{\beta} g\left(h\left(Te_r^*, Te_k^*\right), \bar{e}_i\right)^2.$$

For a mixed totally geodesic submanifold, since the first and last terms of the right hand side in the above equation vanish identically by using Lemma 4.6, then we obtain

$$\|h\|^2 \geq 2\sec^2\theta \sum_{i=1}^{\alpha} \sum_{r,k=1}^{\beta} g\left(h\left(Te_r^*, e_k^*\right), \bar{e}_i\right)^2.$$

Hence from Lemma 4.4, for a mixed totally geodesic submanifold and by considering the fact that $\eta^u(e_i) = 0$, for each $i = 1, 2, \ldots d_2 - 1$ and $1 \leq u \leq s$ for an orthonormal frame, it follows that

$$\|h\|^2 \geq \frac{2}{9}\cos^2\theta \sum_{i=1}^{\alpha} \sum_{r=1}^{\beta} (\bar{e}_i \ln f)^2 g(e_r^*, e_r^*)^2. \tag{40}$$

By adding and subtracting the same term $\sum_{u=1}^{s} \xi_u \ln f$ in (40), it implies that

$$\|h\|^2 \geq \frac{2}{9}\cos^2\theta \sum_{i=1}^{\alpha+1} \sum_{r=1}^{\beta} (\bar{e}_i \ln f)^2 g(e_r^*, e_r^*)^2 - \frac{2}{9}\cos^2\theta \sum_{r=1}^{\beta} \left(\sum_{u=1}^{s} \xi_u \ln f\right)^2 g(e_r^*, e_r^*)^2.$$

We can easily get $\sum_{u=1}^{s} \xi_u \ln f = s$ similar to previous studies for a warped product submanifold of a nearly Kenmotsu f-manifold. From the last equation, it is said that

$$\|h\|^2 \geq \frac{2\beta}{9}\cos^2\theta \left\{\left\|\nabla^{\perp} \ln f\right\|^2 - s^2\right\}.$$

If the equality case holds in the above equation, then from the terms left in (38), we arrive at

$$h\left(D^{\perp}, D^{\perp}\right) = 0,$$

which implies that $M_{\perp}$ is totally geodesic in $\overline{M}$. In a similar way, from the second and third terms in (39), we deduce

$$g\left(h\left(D_{\theta}, D_{\theta}\right), ND_{\theta}\right) = 0 \qquad g\left(h\left(D_{\theta}, D_{\theta}\right), v\right) = 0,$$

which means

$$h\left(D_{\theta}, D_{\theta}\right) \perp ND_{\theta}, \quad h\left(D_{\theta}, D_{\theta}\right) \perp v \implies h\left(D_{\theta}, D_{\theta}\right) \subset \varphi D^{\perp}.$$

This completes the proof. □

Remark 5.2. *For globally frame f−manifold. If we substitute $s = 1$ in Theorem 5.1. Then nearly Kenmotsu f-manifold become nearly Kenmotsu manifold. Then we have*

Theorem 5.3. *Let $M = M_\perp \times_f M_\theta$ be q-dimensional mixed totally geodesic warped product pseudo slant submanifold of a $(2m + 1)$-dimensional nearly Kenmotsu manifold $\overline{M}$ such that $\xi \in \Gamma(TM_\perp)$. Then*
   (i) *The squared norm of the second fundamental form of M is given by*

$$\|h\|^2 \geq \frac{2\beta}{9} \cos^2 \theta \left\{ \left\| \nabla^\perp \ln f \right\|^2 - 1 \right\}. \tag{41}$$

   (ii) *If the equality case holds in (41), then $M_\perp$ is totally geodesic and M_θ is a totally umbilical submanifold into $\overline{M}$.*

Therefore, above Theorem coincide with Theorem 4.1 in [3]. In other words, Theorem 5.1 generalized Theorem 4.1 from [3].

6. Inequality for a warped product pseudo slant submanifold of the form $M_\theta \times_f M_\perp$

In this part, we obtain a another inequality of warped product pseudo slant submanifold in terms of the second fundamental form such that ξ_k is tangent to the slant submanifold for each $1 \leq k \leq s$. By assuming ξ_k is tangent to D_θ, then we can use the last frame.

Theorem 6.1. *Assume that $M = M_\theta \times_f M_\perp$ be a q-dimensional mixed totally geodesic warped product pseudo slant submanifold of a $(2m + s)$-dimensional nearly Kenmotsu f-manifold $\overline{M}$ such that $\xi_k \in \Gamma(TM_\theta)$. Then*
   (i) *The squared norm of the second fundamental form of M is given by*

$$\|h\|^2 \geq \alpha \left\{ \cot^2 \theta \left(\left\| \nabla^\theta \ln f \right\|^2 - s^2 \right) \right\}. \tag{42}$$

where $M_\perp$ is an anti-invariant submanifold of dimension $d_2 = \alpha$ and M_θ is a proper slant submanifold of dimension $d_1 = 2\beta + s$ of $\overline{M}$.
   (ii) *If the equality case holds in (42), then M_θ is totally geodesic and $M_\perp$ is a totally umbilical submanifold of $\overline{M}$.*

Proof. By virtue of definition of second fundamental form, we have

$$\|h\|^2 = \|h(D_\theta, D_\theta)\|^2 + \left\| h(D^\perp, D^\perp) \right\|^2 + 2 \left\| h(D_\theta, D^\perp) \right\|^2.$$

Since M is mixed totally geodesic we derive

$$\|h\|^2 = \|h(D_\theta, D_\theta)\|^2 + \left\| h(D^\perp, D^\perp) \right\|^2. \tag{43}$$

By using (31), then we obtain

$$\|h\|^2 \geq \sum_{l=q+1}^{2m+s} \sum_{r,\,k=1}^{\alpha} g(h(e_r, e_k), e_l)^2.$$

The above equation can be written in the components of $\varphi D^\perp$, ND_θ and v as

$$\|h\|^2 \geq \sum_{l,\,r,\,k=1}^{\alpha} g(h(e_r, e_k), \bar{e}_l)^2 + \sum_{l=\alpha+1}^{2\beta+\alpha} \sum_{r,\,k=1}^{\alpha} g(h(e_r, e_k), \bar{e}_l)^2 + \sum_{l=q}^{2(m-q+s)} \sum_{r,\,k=1}^{\alpha} g(h(e_r, e_k), \bar{e}_l)^2, \tag{44}$$

which gives us

$$\|h\|^2 \geq \sum_{l=1}^{2\beta} \sum_{r,\,k=1}^{\alpha} g\left(h\left(e_r,\,e_k\right),\,\bar{e}_l\right)^2.$$

By taking into account of another adapted frame for ND^θ, we get

$$\|h\|^2 \geq \csc^2\theta \sum_{j=1}^{\beta}\sum_{r=1}^{\alpha} g\left(h\left(e_r,\,e_r\right),\,Ne_j^*\right)^2 + \csc^2\theta\sec^2\theta \sum_{j=1}^{\beta}\sum_{r=1}^{\alpha} g\left(h\left(e_r,\,e_r\right),\,NTe_j^*\right)^2.$$

Thus from Lemma 4.3, for a mixed totally geodesic submanifold and by considering the fact that $\eta^u\left(e_j\right) = 0$, for each $j = 1,\,2,\,\ldots d_1 - 1$ and $1 \leq u \leq s$ for an orthonormal frame, it implies that

$$\|h\|^2 \geq \csc^2\theta \sum_{j=1}^{\beta}\sum_{r=1}^{\alpha} \left(Te_j^*\ln f\right)^2 g\left(e_r,\,e_r\right) + \csc^2\theta\sec^2\theta \sum_{j=1}^{\beta}\sum_{r=1}^{\alpha} \left(e_j^*\ln f\right)^2 g\left(e_r,\,e_r\right).$$

From the hypothesis, we deduce

$$\begin{aligned}
\|h\|^2 &\geq \alpha\csc^2\theta \sum_{j=1}^{\beta}\left(Te_j^*\ln f\right)^2 + \alpha\cot^2\theta \sum_{j=1}^{\beta}\left(e_j^*\ln f\right)^2 \\
&= \alpha\csc^2\theta \sum_{r=1}^{2\beta+s} g\left(e_r^*,\,T\nabla^\theta\ln f\right)^2 - \alpha\csc^2\theta \sum_{r=\beta+1}^{2\beta} g\left(e_r^*,\,T\nabla^\theta\ln f\right)^2 \\
&\quad - \alpha\csc^2\theta \sum_{u=1}^{s}\left(Te_{2\beta+u}^*\ln f\right)^2 + \alpha\cot^2\theta \sum_{r=1}^{\beta}\left(e_r^*\ln f\right)^2.
\end{aligned}$$

As we seen that $Te_{2\beta+u}^* = T\xi_s = 0$. Then using (32), we derive at

$$\|h\|^2 \geq \alpha\csc^2\theta\|T\nabla^\theta\ln f\|^2 - \alpha\csc^2\theta \sum_{r=1}^{\beta} g\left(e_{\beta+r}^*,\,T\nabla^\theta\ln f\right)^2 + \alpha\cot^2\theta \sum_{r=1}^{\beta}\left(e_r^*\ln f\right)^2.$$

From virtue of (29) and the fact that $\sum_{k=1}^{s}\left(\xi_k\ln f\right) = s$, we find that

$$\|h\|^2 \geq \alpha\cot^2\theta\left(\|\nabla^\theta\ln f\|^2 - s^2\right) - \alpha\csc^2\theta \sum_{r=1}^{\beta} g\left(\sec\theta Te_\beta^*,\,T\nabla^\theta\ln f\right)^2 + \alpha\cot^2\theta \sum_{r=1}^{\beta}\left(e_r^*\ln f\right)^2.$$

In view of the trigonometric functions and from (29), we conclude

$$\|h\|^2 \geq \alpha\cot^2\theta\left(\|\nabla^\theta\ln f\|^2 - s^2\right) - \alpha\csc^2\theta\sec^2\theta\cos^4\theta \sum_{r=1}^{\beta}(e_r^*\ln f)^2 + \alpha\cot^2\theta \sum_{r=1}^{\beta}\left(e_r^*\ln f\right)^2,$$

which implies that

$$\|h\|^2 \geq \alpha\cot^2\theta\left(\|\nabla^\theta\ln f\|^2 - s^2\right) - \alpha\cot^2\theta \sum_{r=1}^{\beta}(e_r^*\ln f)^2 + \alpha\cot^2\theta \sum_{r=1}^{\beta}\left(e_r^*\ln f\right)^2.$$

Thus the above equation yields

$$\|h\|^2 \geq \alpha \left\{ \cot^2 \theta \left(\left\| \nabla^\theta \ln f \right\|^2 - s^2 \right) \right\}.$$

If the equality holds, by using the terms left hand side in (43) and (44), we get the following conditions

$$\|h(D, D)\| = 0, \quad g\left(h\left(D^\perp, D^\perp\right), \varphi D^\perp\right) = 0, \quad g\left(h\left(D^\perp, D^\perp\right), v\right) = 0,$$

where $D = D_\theta \oplus \overline{\xi}$, $\overline{\xi} = \sum_{u=1}^s \xi_u$. This implies that M_θ is totally geodesic in $\overline{M}$ and $h\left(D^\perp, D^\perp\right) \subseteq ND_\theta$. On the other hand, using Lemma 4 for a mixed totally geodesic submanifold we get

$$g\left(h\left(Z, W\right), NX\right) = -\left(TX \ln f\right) g\left(Z, W\right),$$

for all vector fields Z, W on $M_\perp$ and X on M_θ. The last equations means that $M_\perp$ is a totally umbilical submanifold of $\overline{M}$ and so the equality case holds. This completes the proof of theorem $\quad\square$

7. Conclusion Remarks

In this paper, we study warped product pseudo-slant submanifolds of nearly Kenmotsu f-manifolds. We generalize some previous results on nearly Kähler manifolds [26] and nearly Kenmotsu manifolds [3]. That is, if we consider $s = 0$ in Theorem 6.1, then by the definition of globally frame f-manifold. It leads that, a nearly Kenmotsu f-maniold turn into nearly Kaehler manifiold. Then we have from Theorem 6.1

Theorem 7.1. *Let $M = M_\theta \times_f M_\perp$ be an q-dimensional mixed totally geodesic warped product pseudo slant submanifold of an $2m$-dimensional nearly Keahler manifold $\overline{M}$. Then*
  (i) The squared norm of the second fundamental form of M is given by

$$\|h\|^2 \geq \alpha \cot^2 \theta \left\| \nabla^\theta \ln f \right\|^2, \tag{45}$$

where $M_\perp$ is an totally real submanifold of dimension $d_2 = \alpha$ and M_θ is a proper slant submanifold of dimension $d_1 = 2\beta$ of $\overline{M}$.
  (ii) If the equality case holds in (45), then M_θ is totally geodesic and $M_\perp$ is a totally umbilical submanifold of $\overline{M}$.

This means that the above Theorem same as Theorem 4.1 in [26] for warped product pseudo-slant submanifold of nearly Kaehler manifold. Hence, Theorem 6.1 is generalized Theorem 4.1 [26].

On the other hand, if we choose $s = 1$ then we give the following theorem as a consequence of Theorem 6.1 as follows

Theorem 7.2. *Let $M = M_\theta \times_f M_\perp$ be an q-dimensional mixed totally geodesic warped product pseudo slant submanifold of a $(2m + 1)$-dimensional nearly Kenmotsu manifold $\overline{M}$ such that $\xi \in \Gamma(TM_\theta)$, where $M_\perp$ is an anti-invariant submanifold of dimension $d_2 = \alpha$ and M_θ is a proper slant submanifold of dimension $d_1 = 2\beta + 1$ of $\overline{M}$. Then we have*
  (i) The squared norm of the second fundamental form of M is given by

$$\|h\|^2 \geq \alpha \left\{ \cot^2 \theta \left(\left\| \nabla^\theta \ln f \right\|^2 - 1 \right) \right\}. \tag{46}$$

  (ii) The equality case holds in (42), if M_θ is totally geodesic and $M_\perp$ is a totally umbilical submanifold of $\overline{M}$.

The above result agrees and modified version with the inequality for a warped product pseudo-slant submanifold of nearly Kenmotsu manifold obtained in [3].

Acknowledgements

References

[1] Ali A., Othman W. A. M., Özel C. and Hajjari T., A geometric inequality for warped product pseudo-slant submanifolds of nearly Sasakian manifolds, Comptes rendus de l'Academie bulgare des Sciences, 70 (2) (2017), 174-182.
[2] Ali A. and Laurian-Ioan P., Geometry of warped product immersions of Kenmotsu space forms and its applications to slant immersions, J. Geom. Phys.,114 (2017), 276-290.
[3] Ali A., Othman W. A. M. and Özel C., Some inequalities for warped product pseudo-slant submanifolds of nearly Kenmotsu manifolds, J. Ineq. Appl., (2015), 2015: 291.
[4] Balkan Y. S., Geometry of nearly C-manifolds (in Turkish), Ph. D. Thesis, Düzce University, 2016.
[5] Balkan Y. S., D-conformal transformations on nearly Kenmotsu f-manifolds, submitted (2018).
[6] Balkan, Y. S., Uddin, S., Stankovic, M. S. Alkhaldi, A. H., *A new class of f-structures satisfying $f^3 - f = 0$,* Filomat 32:17 (2018), 5919-5929.
[7] Bejancu A. *CR*-submanifolds of a Kähler manifold, Proc. Amer. Math. Soc., 69 (1) (1978) 135-142.
[8] Bishop R. L. and O'Neill B., Manifolds of negative curvature, Trans. Amer. Math. Soc., 145 (1969), 1-49.
[9] Blair D. E., Geometry of manifolds with structural group $U(n) \times O(s)$, J. Diff. Geom., 4 (2) (1970) 155-167.
[10] Carriazo A., New developments in slant submanifolds theory, Narosa Publishing House, New Delhi, 2002.
[11] Chen B. Y., Geometry of Submanifolds, M. Dekker, 1973.
[12] Chen B. Y., Geometry of warped product CR-submanifold in Kaehler manifolds, Monatshefte Mat., 133 (3) (2001), 177-195.
[13] Falcitelli M. and Pastore A. M., Almost Kenmotsu f-manifolds, Balkan J. Geom. Appl., 12 (1) (2007) 32-43.
[14] Goldberg S. I. and Yano K., Globally framed f-manifolds, Illinois J. Math., 15 (3) (1971) 456-474.
[15] Goldberg S. I. and Yano K., On normal globally framed f-manifolds, Tohoku Math. J., 22 (3) (1970) 362-370.
[16] Ishihara S., Normal structure f satisfying $f^3 + f = 0$, Kodai Math. Sem. Rep., 18 (1) (1966) 36-47.
[17] Ishihara S. and Yano K., On integrability conditions of a structure f satisfying $f^3 + f = 0$, Quart. J. Math. Oxford Sci., 15 (1) (1964) 217-222.
[18] Kenmotsu K.,A class of almost contact Riemannian manifolds, Tohoku Math. J. 24 (1) (1972) 93-103.
[19] Khan M. A., Uddin S. and Sachdeva R., Semi-invariant warped product submanifolds of cosymplectic manifolds, J. Inequal. Appl., (2012), Article ID 19 (2012).
[20] Mustafa A., Uddin S. and Wong W. R., Generalized inequalities on warped product submanifolds in nearly trans-Sasakian manifolds, J. Inequal. Appl., (2014), Article ID 346 (2014).
[21] Nakagawa H., On framed f-manifolds, Kodai Math. Sem. Rep., 18 (4) (1966) 293-306.
[22] Öztürk H., Murathan C., Aktan N. and Turgut Vanlı A., Almost α-cosymplectic f-manifolds, Annals of Alexandru Ioan Cuza University-Mathematics, 60 (1) (2014) 211-226.
[23] Stong R. E., The rank of f-srtucture, Kodai Math. Sem. Rep., 29 (1-2) (1977) 207-209.
[24] Şahin B., Warped product submanifoldsof a Kaehler manifold with slant factor, Ann. Pol. Math., 95 (2009) 207-226.
[25] Uddin S. and Khan V. A., An inequality for contact CR-warped product submanifolds of cosymplectic manifolds, J. Inequal. Appl., (2012), Article ID 304 (2012).
[26] Uddin S, Al-Solamy F. R. and Khan A. K., Geometry of warped product pseudo-slant submanifolds of nearly Kaehler manifolds, Annals of Alexandru Ioan Cuza University-Mathematics, LXII 2 (3) (2016) 927-938.
[27] Yano K., On a structure f satisfying $f^3 + f = 0$, Technical Report No. 12, University of Washington, Washington-USA, (1961).
[28] Yano K., On a structure defined by a tensor field of f of type (1, 1) satisfying $f^3 + f = 0$, Tensor, 14 (1963) 99-109.