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Convergence Theorems for Composite Viscosity Approaches to Systems Variational Inequalities in Banach Spaces

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Abstract. In this paper, we study a general system of variational inequalities with a hierarchical variational inequality constraint for an infinite family of nonexpansive mappings. We introduce general implicit and explicit iterative algorithms. We prove the strong convergence of the sequences generated by the proposed iterative algorithms to a solution of the studied problems.

1. Introduction

Let X be a Banach space with its dual X^*. Let $\emptyset \neq C \subset X$ be a closed convex set. Let $T : C \to C$ be a mapping. $\mathrm{Fix}(T)$ stands for the set of fixed points of T. Recall that T is nonexpansive if $\|Tu - Tv\| \leq \|u - v\|$, $\forall u, v \in C$. T is said to be a contraction if there exists a constant $\varrho \in [0, 1)$ such that $\|Tu - Tv\| \leq \varrho\|u - v\|$, $\forall u, v \in C$. The normalized dual mapping $J : X \to 2^{X^*}$ is defined as

$$J(x) := \{\phi \in X^* : \langle x, \phi\rangle = \|x\|^2 = \|\phi\|^2\}, \ \forall x \in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between X and X^*.

Let $A, B : C \to X$ be two mappings and ϖ, ς be two positive real numbers. The general system of variational inequalities (GSVI) is to find $(x^*, y^*) \in C \times C$ satisfying

$$\begin{cases} \langle \varpi A y^* + x^* - y^*, J(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \varsigma B x^* + y^* - x^*, J(x - y^*)\rangle \geq 0, & \forall x \in C. \end{cases} \tag{1}$$

The system of variational inequalities plays a crucial tool in science, engineering and economics. Namely, the practical problem can be formulated in the form of a system of variational inequalities; see e.g., [1, 2, 7–9, 19] and the references therein.

Note that problem (1) can be reduced to the following classical variational inequality (VI) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{2}$$

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This problem is a fundamental problem in the variational analysis; in particular, in the optimization theory and mechanics; see e.g., [10–13, 15, 17, 18, 23, 29–39] and the references therein. A large number of algorithms for solving this problem are essentially projection algorithms.

In a Banach space setting and $A = B$, $x^* = y^*$, the VI is defined by

$$\langle Ax^*, J(x - x^*)\rangle \geq 0,\ \forall x \in C. \tag{3}$$

Aoyama, Iiduka and Takahashi [3] proposed an iterative scheme to find the approximate solution of (3). Let $\{T_k\}_{k=1}^{\infty}$ be a sequence of nonexpansive self-mappings. Let $\{\varpi_k\}_{k=1}^{\infty}$ be a sequence in $[0, 1]$. Qin, Cho, Kang and Kang [21] considered the nonexpansive mapping W_k defined by

$$\begin{cases} U_{k,k+1} = I, \\ U_{k,k} = \varpi_k T_k U_{k,k+1} + (1 - \varpi_k)I, \\ U_{k,k-1} = \varpi_{k-1} T_{k-1} U_{k,k} + (1 - \varpi_{k-1})I, \\ \vdots \\ U_{k,i} = \varpi_i T_i U_{k,i+1} + (1 - \varpi_i)I, \\ U_{k,i-1} = \varpi_{i-1} T_{i-1} U_{k,i} + (1 - \varpi_{i-1})I, \\ \vdots \\ U_{k,2} = \varpi_2 T_2 U_{k,3} + (1 - \varpi_2)I, \\ W_k = U_{k,1} = \varpi_1 T_1 U_{k,2} + (1 - \varpi_1)I,\ \forall k \geq 1, \end{cases} \tag{4}$$

and consequently, they proposed the following iterative algorithm

$$\begin{cases} x_0 = x \in C \ \text{ chosen arbitrarily}, \\ y_k = \alpha_k x_k + (1 - \alpha_k) W_k x_k, \\ x_{k+1} = \iota_k u + (1 - \iota_k) y_k,\ \forall k \geq 0. \end{cases} \tag{5}$$

Under some mild assumptions, they derived the strong convergence of the sequence $\{x_k\}$ generated by (5) to a common fixed point of an infinite family of nonexpansive self-mappings $\{T_k\}_{k=1}^{\infty}$ on C in a reflexive and strictly convex Banach space X.

Very recently, in order to solve GSVI (1), Ceng, Gupta and Ansari [7] proved the convergence of implicit and explicit algorithms. For each $t \in (0, 1)$, choose a number $\theta_t \in (0, 1)$ arbitrarily. Then the net $\{z_t\}$ defined by $z_t = tQ_C(I - \varpi A)Q_C(I - \varsigma B)z_t + (1 - t)Q_C(I - \theta_t F)Q_C(I - \varpi A)Q_C(I - \varsigma B)z_t$ converges in norm, as $t \to 0^+$, to the unique solution $x^* \in \mathrm{GSVI}(C, A, B)$ to the following VI:

$$\langle F(x^*), j(x - x^*)\rangle \geq 0,\ \forall x \in \mathrm{GSVI}(C, A, B). \tag{6}$$

In [5], Buong and Phuong introducd a mapping V_k, defined by

$$V_k = V_k^1,\ V_k^i = T^i T^{i+1} \cdots T^k,\ T^i = (1 - \alpha_i)I + \alpha_i T_i,\ i = 1, 2, ..., k, \tag{7}$$

where

$$\alpha_i \in (0, 1) \text{ and } \sum_{i=1}^{\infty} \alpha_i < \infty. \tag{8}$$

Buong and Phuong presented the following iterations

$$x_k = V_k(I - \varpi_k F)x_k,\ \forall k \geq 1, \tag{9}$$

and

$$x_k = \tau_k(I - \varpi_k F)x_k + (I - \tau_k)V_k x_k, \ \forall k \geq 1. \tag{10}$$

The purpose of this paper is to find a solution of GSVI (1) with a HVI constraint for an infinite family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$ in Banach spaces. We introduce general implicit and explicit iterative algorithms, which are based on the viscosity approximation method, hybrid steepest-descent method and Korpelevich's extragradient method. Under some suitable control conditions on the parameter sequences in $[0, 1]$, we prove the strong convergence of the sequences generated by the proposed iterative algorithms to a solution of the GSVI (1) with a HVI constraint by using new V-mappings instead of W-ones. Our results improve and extend the corresponding results in [5, 7, 14].

2. Preliminaries and Algorithms

Let X be a real Banach space. A function $\varrho : [0, \infty) \to [0, \infty)$ is called the modulus of smoothness of X: for $u, v \in X$, $\varrho(\tau) = \sup\{\frac{1}{2}(\|u + v\| + \|u - v\|) - 1 : \ \|u\| = 1, \ \|v\| = \tau\}$.

Lemma 2.1. *([26]) Let X be a q-uniformly smooth Banach space with $1 < q \leq 2$. Then*

$$\|u + v\|^q \leq \|u\|^q + q\langle v, J_q(u)\rangle + 2\|\kappa v\|^q, \quad \forall u, v \in X,$$

where J_q is the generalized duality mapping from X into 2^{X^*} defined by $J_q(u) = \{\phi \in X^* : \langle u, \phi\rangle = \|u\|^q, \ \|\phi\| = \|u\|^{q-1}\}, \ \forall u \in X$.

Lemma 2.2. *([26]) Let X be a real Banach space and J be the normalized duality map on X. Then, for any given $x, y \in X$, the following inequality holds:*

$$\|u + v\|^2 \leq \|u\|^2 + 2\langle v, j(u + v)\rangle, \ \forall j(u + v) \in J(u + v).$$

Let $Q : C \to D$ be a mapping where $D \subset C$ be a set. Q is said to be sunny if $Q[su + (1 - s)Q(u)] = Q(u)$, whenever $su + (1 - s)Q(u) \in C$ for $u \in C$ and $s \geq 0$.

Lemma 2.3. *([22]) Let C be a nonempty closed convex subset of a smooth Banach space X and $\emptyset \neq D \subset C$. Let $Q : C \to D$ be a retraction. Then the following are equivalent*

(i) Q is sunny and nonexpansive;
(ii) $\|Q(u) - Q(v)\|^2 \leq \langle u - v, J(Q(u) - Q(v))\rangle, \ \forall u, v \in C$;
(iii) $\langle u - Q(u), J(v - Q(u))\rangle \leq 0, \ \forall u \in C, v \in D$.

Lemma 2.4. *([28]) Let X be a uniformly smooth Banach space, C be a nonempty closed convex subset of X, $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and $f \in \Omega$. Then $\{u_t\}$ defined by $u_t = sf(u_t) + (1 - s)Tu_t$ converges strongly to a point in $\mathrm{Fix}(T)$ as $s \to 0^+$.*

Recall that a gauge $\varphi : [0, \infty) \to [0, \infty)$ is a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. The duality map $J_\varphi : X \to 2^{X^*}$ is defined by

$$J_\varphi(u) = \{u^* \in X^* : \langle u, u^*\rangle = \|u\|\varphi(\|u\|), \|u^*\| = \varphi(\|u\|)\}, \ \forall u \in X.$$

Lemma 2.5. *([20]) Assume that X has a weakly continuous duality map J_φ with gauge φ.*

(i) For all $u, v \in X$, we have $\Phi(\|u + v\|) \leq \Phi(\|u\|) + \langle v, J_\varphi(u + v)\rangle$.
(ii) Let the sequence $(X \ni) u_k \rightharpoonup u$. Then,

$$\limsup_{k\to\infty} \Phi(\|u_k - v\|) = \limsup_{k\to\infty} \Phi(\|u_k - u\|) + \Phi(\|v - u\|), \ \forall v \in X.$$

Lemma 2.6. *([16]) Let X be a reflexive Banach space which have a weakly continuous duality map J_φ with gauge φ. Let $\emptyset \neq C \subset X$ be a closed convex set. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and let $f : C \to C$ be a contraction. Then u_t defined by $u_t = sf(u_t) + (1-s)Tu_t$ converges strongly to a point in $\mathrm{Fix}(T)$ as $s \to 0^+$.*

A mapping F with domain $D(F)$ and range $R(F)$ in a real Banach space X is called

(i) ν-strongly accretive if for each $u, v \in D(F)$, there exists $j(u-v) \in J(u-v)$ such that

$$\langle Fu - Fv, j(u-v)\rangle \geq \nu\|u-v\|^2 \quad \text{for some } \nu \in (0,1).$$

(ii) υ-strictly pseudocontractive [4] if for each $u, v \in D(F)$, there exists $j(u-v) \in J(u-v)$ such that

$$\langle Fu - Fv, j(u-v)\rangle \leq \|u-v\|^2 - \upsilon\|u-v-(Fu-Fv)\|^2 \quad \text{for some } \upsilon \in (0,1).$$

It is easy to see that last inequality is equivalent to

$$\langle (I-F)u - (I-F)v, j(u-v)\rangle \geq \upsilon\|(I-F)u - (I-F)v\|^2. \tag{11}$$

Lemma 2.7. *([7]) Let C be a nonempty closed convex subset of a real smooth Banach space X. Let $Q_C : X \to$ be a sunny nonexpansive retraction. Let $A, B : C \to X$ be two nonlinear mappings and ϖ, ς be two positive numbers. For given $u^*, v^* \in C$, (u^*, v^*) is a solution of the GSVI (1) if and only if $u^* \in \mathrm{GSVI}(C, A, B)$ where $\mathrm{GSVI}(C, A, B)$ is the set of fixed points of the mapping $G := Q_C(I - \varpi A)Q_C(I - \varsigma B)$ and $v^* = Q_C(u^* - \varsigma Bu^*)$.*

Lemma 2.8. *([5]) Let C be a nonempty closed convex subset of a strictly convex Banach space X and let $\{T_i\}_{i=1}^k$, be k nonexpansive self-mappings on C such that the set of common fixed points $\mathcal{F} := \bigcap_{i=1}^k \mathrm{Fix}(T_i) \neq \emptyset$. Let a, b and α_i, $i = 1, 2, ..., k$, be real numbers such that $0 < a \leq \alpha_i \leq b < 1$, and let V_k be a mapping, defined by (7) for all $k \geq 1$. Then, $\mathrm{Fix}(V_k) = \mathcal{F}$.*

Lemma 2.9. *([5]) Let C be a nonempty closed convex subset of a Banach space X and let $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive self-mappings on C such that the set of common fixed points $\mathcal{F} := \bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \neq \emptyset$. Let V_k be a mapping, defined by (7), and let α_i satisfy (8). Then, for each $x \in C$ and $i \geq 1$, $\lim_{k\to\infty} V_k^i x$ exists.*

Lemma 2.10. *([5]) Let C be a nonempty closed convex subset of a strictly convex Banach space X and let $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive self-mappings on C such that the set of common fixed points $\mathcal{F} := \bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \neq \emptyset$. Let α_i satisfy the first condition in (8). Then, $\mathrm{Fix}(V) = \mathcal{F}$.*

Lemma 2.11. *([5]) Let C be a nonempty closed convex subset of a strictly convex and smooth Banach space X. Let $Q_C : X \to C$ be a sunny nonexpansive retraction. Let $A, B : C \to X$ be two mappings and ϖ, ς be two positive numbers such that the mapping $G : C \to C$ is a nonexpansive mapping where $G := Q_C(I - \varpi A)Q_C(I - \varsigma B)$. Let $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive self-mappings on C such that $\mathcal{F} := \bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, A, B) \neq \emptyset$. Let α_i satisfy the first condition in (8). Then, $\mathrm{Fix}(V \circ G) = \mathcal{F}$.*

Lemma 2.12. *([6]) Let C be a nonempty closed convex subset of a real smooth Banach space X and $F : C \to X$ be a mapping.*

(a) If F is υ-strictly pseudocontractive, then F is Lipschitz continuous with constant $1 + \frac{1}{\upsilon}$.
(b) If F is ν-strongly accretive and υ-strictly pseudocontractive with $\nu + \upsilon > 1$, then $I - F$ is contractive with constant $\sqrt{\frac{1-\nu}{\upsilon}} \in (0,1)$.
(c) If F is ν-strongly accretive and υ-strictly pseudocontractive with $\nu + \upsilon > 1$, then for any fixed number $\varpi \in (0,1)$, $I - \varpi F$ is contractive with constant $1 - \varpi(1 - \sqrt{\frac{1-\nu}{\upsilon}}) \in (0,1)$.

Recall that X satisfies Opial's property provided, for each sequence $u_k \in X$, the condition $u_k \rightharpoonup u$ implies $\limsup_{k\to\infty} \|u_k - u\| < \limsup_{k\to\infty} \|u_k - v\|$, $\forall v \in X$, $v \neq u$. Denote by $\omega_w(u_k)$ the weak ω-limit set of $\{u_k\}$, i.e.,

$$\omega_w(u_k) = \{\bar{u} \in X : u_{k_i} \rightharpoonup \bar{u} \text{ for some subsequence } \{u_{k_i}\} \text{ of } \{u_k\}\}. \tag{12}$$

Lemma 2.13. *([25]) Let $\{a_k\}$ and $\{b_k\}$ be sequences of nonnegative real numbers such that $\sum_{k=1}^{\infty} b_k < \infty$ and $a_{k+1} \le a_k + b_k$ for all $k \ge 1$. Then $\lim_{k\to\infty} a_k$ exists.*

Lemma 2.14. *([27]) Assume that $\{a_k\}$ is a sequence of nonnegative real numbers satisfying the condition:*

$$a_{k+1} \le (1 - \varsigma_k)a_k + \varsigma_k \nu_k + \epsilon_k, \quad \forall k \ge 1,$$

where $\{\varsigma_k\} \subset [0, 1]$, $\{\nu_k\} \subset \mathbf{R}$ and $\{\epsilon_k\} \subset [0, \infty)$ such that

(i) $\sum_{k=1}^{\infty} \varsigma_k = \infty$;
(ii) $\limsup_{k\to\infty} \nu_k \le 0$ *or* $\sum_{k=1}^{\infty} |\varsigma_k \nu_k| < \infty$;
(iii) $\sum_{k=1}^{\infty} \epsilon_k < \infty$.

Then, $\lim_{k\to\infty} a_k = 0$.

Lemma 2.15. *([24]) Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\alpha_k\}$ be a sequence in $[0, 1]$ such that $0 < \liminf_{k\to\infty} \alpha_k \le \limsup_{k\to\infty} \alpha_k < 1$. Suppose that $x_{k+1} = \alpha_k x_k + (1 - \alpha_k)z_k, \forall k \ge 1$, and*

$$\limsup_{k\to\infty}(\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\|) \le 0.$$

Then $\lim_{k\to\infty} \|z_k - x_k\| = 0$.

3. Main results

In this section, we introduce our iterative algorithms and give the convergence analysis.

Theorem 3.1. *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space X. Assume, in addition, X either is uniformly smooth or has a weakly continuous duality map J_φ with gauge φ. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $A, B : C \to X$ be two nonlinear mappings and ϖ, ς be two positive numbers such that the mapping $G : C \to C$ is a nonexpansive mapping where $G := Q_C(I - \varpi A)Q_C(I - \varsigma B)$. Let $F : C \to X$ be ν-strongly accretive and υ-strictly pseudocontractive with $\nu + \upsilon > 1$. Let $T_i : C \to C$ be a nonexpansive mapping for each $i = 1, 2, ...$ such that $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, A, B) \ne \emptyset$, and $f \in \Omega$ with contractive constant $\varrho \in (0, 1)$. Let $\{V_k\}_{k=1}^{\infty}$ be defined by (7) and (8). Given sequences $\{\iota_k\}_{k=1}^{\infty}, \{\tau_k\}_{k=1}^{\infty}$ in $[0, 1]$ and $\{\varpi_k\}_{k=1}^{\infty}$ in $(0, 1]$, the following conditions are satisfied:*

(C1) $0 < \tau_k \le 1 - \varrho$, $\forall k \ge 1$;
(C2) $\limsup_{k\to\infty} \iota_k < 1$ *and* $\lim_{k\to\infty}(\iota_k\varpi_k)/\tau_k = 0$.

For given $x_0 \in C$, define a sequence $\{x_k\}$ iteratively by

$$\begin{cases} y_k = [(1 - \iota_k)V_k + \iota_k Q_C(I - \varpi_k F)]Gx_k, \\ x_k = \tau_k f(x_k) + (1 - \tau_k)y_k, \quad \forall k \ge 1. \end{cases} \tag{13}$$

Then,

$$x_k \to X^\dagger \in \mathcal{F} \iff \tau_k\|(I - f)x_k\| + \iota_k\|(I - G)x_k\| \to 0.$$

(i) If X is uniformly smooth, then $X^\dagger \in \mathcal{F}$ solves the VI

$$\langle (I - f)X^\dagger, J(X^\dagger - p)\rangle \ge 0, \quad f \in \Omega, p \in \mathcal{F};$$

(ii) If X has a weakly continuous duality map J_φ with gauge φ, then $X^\dagger \in \mathcal{F}$ solves the VI

$$\langle (I - f)X^\dagger, J_\varphi(X^\dagger - p)\rangle \ge 0, \quad f \in \Omega, p \in \mathcal{F}.$$

Proof. Without loss of generality, we assume that $\{\iota_k\}_{k=1}^{\infty} \subset [0,b] \subset [0,1)$. Note that $\lim_{k\to\infty} \iota_k\varpi_k = 0$. Let the mapping $G_k : C \to C$ be defined as $G_k x := Q_C(I - \varpi_k F)x$ where $0 < \varpi_k \leq 1$. In terms of Lemma 2.12 we know that G_k is a contraction with contractive constant $1 - \varpi_k\tau$, where $\tau = 1 - \sqrt{\frac{1-\nu}{\upsilon}} \in (0,1)$. Then, (13) can be rewritten as

$$x_k = \tau_k f(x_k) + (1-\tau_k)[\iota_k G_k G x_k + (1-\iota_k)V_k G x_k], \quad \forall k \geq 1. \tag{14}$$

Define a mapping $U_k x = \tau_k f(x) + (1-\tau_k)[\iota_k G_k G x + (1-\iota_k)V_k G x]$, $\forall x \in C$. Utilizing Lemma 2.12, we have

$$\begin{aligned}
\|U_k x - U_k y\| &\leq \tau_k\|f(x) - f(y)\| + (1-\tau_k)[\iota_k\|G_k G x - G_k G y\| + (1-\iota_k)\|V_k G x - V_k G y\|] \\
&\leq \tau_k\varrho\|x-y\| + (1-\tau_k)[\iota_k(1-\varpi_k\tau)\|Gx - Gy\| + (1-\iota_k)\|Gx - Gy\|] \\
&= \tau_k\varrho\|x-y\| + (1-\tau_k)(1-\iota_k\varpi_k\tau)\|Gx - Gy\| \\
&\leq \tau_k\varrho\|x-y\| + (1-\tau_k)\|x-y\| \\
&= (1-(1-\varrho)\tau_k)\|x-y\|.
\end{aligned}$$

This means that $U_k : C \to C$ is a contraction. Thus, the fixed point equation (14) has a unique solution $x_k \in C$ for each $k \geq 1$.

Let $p \in \mathcal{F}$. Then, $Gp = p$ and $p = V_k p$ for all $k \geq 1$. From Lemma2.12, it follows that for each $k \geq 1$,

$$\begin{aligned}
\|x_k - p\| &\leq \tau_k\|f(x_k) - p\| + (1-\tau_k)\|\iota_k G_k G x_k + (1-\iota_k)V_k G x_k - p\| \\
&\leq \tau_k(\varrho\|x_k - p\| + \|f(p) - p\|) + (1-\tau_k)[\iota_k\|G_k G x_k - G_k p + G_k p - p\| + (1-\iota_k)\|x_k - p\|] \\
&\leq \tau_k(\varrho\|x_k - p\| + \|f(p) - p\|) + (1-\tau_k)[(1-\iota_k\varpi_k\tau)\|x_k - p\| + \iota_k\varpi_k\|F(p)\|] \\
&\leq \tau_k(\varrho\|x_k - p\| + \|f(p) - p\|) + (1-\tau_k)\|x_k - p\| + \iota_k\varpi_k\|F(p)\| \\
&= (1-(1-\varrho)\tau_k)\|x_k - p\| + \tau_k\|f(p) - p\| + \iota_k\varpi_k\|F(p)\|.
\end{aligned} \tag{15}$$

Therefore,

$$\|x_k - p\| \leq \frac{\|f(p) - p\|}{1-\varrho} + \frac{\iota_k\varpi_k}{\tau_k}\cdot\frac{\|F(p)\|}{1-\varrho},$$

which together with $\lim_{k\to\infty}(\iota_k\varpi_k)/\tau_k = 0$, implies that

$$\limsup_{k\to\infty}\|x_k - p\| \leq \frac{\|f(p) - p\|}{1-\varrho}.$$

This shows that $\{x_k\}_{k=1}^{\infty}$ is bounded. So, the sequences $\{F(Gx_k)\}_{k=1}^{\infty}$, $\{V_k G x_k\}_{k=1}^{\infty}$, $\{G_k G x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$, where $y_k = \iota_k G_k G x_k + (1-\iota_k)V_k G x_k$, are also bounded.

Suppose that $x_k \to X^\dagger \in \mathcal{F}$ as $k \to \infty$. Then $X^\dagger = G(X^\dagger)$ and $X^\dagger = V_k X^\dagger$ for all $k \geq 1$. It is clear that as $k \to \infty$,

$$\|\iota_k(I-G)x_k\| \leq \|x_k - Gx_k\| \leq \|x_k - X^\dagger\| + \|Gx_k - X^\dagger\| \leq 2\|x_k - X^\dagger\| \to 0.$$

That is, $\lim_{k\to\infty}\|\iota_k(I-G)x_k\| = 0$. From (13) and $\lim_{k\to\infty}\iota_k\varpi_k = 0$ it follows that

$$\begin{aligned}
\|y_k - X^\dagger\| &\leq \iota_k\|G_k G x_k - X^\dagger\| + (1-\iota_k)\|V_k G x_k - X^\dagger\| \\
&= \iota_k\|G_k G x_k - G_k X^\dagger + G_k X^\dagger - X^\dagger\| + (1-\iota_k)\|V_k G x_k - X^\dagger\| \\
&\leq \iota_k[(1-\varpi_k\tau)\|x_k - X^\dagger\| + \|(I-\varpi_k F)X^\dagger - X^\dagger\|] + (1-\iota_k)\|x_k - X^\dagger\| \\
&\leq \iota_k[\|x_k - X^\dagger\| + \varpi_k\|F(X^\dagger)\|] + (1-\iota_k)\|x_k - X^\dagger\| \\
&= \|x_k - X^\dagger\| + \iota_k\varpi_k\|F(X^\dagger)\| \to 0 \quad \text{as} \quad k \to \infty,
\end{aligned}$$

that is, $\lim_{k\to\infty} \|y_k - X^\dagger\| = 0$. Again from (13) we obtain

$$\begin{aligned}\|\tau_k(f(x_k) - x_k)\| &= \| - (1 - \tau_k)(y_k - x_k)\| \\ &\le \|y_k - x_k\| \le \|x_k - X^\dagger\| + \|y_k - X^\dagger\| \to 0 \quad \text{as } k \to \infty,\end{aligned}$$

that is, $\lim_{k\to\infty} \|\tau_k(I - f)x_k\| = 0$.

Conversely, suppose that $\|\tau_k(I - f)x_k\| + \|\iota_k(I - G)x_k\| \to 0$ as $k \to \infty$. From (13) we observe that

$$0 = \tau_k(f(x_k) - x_k) + (1 - \tau_k)(y_k - x_k),$$

and

$$y_k - x_k = \iota_k(G_kGx_k - x_k) + (1 - \iota_k)(V_kGx_k - x_k).$$

Then from $\|\tau_k(f(x_k) - x_k)\| \to 0$, $\iota_k\varpi_k \to 0$ and $0 < \tau_k \le 1 - \varrho$ it follows that as $k \to \infty$,

$$\varrho\|y_k - x_k\| \le (1 - \tau_k)\|y_k - x_k\| = \|\tau_k(f(x_k) - x_k)\| \to 0,$$

and

$$\begin{aligned}(1 - b)\|V_kGx_k - x_k\| &\le \|(1 - \iota_k)(V_kGx_k - x_k)\| \\ &\le \|y_k - x_k\| + \iota_k\|G_kGx_k - x_k\| \\ &\le \|y_k - x_k\| + \iota_k\|(I - \varpi_kF)Gx_k - Gx_k\| + \iota_k\|Gx_k - x_k\| \\ &\le \|y_k - x_k\| + \iota_k\varpi_k\|F(Gx_k)\| + \iota_k\|Gx_k - x_k\| \to 0.\end{aligned}$$

That is,

$$\lim_{k\to\infty} \|y_k - x_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|V_kGx_k - x_k\| = 0. \tag{16}$$

Furthermore, we deduce that if D is a nonempty and bounded subset of C, then, for $\varepsilon > 0$, there exists $m_0 > i$ such that for all $k > m_0$

$$\sup_{x\in D} \|V_k^i x - V_\infty^i x\| \le \varepsilon. \tag{17}$$

Set $D = \{Gx_k : k \ge 1\}$. From (17) we have

$$\|V_kGx_k - VGx_k\| \le \sup_{x\in D} \|V_kx - Vx\| \le \varepsilon,$$

which immediately imply that

$$\lim_{k\to\infty} \|V_kGx_k - VGx_k\| = 0. \tag{18}$$

Taking into account that $\|x_k - VGx_k\| \le \|x_k - V_kGx_k\| + \|V_kGx_k - VGx_k\|$, from (16) and (18) we get

$$\lim_{k\to\infty} \|x_k - VGx_k\| = 0. \tag{19}$$

By virtue of Lemma 2.10, we know that $\mathrm{Fix}(V \circ G) = \mathcal{F}$. Let z_t be the unique fixed point of T_t given by $T_tx = tf(x) + (1 - t)VGx$, $t \in (0, 1)$. Define $X^\dagger := \lim_{t\to 0^+} z_t$ and $X^\dagger \in \mathrm{Fix}(V \circ G) = \mathcal{F}$. We consider two cases.

(i) X is uniformly smooth. By Lemma 2.4, $X^\dagger$ solves the VI $\langle (I - f)X^\dagger, J(X^\dagger - p)\rangle \le 0$, $\forall p \in \mathcal{F}$. Next, we show

$$\limsup_{k\to\infty} \langle f(z) - z, J(x_k - z)\rangle \le 0, \tag{20}$$

where $z = X^\dagger$.

Applying Lemma 2.2 we derive

$$\begin{aligned}\|z_t - x_k\|^2 &\le (1-t)^2\|VGz_t - x_k\|^2 + 2t\langle f(z_t) - x_k, J(z_t - x_k)\rangle \\ &\le (1-t)^2(\|VGz_t - VGx_k\| + \|VGx_k - x_k\|)^2 + 2t\langle f(z_t) - z_t, J(z_t - x_k)\rangle + 2t\|z_t - x_k\|^2 \\ &\le (1-t)^2\|z_t - x_k\|^2 + a_k(t) + 2t\langle f(z_t) - z_t, J(z_t - x_k)\rangle + 2t\|z_t - x_k\|^2,\end{aligned}$$

where

$$a_k(t) = \|VGx_k - x_k\|(2\|z_t - x_k\| + \|Vx_k - x_k\|) \to 0 \quad (\text{due} \quad \text{to} \quad (19))$$

The last inequality implies

$$\langle z_t - f(z_t), J(z_t - x_k)\rangle \le \frac{t}{2}\|z_t - x_k\|^2 + \frac{1}{2t}a_k(t).$$

It follows that

$$\limsup_{k\to\infty}\langle z_t - f(z_t), J(z_t - x_k)\rangle \le M\frac{t}{2}, \tag{21}$$

where $M > 0$ is a constant such that $M \ge \|z_t - x_k\|^2$ for all $k \ge 1$ and small enough t in $(0,1)$. Taking the limsup as $t \to 0^+$ in (21) and noticing the fact that the two limits are interchangeable due to the fact that the duality map J is uniformly norm-to-norm continuous on any bounded subset of X, we obtain (20).

Note that

$$x_k - z = \tau_k(f(x_k) - z) + (1-\tau_k)(1-\iota_k)(V_kGx_k - z) + (1-\tau_k)\iota_k(G_kGx_k - z).$$

Then, applying Lemma 2.2, we get

$$\begin{aligned}\|x_k - z\|^2 &\le \|(1-\tau_k)(1-\iota_k)(V_kGx_k - z) + (1-\tau_k)\iota_k(G_kGx_k - z)\|^2 + 2\tau_k\langle f(x_k) - z, J(x_k - z)\rangle \\ &\le [(1-\tau_k)(1-\iota_k)\|V_kGx_k - z\| + (1-\tau_k)\iota_k\|G_kGx_k - z\|]^2 + 2\tau_k\langle f(x_k) - f(z), J(x_k - z)\rangle \\ &\quad + 2\tau_k\langle f(z) - z, J(x_k - z)\rangle \\ &\le [(1-\tau_k)(1-\iota_k)\|x_k - z\| + (1-\tau_k)\iota_k(\|x_k - z\| + \varpi_k\|F(z)\|)]^2 + 2\tau_k\varrho\|x_k - z\|^2 + 2\tau_k\langle f(z) - z, J(x_k - z)\rangle \\ &\le (1-\tau_k)^2[\|x_k - z\|^2 + \iota_k\varpi_k\|F(z)\|(2\|x_k - z\| + \iota_k\varpi_k\|F(z)\|)] + 2\tau_k\varrho\|x_k - z\|^2 + 2\tau_k\langle f(z) - z, J(x_k - z)\rangle \\ &\le (1-\tau_k(1-\varrho))\|x_k - z\|^2 + \iota_k\varpi_k\|F(z)\|(2\|x_k - z\| + \iota_k\varpi_k\|F(z)\|) + 2\tau_k\langle f(z) - z, J(x_k - z)\rangle,\end{aligned}$$

which immediately implies that

$$\|x_k - z\|^2 \le \frac{\iota_k\varpi_k}{\tau_k}\cdot\frac{\|F(z)\|(2\|x_k - z\| + \iota_k\varpi_k\|F(z)\|)}{1-\varrho} + \frac{2}{1-\varrho}\langle f(z) - z, J(x_k - z)\rangle. \tag{22}$$

Since $\lim_{k\to\infty}(\iota_k\varpi_k)/\tau_k = 0$ and $\{x_k\}$ is bounded, we deduce from (20) and (22) that

$$\limsup_{k\to\infty}\|x_k - z\|^2 \le 0.$$

That is, $\lim_{k\to\infty}\|x_k - z\| = 0$.

(ii) X has a weakly continuous duality map J_φ with gauge φ. By Lemma 2.6, $X^\dagger$ solves the VI

$$\langle (I - f)X^\dagger, J_\varphi(X^\dagger - p)\rangle \le 0, \quad \forall p \in \mathcal{F}. \tag{23}$$

Let us show that

$$\limsup_{k\to\infty}\langle f(z) - z, J_\varphi(x_k - z)\rangle \le 0, \tag{24}$$

where $z = X^{\dagger}$. We take a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that

$$\limsup_{k\to\infty}\langle f(z)-z, J_\varphi(x_k-z)\rangle = \limsup_{i\to\infty}\langle f(z)-z, J_\varphi(x_{k_i}-z)\rangle. \tag{25}$$

Since X is reflexive and $\{x_k\}$ is bounded, we may further assume that $x_{k_i} \rightharpoonup \bar{x}$ for some $\bar{x} \in C$. Since J_φ is weakly continuous, utilizing Lemma 2.5 (ii), we have

$$\limsup_{i\to\infty}\Phi(\|x_{k_i}-x\|) = \limsup_{i\to\infty}\Phi(\|x_{k_i}-\bar{x}\|) + \Phi(\|x-\bar{x}\|), \quad \forall x \in X.$$

Set $\Gamma(x) = \limsup_{i\to\infty}\Phi(\|x_{k_i}-x\|)$, $\forall x \in X$. It follows that $\Gamma(x) = \Gamma(\bar{x}) + \Phi(\|x-\bar{x}\|)$, $\forall x \in X$. From (19), we have

$$\begin{aligned}\Gamma(VG\bar{x}) &= \limsup_{i\to\infty}\Phi(\|x_{k_i}-VG\bar{x}\|) = \limsup_{i\to\infty}\Phi(\|VGx_{k_i}-VG\bar{x}\|)\\ &\le \limsup_{i\to\infty}\Phi(\|x_{k_i}-\bar{x}\|) = \Gamma(\bar{x}).\end{aligned} \tag{26}$$

Furthermore, observe that

$$\Gamma(VG\bar{x}) = \Gamma(\bar{x}) + \Phi(\|VG\bar{x}-\bar{x}\|). \tag{27}$$

Combining (26) with (27), we obtain $\Phi(\|VG\bar{x}-\bar{x}\|) \le 0$. Hence, $VG\bar{x} = \bar{x}$ and $\bar{x} \in \mathrm{Fix}(V \circ G) = \mathcal{F}$ (by Lemma 2.11). Thus, from (23) and (25), it is easy to see that

$$\limsup_{k\to\infty}\langle f(z)-z, J_\varphi(x_k-z)\rangle = \langle f(z)-z, J_\varphi(\bar{x}-z)\rangle \le 0.$$

Therefore, we deduce that (24) holds.

Applying Lemma 2.5, we obtain

$$\begin{aligned}\Phi(\|y_k-z\|) &= \Phi(\|\iota_k(G_kGx_k-G_kz) + (1-\iota_k)(V_kGx_k-z) + \iota_k(G_kz-z)\|)\\ &\le \Phi(\|\iota_k(G_kGx_k-G_kz) + (1-\iota_k)(V_kGx_k-z)\|) + \iota_k\langle G_kz-z, J_\varphi(y_k-z)\rangle\\ &\le \Phi(\iota_k(1-\varpi_k\tau)\|x_k-z\| + (1-\iota_k)\|x_k-z\|) + \iota_k\|G_kz-z\|\varphi(\|y_k-z\|)\\ &\le \Phi(\iota_k\|x_k-z\| + (1-\iota_k)\|x_k-z\|) + \iota_k\|(I-\varpi_kF)z-z\|\varphi(\|y_k-z\|)\\ &= \Phi(\|x_k-z\|) + \iota_k\varpi_k\|F(z)\|\varphi(\|y_k-z\|),\end{aligned}$$

and hence

$$\begin{aligned}\Phi(\|x_k-z\|) &= \Phi(\|\tau_k(f(y_k)-f(z)) + (1-\tau_k)(y_k-z) + \tau_k(f(x_k)-f(y_k)) + \tau_k(f(z)-z)\|)\\ &\le \Phi(\|\tau_k(f(y_k)-f(z)) + (1-\tau_k)(y_k-z)\|) + \tau_k\langle f(x_k)-f(y_k), J_\varphi(x_k-z)\rangle\\ &\quad + \tau_k\langle f(z)-z, J_\varphi(x_k-z)\rangle\\ &\le (1-(1-\varrho)\tau_k)\Phi(\|y_k-z\|) + \tau_k\varrho\|x_k-y_k\|\varphi(\|x_k-z\|) + \tau_k\langle f(z)-z, J_\varphi(x_k-z)\rangle\\ &\le (1-(1-\varrho)\tau_k)[\Phi(\|x_k-z\|) + \iota_k\varpi_k\|F(z)\|\varphi(\|y_k-z\|)] + \tau_k\varrho\|x_k-y_k\|\varphi(\|x_k-z\|)\\ &\quad + \tau_k\langle f(z)-z, J_\varphi(x_k-z)\rangle\\ &\le (1-(1-\varrho)\tau_k)\Phi(\|x_k-z\|) + \iota_k\varpi_k\|F(z)\|\varphi(\|y_k-z\|) + \tau_k\varrho\|x_k-y_k\|\varphi(\|x_k-z\|)\\ &\quad + \tau_k\langle f(z)-z, J_\varphi(x_k-z)\rangle,\end{aligned}$$

which immediately yields

$$\Phi(\|x_k-z\|) \le \frac{\iota_k\varpi_k}{\tau_k}\cdot\frac{\|F(z)\|\varphi(\|y_k-z\|)}{1-\varrho} + \frac{\varrho}{1-\varrho}\|x_k-y_k\|\varphi(\|x_k-z\|) + \frac{1}{1-\varrho}\langle f(z)-z, J_\varphi(x_k-z)\rangle,$$

Since $\lim_{k\to\infty}(\iota_k\varpi_k)/\tau_k = 0$ and the sequences $\{x_k\}$ and $\{y_k\}$ are bounded, we conclude from (16) and (24) that $\lim_{k\to\infty}\Phi(\|x_k-z\|) = 0$ which implies that $\lim_{k\to\infty}\|x_k-z\| = 0$. This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space X. Assume, in addition, X either is uniformly smooth or has a weakly continuous duality map J_φ with gauge φ. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $A, B : C \to X$ be two nonlinear mappings and ϖ, ς be two positive numbers such that the mapping $G : C \to C$ is a nonexpansive mapping where $G := Q_C(I - \varpi A)Q_C(I - \varsigma B)$. Let $F : C \to X$ be ν-strongly accretive and υ-strictly pseudocontractive with $\nu + \upsilon > 1$. Let $T_i : C \to C$ be a nonexpansive mapping for each $i = 1, 2, ...$ such that $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, A, B) \neq \emptyset$, and $f \in \Omega$ with contractive constant $\varrho \in (0, 1)$. Let $\{V_k\}_{k=1}^{\infty}$ be defined by (7) and (8). Given sequences $\{\iota_k\}_{k=1}^{\infty}, \{\tau_k\}_{k=1}^{\infty}$ in $[0, 1]$ and $\{\varpi_k\}_{k=1}^{\infty}$ in $(0, 1]$, the following conditions are satisfied:*

(C1) $0 < \tau_k \leq 1 - \varrho$, $\forall k \geq k_0$ *for some* $k_0 \geq 1$, *and* $\sum_{k=1}^{\infty} \tau_k = \infty$;
(C2) $0 < \liminf_{k\to\infty} \iota_k \leq \limsup_{k\to\infty} \iota_k < 1$;
(C3) $\lim_{k\to\infty}(\frac{\tau_{k+1}}{1-(1-\tau_{k+1})\iota_{k+1}} - \frac{\tau_k}{1-(1-\tau_k)\iota_k}) = 0$;
(C4) $\sum_{k=1}^{\infty} \varpi_k < \infty$ *or* $\lim_{k\to\infty} \varpi_k/\tau_k = 0$.

For an arbitrary $x_1 \in C$, let $\{x_k\}_{k=1}^{\infty}$ be generated by

$$\begin{cases} y_k = [\iota_k Q_C(I - \varpi_k F) + (1 - \iota_k)V_k]Gx_k, \\ x_{k+1} = \tau_k f(x_k) + (1 - \tau_k)y_k, \quad \forall k \geq 1. \end{cases} \tag{28}$$

Then,

$$x_k \to X^\dagger \in \mathcal{F} \;\Leftrightarrow\; \tau_k\|(I - f)x_k\| + \|(I - G)x_k\| \to 0.$$

(i) If X is uniformly smooth, then $X^\dagger \in \mathcal{F}$ solves the VI

$$\langle (I - f)X^\dagger, J(X^\dagger - p)\rangle \geq 0, \quad f \in \Omega, p \in \mathcal{F};$$

(ii) If X has a weakly continuous duality map J_φ with gauge φ, then $X^\dagger \in \mathcal{F}$ solves the VI

$$\langle (I - f)X^\dagger, J_\varphi(X^\dagger - p)\rangle \geq 0, \quad f \in \Omega, p \in \mathcal{F}.$$

Proof. Without loss of generality, we assume that $\{\iota_k\}_{k=1}^{\infty} \subset [a, b] \subset (0, 1)$. Observe that $\lim_{k\to\infty} \varpi_k = 0$. Define $G_k x := Q_C(I - \varpi_k F)x$ where $0 < \varpi_k \leq 1$. According to Lemma 2.12, G_k is a contraction with contractive constant $1 - \varpi_k\tau$, where $\tau = 1 - \sqrt{\frac{1-\nu}{\upsilon}} \in (0, 1)$. Then, (28) can be rewritten as

$$\begin{cases} y_k = \iota_k G_k G x_k + (1 - \iota_k)V_k G x_k, \\ x_{k+1} = \tau_k f(x_k) + (1 - \tau_k)y_k, \quad \forall k \geq 1. \end{cases} \tag{29}$$

Let $p \in \mathcal{F}$. Then, $p = Gp$ and $p = V_k p$ for all $k \geq 1$. Hence,

$$\begin{aligned} \|y_k - p\| &\leq \iota_k\|G_k G x_k - p\| + (1 - \iota_k)\|V_k G x_k - p\| \\ &\leq \iota_k\|G_k G x_k - G_k p + G_k p - p\| + (1 - \iota_k)\|x_k - p\| \\ &\leq \iota_k[(1 - \varpi_k\tau)\|x_k - p\| + \|(I - \varpi_k F)p - p\|] + (1 - \iota_k)\|x_k - p\| \\ &\leq \max\{\|x_k - p\|, \frac{\|F(p)\|}{\tau}\}. \end{aligned}$$

By (28), we get

$$\begin{aligned} \|x_{k+1} - p\| &\leq \tau_k(\|f(x_k) - f(p)\| + \|f(p) - p\|) + (1 - \tau_k)\|y_k - p\| \\ &\leq \tau_k(\varrho\|x_k - p\| + \|f(p) - p\|) + (1 - \tau_k)\max\{\|x_k - p\|, \frac{\|F(p)\|}{\tau}\} \\ &\leq \tau_k\varrho\max\{\|x_k - p\|, \frac{\|F(p)\|}{\tau}\} + \tau_k\|f(p) - p\| + (1 - \tau_k)\max\{\|x_k - p\|, \frac{\|F(p)\|}{\tau}\} \\ &= (1 - \varrho)\tau_k\frac{\|f(p) - p\|}{1 - \varrho} + (1 - (1 - \varrho)\tau_k)\max\{\|x_k - p\|, \frac{\|F(p)\|}{\tau}\} \\ &\leq \max\{\|x_k - p\|, \frac{\|f(p) - p\|}{1 - \varrho}, \frac{\|F(p)\|}{\tau}\}. \end{aligned}$$

By induction, we have

$$\|x_k - p\| \le \max\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1-\varrho}, \frac{\|F(p)\|}{\tau}\}, \quad \forall k \ge 1.$$

This implies that $\{x_k\}_{k=1}^\infty$ is bounded. So, the sequences $\{F(Gx_k)\}_{k=1}^\infty$, $\{V_k Gx_k\}_{k=1}^\infty$, $\{G_k Gx_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$, where $y_k = \iota_k G_k Gx_k + (1-\iota_k)V_k Gx_k$, are also bounded.

Suppose that $x_k \to X^\dagger \in \mathcal{F}$ as $k \to \infty$. Then $X^\dagger = G(X^\dagger)$ and $X^\dagger = V_k X^\dagger$ for all $k \ge 1$. It is clear that $\|x_k - Gx_k\| \to 0$. From (29), we deduce

$$\begin{aligned}\|y_k - X^\dagger\| &\le \iota_k \|G_k Gx_k - X^\dagger\| + (1-\iota_k)\|V_k Gx_k - X^\dagger\| \\ &\le \iota_k[(1-\varpi_k\tau)\|x_k - X^\dagger\| + \|(I-\varpi_k F)X^\dagger - X^\dagger\|] + (1-\iota_k)\|x_k - X^\dagger\| \\ &\le \|x_k - X^\dagger\| + \varpi_k \|F(X^\dagger)\| \to 0 \quad \text{as } k \to \infty,\end{aligned}$$

that is, $\lim_{k\to\infty} \|y_k - X^\dagger\| = 0$. Again from (29) we obtain that

$$\begin{aligned}\|\tau_k(f(x_k) - x_k)\| &= \|x_{k+1} - x_k - (1-\tau_k)(y_k - x_k)\| \\ &\le \|x_{k+1} - x_k\| + (1-\tau_k)\|y_k - x_k\| \\ &\le \|x_{k+1} - x_k\| + \|x_k - X^\dagger\| + \|y_k - X^\dagger\| \to 0.\end{aligned}$$

Conversely, suppose that $\|\tau_k(I-f)x_k\| + \|(I-G)x_k\| \to 0$ as $k \to \infty$. Set $\nu_k = (1-\tau_k)\iota_k$, $\forall k \ge 1$. Then, it follows from (C1) and (C2) that $\iota_k \ge \nu_k = (1-\tau_k)\iota_k \ge (1-(1-\varrho))\iota_k = \varrho\iota_k$, $\forall k \ge k_0$, and hence

$$0 < \liminf_{k\to\infty} \nu_k \le \limsup_{k\to\infty} \nu_k < 1. \tag{30}$$

Without loss of generality, we assume that $\{\nu_k\}_{k=1}^\infty \subset [c, d] \subset (0,1)$. Define z_k by

$$x_{k+1} = \nu_k x_k + (1-\nu_k) z_k. \tag{31}$$

Observe that

$$\begin{aligned}z_{k+1} - z_k &= \frac{x_{k+2} - \nu_{k+1}x_{k+1}}{1-\nu_{k+1}} - \frac{x_{k+1} - \nu_k x_k}{1-\nu_k} \\ &= (\frac{\tau_{k+1}}{1-\nu_{k+1}} - \frac{\tau_k}{1-\nu_k})(f(x_{k+1}) - V_{k+1}Gx_{k+1}) + \frac{\tau_k}{1-\nu_k}(f(x_{k+1}) - f(x_k)) \\ &\quad + \frac{1-\nu_k-\tau_k}{1-\nu_k}(V_{k+1}Gx_{k+1} - V_k Gx_k) + \frac{\nu_{k+1}(G_{k+1}Gx_{k+1} - x_{k+1})}{1-\nu_{k+1}} - \frac{\nu_k(G_k Gx_k - x_k)}{1-\nu_k}.\end{aligned}$$

It follows that

$$\begin{aligned}\|z_{k+1} - z_k\| &\le |\frac{\tau_{k+1}}{1-\nu_{k+1}} - \frac{\tau_k}{1-\nu_k}|\|f(x_{k+1}) - V_{k+1}Gx_{k+1}\| + \frac{\tau_k}{1-\nu_k}\|f(x_{k+1}) - f(x_k)\| \\ &\quad + \frac{1-\nu_k-\tau_k}{1-\nu_k}\|V_{k+1}Gx_{k+1} - V_k Gx_k\| + \frac{\nu_{k+1}\|G_{k+1}Gx_{k+1} - x_{k+1}\|}{1-\nu_{k+1}} + \frac{\nu_k\|G_k Gx_k - x_k\|}{1-\nu_k} \\ &\le |\frac{\tau_{k+1}}{1-\nu_{k+1}} - \frac{\tau_k}{1-\nu_k}|(\|f(x_{k+1})\| + \|V_{k+1}Gx_{k+1}\|) + \frac{\varrho\tau_k}{1-\nu_k}\|x_{k+1} - x_k\| + \frac{1-\nu_k-\tau_k}{1-\nu_k}(\|x_{k+1} - x_k\| \\ &\quad + \alpha_{k+1}\|T_{k+1}Gx_k - Gx_k\|) + \frac{d(\varpi_{k+1}\|F(Gx_{k+1})\| + \|(I-G)x_{k+1}\|)}{1-d} + \frac{d(\varpi_k\|F(x_k)\| + \|(I-G)x_k\|)}{1-d} \\ &\le \|x_{k+1} - x_k\| + |\frac{\tau_{k+1}}{1-\nu_{k+1}} - \frac{\tau_k}{1-\nu_k}|(\|f(x_{k+1})\| + \|V_{k+1}Gx_{k+1}\|) + \alpha_{k+1}\|T_{k+1}Gx_k - Gx_k\| \\ &\quad + \frac{d(\varpi_{k+1}\|F(Gx_{k+1})\| + \|(I-G)x_{k+1}\|)}{1-d} + \frac{d(\varpi_k\|F(Gx_k)\| + \|(I-G)x_k\|)}{1-d}.\end{aligned} \tag{32}$$

It follows that

$$\limsup_{k\to\infty}(\|z_{k+1}-z_k\|-\|x_{k+1}-x_k\|)\le 0.$$

This together with Lemma 2.15 implies that $\lim_{k\to\infty}\|z_k-x_k\|=0$. It follows from (30) and (31) that

$$\lim_{k\to\infty}\|x_{k+1}-x_k\|=\lim_{k\to\infty}(1-\nu_k)\|z_k-x_k\|=0.$$

Note that

$$\begin{aligned}\varrho\|y_k-x_k\| &\le (1-\tau_k)\|y_k-x_k\|\\ &=\|x_{k+1}-x_k-\tau_k(f(x_k)-x_k)\|\\ &\le \|x_{k+1}-x_k\|+\|\tau_k(I-f)x_k\|.\end{aligned}$$

Since $x_{k+1}-x_k\to 0$ and $\|\tau_k(I-f)x_k\|\to 0$, we get

$$\lim_{k\to\infty}\|y_k-x_k\|=0. \tag{33}$$

Observe that

$$y_k-x_k=\iota_k(G_kGx_k-x_k)+(1-\iota_k)(V_kGx_k-x_k). \tag{34}$$

So, $(1-b)\|V_kGx_k-x_k\|\le\|y_k-x_k\|+\varpi_k\|F(Gx_k)\|+\|(I-G)x_k\|\to 0$. Consequently,

$$\lim_{k\to\infty}\|V_kGx_k-VGx_k\|=0. \tag{35}$$

Taking into account that $\|x_k-VGx_k\|\le\|x_k-V_kGx_k\|+\|V_kGx_k-VGx_k\|$, from (35) we get

$$\lim_{k\to\infty}\|x_k-VGx_k\|=0. \tag{36}$$

In the light of Lemma 2.11, we conclude that $\mathrm{Fix}(V\circ G)=\mathcal{F}$. Let z_t be the unique fixed point of T_t given by $T_tx=tf(x)+(1-t)VGx,\ t\in(0,1)$. Set $X^\dagger:=\lim_{t\to 0^+}z_t$ and note that $X^\dagger\in\mathrm{Fix}(V\circ G)=\mathcal{F}$. We consider two cases.

(i) X is uniformly smooth. By Lemma 2.4, $X^\dagger$ solves the VI $\langle(I-f)X^\dagger,J(X^\dagger-p)\rangle\le 0,\ \forall p\in\mathcal{F}$. Repeating the same arguments as those of (20) in the proof of Theorem 3.1, we obtain

$$\limsup_{k\to\infty}\langle f(z)-z,J(x_k-z)\rangle\le 0, \tag{37}$$

where $z=X^\dagger$.

From (29), we have

$$\begin{aligned}\|y_k-z\|^2 &\le \iota_k\|G_kGx_k-z\|^2+(1-\iota_k)\|V_kGx_k-z\|^2\\ &\le \iota_k[(1-\varpi_k\tau)\|x_k-z\|+\|(I-\varpi_kF)z-z\|]^2+(1-\iota_k)\|x_k-z\|^2\\ &\le \iota_k[(1-\varpi_k\tau)\|x_k-z\|^2+\varpi_k\frac{\|F(z)\|^2}{\tau}]+(1-\iota_k)\|x_k-z\|^2\\ &\le \|x_k-z\|^2+\varpi_k\frac{\|F(z)\|^2}{\tau}.\end{aligned}$$

By Lemma 2.2, we derive

$$\begin{aligned}\|x_{k+1}-z\|^2 &\le (1-\tau_k)^2\|y_k-z\|^2+2\tau_k\langle f(x_k)-z,J(x_{k+1}-z)\rangle\\ &\le (1-\tau_k)^2\|x_k-z\|^2+2\varrho\tau_k\|x_k-z\|\|x_{k+1}-z\|+2\tau_k\langle f(z)-z,J(x_{k+1}-z)\rangle+\varpi_k\frac{\|F(z)\|^2}{\tau}\\ &\le (1-\tau_k)^2\|x_k-z\|^2+\varrho\tau_k(\|x_k-z\|^2+\|x_{k+1}-z\|^2)+2\tau_k\langle f(z)-z,J(x_{k+1}-z)\rangle+\varpi_k\frac{\|F(z)\|^2}{\tau}.\end{aligned}$$

It follows that, for all $k \geq k_0$, we have

$$\begin{aligned}\|x_{k+1}-z\|^2 &\leq \frac{1-(2-\varrho)\tau_k+\tau_k^2}{1-\varrho\tau_k}\|x_k-z\|^2+\frac{2\tau_k}{1-\varrho\tau_k}\langle f(z)-z, J(x_{k+1}-z)\rangle+\frac{\varpi_k}{1-\varrho\tau_k}\frac{\|F(z)\|^2}{\tau}\\ &\leq (1-\frac{(1-\varrho)\tau_k}{1-\varrho\tau_k})\|x_k-z\|^2+\frac{2\tau_k}{1-\varrho\tau_k}\langle f(z)-z, J(x_{k+1}-z)\rangle+\frac{\varpi_k}{1-\varrho\tau_k}\frac{\|F(z)\|^2}{\tau}.\end{aligned}$$

Set $\varsigma_k = \frac{(1-\varrho)\tau_k}{1-\varrho\tau_k}$ and $\nu_k = \frac{2}{1-\varrho}\langle f(z)-z, J(x_{k+1}-z)\rangle$. Then,

$$\|x_{k+1}-z\|^2 \leq (1-\varsigma_k)\|x_k-z\|^2+\varsigma_k\nu_k+\sigma_k, \quad \forall k \geq k_0, \tag{38}$$

where $\sigma_k = \frac{\varpi_k}{1-\varrho\tau_k}\frac{\|F(z)\|^2}{\tau}$. Applying Lemma 2.14 to (38), we conclude that $x_k \to z$.

(ii) X has a weakly continuous duality map J_φ with gauge φ. By Lemma 2.6, $X^\dagger$ solves the VI

$$\langle (I-f)X^\dagger, J_\varphi(X^\dagger-p)\rangle \leq 0, \quad \forall p \in \mathcal{F}. \tag{39}$$

Repeating the same arguments as those of (24) in the proof of Theorem 3.1, we obtain

$$\limsup_{k\to\infty}\langle f(z)-z, J_\varphi(x_k-z)\rangle \leq 0, \tag{40}$$

where $z = X^\dagger$.

Note that

$$\begin{aligned}\Phi(\|y_k-z\|) &= \Phi(\|\iota_k(G_kGx_k-G_kz)+(1-\iota_k)(V_kGx_k-z)+\iota_k(G_kz-z)\|)\\ &\leq \Phi(\|\iota_k(G_kGx_k-G_kz)+(1-\iota_k)(V_kGx_k-z)\|)+\iota_k\langle G_kz-z, J_\varphi(y_k-z)\rangle\\ &\leq \Phi(\iota_k(1-\varpi_k\tau)\|x_k-z\|+(1-\iota_k)\|x_k-z\|)+\iota_k\|(I-\varpi_kF)z-z\|\varphi(\|y_k-z\|)\\ &\leq \Phi(\|x_k-z\|)+\iota_k\varpi_k\|F(z)\|\varphi(\|y_k-z\|).\end{aligned}$$

and hence

$$\begin{aligned}\Phi(\|x_{k+1}-z\|) &= \Phi(\|\tau_k(f(y_k)-f(z))+(1-\tau_k)(y_k-z)+\tau_k(f(x_k)-f(y_k))+\tau_k(f(z)-z)\|)\\ &\leq \Phi(\|\tau_k(f(y_k)-f(z))+(1-\tau_k)(y_k-z)\|)+\tau_k\langle f(x_k)-f(y_k), J_\varphi(x_{k+1}-z)\rangle\\ &\quad +\tau_k\langle f(z)-z, J_\varphi(x_{k+1}-z)\rangle\\ &\leq (1-(1-\varrho)\tau_k)[\Phi(\|x_k-z\|)+\iota_k\varpi_k\|F(z)\|\varphi(\|y_k-z\|)]+\tau_k\varrho\|x_k-y_k\|\varphi(\|x_{k+1}-z\|)\\ &\quad +\tau_k\langle f(z)-z, J_\varphi(x_{k+1}-z)\rangle\\ &\leq (1-(1-\varrho)\tau_k)\Phi(\|x_k-z\|)+\tau_k\varrho\|x_k-y_k\|\varphi(\|x_{k+1}-z\|)+\tau_k\langle f(z)-z, J_\varphi(x_{k+1}-z)\rangle\\ &\quad +\varpi_k\|F(z)\|\varphi(\|y_k-z\|).\end{aligned}$$

Set $\varsigma_k = (1-\varrho)\tau_k$ and $\nu_k = \frac{\varrho}{1-\varrho}\|x_k-y_k\|\varphi(\|x_{k+1}-z\|)+\frac{1}{1-\varrho}\langle f(z)-z, J_\varphi(x_{k+1}-z)\rangle$, $\forall k \geq 1$. Then,

$$\|x_{k+1}-z\|^2 \leq (1-\varsigma_k)\|x_k-z\|^2+\varsigma_k\nu_k+\sigma_k, \quad \forall k \geq 1, \tag{41}$$

where $\sigma_k = \varpi_k\|F(z)\|\varphi(\|y_k-z\|)$.

Applying Lemma 2.14 to (41), we conclude that $\lim_{k\to\infty}\Phi(\|x_k-z\|) = 0$, which implies that $\lim_{k\to\infty}\|x_k-z\| = 0$. This completes the proof. □

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References

[1] A. S. M. Alofi, A. Latif, A. E. Al-Mazrooei, J. C. Yao, *Composite viscosity iterative methods for general systems of variational inequalities and fixed point problem in Hilbert sSpaces*, J. Nonlinear Convex Anal., **17**(2016), 669–682.

[2] Q. H. Ansari, J. C. Yao, *Systems of generalized variational inequalities and their applications*, Appl. Anal., **76**(2000), 203–217.

[3] K. Aoyama, H. Iiduka, W. Takahashi, *Weak convergence of an iterative sequence for accretive operators in Banach spaces*, Fixed Point Theory Appl., **2006**(2006:35390), 13 pages.

[4] F. E. Browder, W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl., **20**(1967), 197–228.

[5] N. Buong, N.T.H. Phuong, *Strong convergence to solutions for a class of variational inequalities in Banach spaces by implicit iteration methods*, J. Optim. Theory Appl., **159**(2013), 399–411.

[6] L. C. Ceng, Q. H. Ansari, J. C. Yao, *Mann-type steepest-descent and modified steepest-descent methods for variational inequalities in Banach spaces*, Numer. Funct. Anal. Optim., **29(9-10)**, 987–1033.

[7] L. C. Ceng, H. Gupta, Q. H. Ansari, *Implicit and explicit algorithms for a system of nonlinear variational inequalities in Banach spaces*, J. Nonlinear Convex Anal., **16** (2015), 1–20.

[8] L. C. Ceng, A. Latif, A. E. Al-Mazrooei, *Hybrid viscosity methods for equilibrium problems, variational inequalities, and fixed point problems, Applicable Analysis*, **95**(2016), 1088–1117.

[9] L. C. Ceng, A. Latif, A. E. Al-Mazrooei, *Composite viscosity methods for common solutions of general mixed equilibrium problem, variational inequalities and common fixed points*, J. Inequal. Appl., **2015**(2015), Art. ID 217.

[10] L. C. Ceng, A. Petrusel, J. C. Yao, *Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of Lipschitz pseudocontractive mappings*, J. Math. Inequal., **1**(2007), 243–258.

[11] L. C. Ceng, A. Petrusel, J. C. Yao, Y. Yao, *Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces*, Fixed Point Theory, **19**(2018), 487–502.

[12] L. C. Ceng, A. Petrusel, J. C. Yao, Y. Yao, *Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions*, Fixed Point Theory, **20**(2019), 113–134.

[13] L. C. Ceng, C. Y. Wang, J. C. Yao, *Strong convergence theorems by a related extragradient method for a general system of variational inequalities*, Math. Meth. Oper. Res., **67**(2008),375–390.

[14] L. C. Ceng, N. C. Wong, J. C. Yao, *Strong and weak convergence theorems for an infinite family of nonexpansive mappings and applications*, Fixed Point Theory Appl., **2012**, (2012:117), 21 pages.

[15] Y. Censor, A. Gibali, S. Reich, *Two extensions of Korpelevich's extragradient method for solving the variational inequality problem in Euclidean space*, Technical Report, (2010).

[16] R. D. Chen, Z. C. Zhu, *Viscosity approximation of fixed points of nonexpansive and m-accretive operators*, Fixed Point Theory Appl. (2006), 10 pages.

[17] S. Y. Cho, X. Qin, J. C. Yao, Y. Yao, *Viscosity approximation splitting methods for monotone and nonexpansive operators in Hilbert spaces*, J. Nonlinear Convex Anal., **19**(2018), 251–264.

[18] G. M. Korpelevich, *An extragradient method for finding saddle points and for other problems*, Ekonom. Mat. Metody., **12**(1976), 747–756.

[19] A. Latif, A. S. M. Alofi, A. E. Al-Mazrooei, J. C Yao, *General composite iterative methods for general systems of variational inequalities*, Fixed Point Theory, **19**(2018), 287–300.

[20] T. C. Lim, H. K. Xu, *Fixed point theorems for asymptotically nonexpansive mappings*, Nonlinear. Anal., **22**(1994),1345–1355.

[21] X. L. Qin, Y. J. Cho, J. I. Kang, S. M. Kang, *Strong convergence theorems for an infinite family of nonexpansive mappings in Banach spaces*, J. Comput. Appl. Math., **230**(2009), 121–127.

[22] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **67**(1979) 274–276.

[23] M. V. Solodov, B. F. Svaiter, *A new projection method for variational inequality problems*, SIAM J. Control Optim., **37**(1999),765–776.

[24] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl., **305**(2005), 227–239.

[25] K. K. Tan, H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl., **178**(1993), 301–308.

[26] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16**(1991), 1127–1138.

[27] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66**(2002), 240–256.

[28] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298**(2004), 279–291.

[29] Y. Yao, R. Chen, H. K. Xu, *Schemes for finding minimum-norm solutions of variational inequalities*, Nonlinear Anal., **72**(2010), 3447–3456.

[30] Y. Yao, Y. C. Liou, S. M. Kang, *Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method*, Comput. Math. Appl., **59**(2010), 3472–3480.

[31] Y. Yao, Y. C. Liou, J. C. Yao, *Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations*, J. Nonlinear Sci. Appl., **10**(2017), 843–854.

[32] Y. Yao, M. Postolache, Y.C. Liou, Z.S. Yao, *Construction algorithms for a class of monotone variational inequalities*, Optim. Lett., **10**(2016), 1519–1528.

[33] Y. Yao, M. Postolache, J. C. Yao, *An iterative algorithm for solving the generalized variational inequalities and fixed points problems*, Mathematics, **7**(2019), Art. No. 61, doi:10.3390/math7010061.

[34] Y. Yao, M. Postolache, J.C. Yao, *Iterative algorithms for the generalized variational inequalities*, U.P.B. Sci. Bull., Series A, in press.

[35] Y. Yao, X. Qin, J. C. Yao, *Projection methods for firmly type nonexpansive operators*, J. Nonlinear Convex Anal., **19**(2018), 407–415.

[36] Y. Yao, N. Shahzad, *Strong convergence of a proximal point algorithm with general errors*, Optim. Lett. **6**(2012), 621–628.

[37] H. Zegeye, N. Shahzad, Y. Yao, *Minimum-norm solution of variational inequality and fixed point problem in Banach spaces*, Optim., **64**(2015), 453–471.

[38] L. C. Zeng, J. C. Yao, *Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings*, Nonlinear Anal., **64**(2006), 2507–2515.

[39] C. Zhang, Z. Zhu, Y. Yao, Q. Liu, *Homotopy method for solving mathematical programs with bounded box-constrained variational inequalities*, Optim., **68**(2019), 2293–2312.