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# Centralizing b-Generalized Derivations on Multilinear Polynomials

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**Abstract.** Let $R$ be a prime ring of characteristic different from 2 and $F$ a $b$-generalized derivation on $R$. Let $U$ be Utumi quotient ring of $R$ with extended centroid $C$ and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $C$ which is not central valued on $R$. Suppose that $d$ is a non zero derivation on $R$ such that

$$d([F(f(r)), f(r)]) \in C$$

for all $r = (r_1, \ldots, r_n) \in R^n$; then one of the following holds:

(1) there exist $a \in U$, $\lambda \in C$ such that $F(x) = ax + \lambda x + xa$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$,

(2) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$.

## 1. Introduction

Throughout the paper $R$ denotes an associative ring with center $Z(R)$. The Utumi quotient ring of a prime ring $R$ is denoted by $U$. The definition and axiomatic formulation of Utumi quotient ring $U$ can be found in [11] and [5]. We notice that $U$ is a prime ring with unity and the center of $U$ is called the extended centroid of $R$, denoted by $C$. For $x, y \in R$, the commutator of $x$ and $y$ is equal to $xy - yx$ and it is denoted by $[x, y]$. Sometimes commutator of $x$ and $y$ is called Lie product of $x$ and $y$. Let $S \subseteq R$. A function $f$ on $R$ is called a centralizing (or commuting) function on $S$ if $[f(s), s] \in Z(R)$ (or $[f(s), s] = 0$) for all $s \in S$. In this direction, Divinsky [17] studied the commuting automorphism on rings. More precisely, he proved that a simple artinian ring is commutative if it has a commuting automorphism different from the identity mapping. Further, Posner [9] studied the centralizing derivations on prime rings. More precisely, he proved that there does not exist any non zero centralizing derivation on non commutative prime ring. This was the starting point for the research by several authors. By a derivation of $R$, we mean an additive mapping $d$ on $R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $a \in R$, define a mapping $f$ on $R$ such that $f(x) = [a, x]$ for all $x \in R$. Here, we notice that $f$ is a derivation on $R$. This kind of derivations is called an inner derivation induced by an element $a$. A derivation is called outer if it not inner.

Brešar [13] extended the Posner's [9] result by taking two derivations and proved that if $d$ and $\delta$ are two derivations of $R$ with at least one derivation is non zero, such that $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$, then $R$ is

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commutative. Latter on, many mathematicians extended these results on some appropriate subsets of prime ring $R$. A natural question will arise that what will happen if we replace $x$ with multilinear polynomial in Posner's theorem [9] as well as Brešar's theorem [13]. The definition of multilinear polynomials is given below.

Let $\mathbb{Z}\langle X\rangle$ be the free algebra on $X$ over $\mathbb{Z}$, where $X = \{x_1, x_2, \ldots\}$ is a countable set. Let $f = f(x_1, \ldots, x_n) \in \mathbb{Z}\langle X\rangle$ be a polynomial such that at least one of its monomials of highest degree has a coefficient equal to 1. Let $R$ be a nonempty subset of a ring $A$. We say that $f$ is a polynomial identity on $R$ if $f(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$.

**Definition 1.1.** *A polynomial $f = f(x_1, \ldots, x_n) \in \mathbb{Z}\langle X\rangle$ is said to be multilinear if every $x_i, 1 \leq i \leq n$, appears exactly once in each of the monomials of $f$.*

The answer of above question was given by Lee and Shiue [21] and they proved that if $R$ is a prime ring, $f(x_1, \ldots, x_n)$ a polynomial over $C$ which is not central valued on $R$, $d$ and $g$ are two derivations of $R$ such that

$$d(f(x_1, \ldots, x_n))f(x_1 \ldots, x_n) - f(x_1, \ldots, x_n)g(f(x_1 \ldots, x_n)) \in C$$

for all $x_1, \ldots, x_n \in R$, then either $d = 0 = g$ or $d = -gg$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$, except when $\mathrm{char}(R) = 2$ and $\dim_C(RC) = 4$.

It is natural to ask that what will happen if the derivations are replaced by generalized derivations. The notion of generalized derivation introduced by Brešar in [12] which is a generalization of derivation. The definition of generalized derivation is given below.

**Definition 1.2.** *Let $R$ be a ring and $F$ be an additive mapping on $R$. Then mapping $F$ on $R$ is said to be a generalized derivation if there exists a derivation $d$ on $R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.*

Here, we notice that every derivation is a generalized derivation but converse is not true in general. The following example confirms our claim. Simplest example is an identity function on $R$. Here, we shall give non trivial example.

**Example 1.3.** *Let $\mathbb{Z}$ be the set of integers. Suppose $R = \left\{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} | x, y, z \in \mathbb{Z}\right\}$. Define $d : R \longrightarrow R$ as $d\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Then $d$ is a derivation on $R$. Define a mapping $F$ on $R$ such that $F\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix}$. Then $F$ is a generalized derivation associated with a non zero derivation $d$ on $R$. Here, we see that $F$ is not a derivation on $R$.*

Note that if $R$ is a prime or a semiprime ring then the derivation $d$ is uniquely determined by $F$ and $d$ is called the associated derivation of $F$.

Next, Argac and De Filippis [16] gave the partial generalization of Posner's theorem [9] that is they proved for commuting case only. More precisely, they describe the structure of additive mapping satisfying the identity $F(x)x - xG(x) = 0$ for all $x \in f(R)$, where $f$ is a multinear polynomial over extended centroid $C$ of Utumi ring of quotient $U$ and $G$, $G$ are two generalized derivations on prime ring $R$. In 2018, Tiwari [18] studied the commuting generalized derivations on prime ring, which is generalization of the work of Argac and De Filippis [16].

In 2016, Dhara et. al. in [1], generalize the Posner's [9] result. More precisely, they proved the following. Let $R$ be a non commutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ which is not central valued on $R$. Suppose that $F$ and $G$ are two generalized derivations of $R$ and $d$ is a non zero derivation on $R$ such that $d\{F(u)u - uG(u)\} = 0$ for all $u \in f(R) = \{f(r_1, \ldots, r_n) \mid r_i \in R\}$, then one of the following holds:

(i) there exist $\lambda \in C$ and $a, b, q, c \in U$ such that $F(x) = (a + \lambda)x + xb$, $G(x) = bx + xq$, $d(x) = [c, x]$ for all $x \in R$ with $[c, a - q] = 0$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$,

(ii) there exists $a \in U$ such that $F(x) = xa$ and $G(x) = ax$ for all $x \in R$,

(iii) there exist $\lambda \in C$, $a, b, c \in U$ such that $F(x) = x(\lambda + a) - bx$, $G(x) = ax + xb$ and $d(x) = [c, x]$ for all $x \in R$ with $b + \alpha c \in C$ for some $\alpha \in C$,

(iv) $R$ satisfies $S_4$ and there exist $a, b \in U$, $\lambda \in C$ such that $F(x) = x(\lambda + a) - xb$ and $G(x) = ax + xb$ for all $x \in R$,

(v) there exist $a, b, c \in U$ and $\delta$ a derivation of $R$ such that $F(x) = ax + xb - \delta(x)$, $G(x) = bx + \delta(x)$ and $d(x) = [c, x]$ for all $x \in R$ with $[c, a] = 0$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$.

More recently, in 2018, Dhara [3], studied the identity $F(u)G(u) = H(u^2)$ for all $u = f(r_1, \ldots, r_n) \in f(R)$, where $F$, $G$ and $H$ are three generalized derivations on prime ring $R$ and gave the complete description of these additive mappings.

## 2. *b*-generalized derivation

One more generalization of derivation and generalized derivations is $b$-generalized derivation. The definition of $b$-generalized derivation is given below. Let $R$ be a semiprime ring and $Q$ be its Martindale ring of quotient. Let $b \in Q$.

**Definition 2.1.** *A mapping $F : R \to Q$ is called a $b$-generalized derivation of $R$ if $F(x + y) = F(x) + F(y)$ and $F(xy) = F(x)y + bxd(y)$ for all $x, y \in R$, where $d : R \to Q$ is an additive map.*

Note that, in [14] Košan and Lee, proved that if $R$ is a prime ring and $b \neq 0$, then the associated map $d$ must be a derivation of $R$. Here, we see that a 1-generalized derivation is a generalized derivation. For some $a, b, c \in Q$, define a map $F : R \to Q$ such that $F(x) = ax + bxc$ for all $x \in R$. This is a $b$-generalized derivation which is called an inner $b$-generalized derivation. A $b$-generalized derivation is also an extension of generalized $\alpha$-derivation, provided associated automorphism is an inner. We will see the below.

Let $\alpha$ is an automorphism on $R$. Then $\alpha$ is said to be an inner automorphism of $R$, if there exists an invertible element $p \in Q$ such that $\alpha(x) = pxp^{-1}$ for all $x \in R$ otherwise it is called outer automorphism. Before going to define generalized $\alpha$-derivation, first we shall define $\alpha$-derivation. An additive mapping on $R$ is said to an $\alpha$-derivation on $R$ if $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$, where $\alpha$ is called associated automorphism on $R$. An additive mapping $F$ on $R$ is called generalized $\alpha$-derivation if there exists an $\alpha$-derivation on $R$ such that $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x, y \in R$. Let 1 be an identity mapping on $R$. Then generalized 1-derivation becomes a generalized derivation on $R$.

Let $\alpha$ is an inner automorphism on $R$ that is there exists invertible element $p \in Q$ such that $\alpha(x) = pxp^{-1}$ for all $x \in R$. Now by definition of generalized $\alpha$-derivation, we have $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x, y \in R$. If $d$ is an inner $\alpha$-derivation, then we know that $d(x) = ax - \alpha(x)a = ax - pxp^{-1}a$. Thus we have $F(xy) = F(x)y + pxp^{-1}(ay - pyp^{-1}a)$, which implies that $F(xy) = F(x)y + pxp^{-1}ay - pxp^{-1}pyp^{-1}a = F(x)y + px\{p^{-1}ay - yp^{-1}a\}$. This gives that $F(xy) = F(x)y + pxd(y)$, where $d(y) = [p^{-1}a, y]$ for all $y \in R$ that is $d$ is an inner derivation induced by $p^{-1}a$. This implies that it is a $p$-generalized derivation on $R$. Thus, if $\alpha$ is an inner automorphism on $R$, then every generalized $\alpha$-derivation on $R$ is a $b$-generalized derivation.

Recently, Liu [7] generalized the result of Posner's [9] by considering generalized $b$-derivation with engel conditions on prime ring $R$.

More recently, Dhara [2] studied the centralizing $b$-generalized derivations on prime ring with multiliear polynomial over $C$. More precisely, prove the following.

Let $R$ be a prime ring of characteristic different from 2 and $F$ be a $b$-generalized derivation on $R$. Let $U$ be Utumi quotient ring of $R$ with extended centroid $C$ and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $C$ which is not central valued on $R$. Suppose that $d$ is a non zero derivation on $R$ such that

$$d([F(f(r)), f(r)]) = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(1) there exist $a \in U$, $\lambda \in C$ such that $F(x) = ax + \lambda x + xa$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$,

(2) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$.

It is natural to ask what will be the structure of these additive mappings when identity studied by Dhara [2] is in the center. Our main motive is to give the answer of this question. The statement of our main theorem is the following.

**Main Theorem:** Let $R$ be a prime ring of characteristic different from 2 and $F$ be a $b$-generalized derivation on $R$. Let $U$ be Utumi quotient ring of $R$ with extended centroid $C$ and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $C$ which is not central valued on $R$. Suppose that $d$ is a non zero derivation on $R$ such that

$$d([F(f(r)), f(r)]) \in Z(R)$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(1) there exist $a \in U$, $\lambda \in C$ such that $F(x) = ax + \lambda x + xa$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$,

(2) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$.

The following corollaries are particular cases of our Main Theorem.

**Corollary 2.2.** *[23, Theorem] Let $K$ be a commutative ring with unity, $R$ be a prime algebra over $K$ and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $K$, not central valued on $R$. Suppose that $d$ is a non zero derivation and $F$ is a non zero generalized derivation of $R$ such that*

$$d([F(f(r)), f(r)]) = 0$$

*for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

*(1) there exist $a \in U$, $\lambda \in C$ such that $F(x) = ax + \lambda x + xa$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$,*

*(2) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$.*

Similarly, we have the following corollary.

**Corollary 2.3.** *Let $K$ be a commutative ring with unity, $R$ be a prime algebra over $K$ and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $K$, not central valued on $R$. Suppose that $d$ is a non zero derivation and $F$ is a non zero generalized derivation of $R$ such that*

$$[F(f(r)), f(r)] \in Z(R)$$

*for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

*(1) there exist $a \in U$, $\lambda \in C$ such that $F(x) = ax + \lambda x + xa$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$,*

*(2) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$.*

As an application of above theorem, we have the following corollary which is a generalization of particular result of Filippis [24].

**Corollary 2.4.** *Let $R$ be a non commutative prime ring of characteristic different from 2, $Q$ be its maximal right ring of quotients and $C$ be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a non central multilinear polynomial over $C$, $F$ a $b$-generalized derivation of $R$ and $d$ is a non zero derivation of $R$ such that*

$$[F(f(r)), f(r)] \in Z(R)$$

*for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

*(1) there exist $a \in U$, $\lambda \in C$ such that $F(x) = ax + \lambda x + xa$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$,*

*(2) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$.*

If we take $F = d$, a derivation, then we get a famous Posner's theorem [9]. That is we have following.

**Corollary 2.5.** *[9, Theorem 2] Let $R$ be a prime ring and $d$ is a derivation on $R$ such that $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or $R$ is commutative.*

## 3. Preliminaries and Notations

We will use the following notation:

$$f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \gamma_\sigma x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)},$$

where $\gamma_\sigma \in C$ and $S_n$ is the symmetric group of degree $n$.

Further, we will frequently use some important theory of generalized polynomial identities and differential identities. We recall some of the remarks.

**Remark 3.1.** *If $I$ is a two-sided ideal of $R$, then $R$, $I$ and $U$ satisfy the same differential identities ([20]).*

**Remark 3.2.** *If $I$ is a two-sided ideal of $R$, then $R$, $I$ and $U$ satisfies the same generalized polynomial identities with coefficients in $U$ ([5]).*

**Remark 3.3.** *(Kharchenko [25, Theorem 2] Let $R$ be a prime ring, $d$ a nonzero derivation on $R$ and $I$ a nonzero ideal of $R$. If $I$ satisfies the differential identity*

$$f(r_1, \ldots, r_n, d(r_1), \ldots, d(r_n)) = 0$$

*for any $r_1, \ldots, r_n \in I$, then either*

*(i) $I$ satisfies the generalized polynomial identity*

$$f(r_1, \ldots, r_n, x_1, \ldots, x_n) = 0$$

*or*

*(ii) $d$ is $Q$-inner i.e., for some $q \in Q$, $d(x) = [q, x]$ and $I$ satisfies the generalized polynomial identity*

$$f(r_1, \ldots, r_n, [q, r_1], \ldots, [q, r_n]) = 0.$$

**Lemma 3.4.** *[4, Theorem 1] Let $R$ be a prime ring, $d$ and $\delta$ two non zero derivations of $R$ and $\rho$ a right ideal of $R$ such that $\delta d([\rho, \rho]) = 0$ and $[\rho, \rho]\rho = 0$. Then either $\delta = \alpha d$ for some $\alpha \in C$ and $d^2 = 0$, or there exist $p, q \in Q$ such that $\delta = ad(q)$, $d = ad(p)$ with $p\rho = 0 = q\rho$ and $pq = 0$, except when $char(R) = 2$ and $\rho C = eRC$ for some idempotent $e$ in the socle of $RC$ such that $dim_C eRCe = 4$.*

**Lemma 3.5.** *[4, Theorem 2] If $R$ is a prime ring, $d$ and $\delta$ are two non zero derivations of $R$, and $\rho$ a right ideal of $R$ such that $0 \neq \delta d([\rho, \rho]) \subseteq Z(R)$ and $[\rho, \rho]\rho = 0$. Then $char(R) = 2$ and $dim_C RC = 4$.*

The following Lemma is a particular case of Lemma 3.4 and Lemma 3.5.

**Lemma 3.6.** *Let $R$ be a non commutative prime ring with characteristic different from 2 and $I$ a non zero ideal of $R$. Suppose that $d$ and $\delta$ are two derivations on $R$ such that $d(\delta[x, y]) \in Z(R)$ for all $x, y \in I$, then either $d = 0$ or $\delta = 0$.*

## 4. *F* is an inner *b*-generalized derivation and *d* is an inner derivation

In this section, we study the situation when $F$ is $b$-inner generalized derivation and $d$ is an inner derivation of $R$. Let $F(x) = ax + bxu$ and $d(x) = [P, x]$ for all $x \in R$, for some $a, b, u, P \in U$. Then we prove the following proposition:

**Proposition 4.1.** *Let $R$ be a prime ring of characteristic different from 2, $U$ be Utumi ring of quotient of $R$, $C$ extended centroid of $R$, $F$ be an inner $b$-generalized derivation and $d$ be a non zero inner derivation, defined as $F(x) = ax + bxu$ and $d(x) = [P, x]$ for all $x \in R$ and for some $a, b, u, P \in U$. Let $f(x_1, \ldots, x_n)$ be a non central multilinear polynomial over $C$. If*

$$d([F(f(r)), f(r)]) \in C$$

*for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

*(1) there exist $a \in U$, $\lambda \in C$ such that $F(x) = ax + \lambda x + xa$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$,*

*(2) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$.*

For proof of this proposition, we need the following.

**Lemma 4.2.** *[23, Lemma 1] Let $C$ be an infinite field and $m \geq 2$. If $A_1, \ldots, A_k$ are non scalar matrices in $M_m(C)$ then there exists some invertible matrix $P \in M_m(C)$ such that each matrix $PA_1P^{-1}, \ldots, PA_kP^{-1}$ has all non zero entries.*

**Proposition 4.3.** *Let $R = M_k(C)$ be the ring of all $k \times k$ matrices over the infinite field $C$, where $k \geq 2$ with characteristic different from 2. Let $a, b, u, P, a', b', c' \in R$ such that*

$$a'x^2 + b'xux - Pxax - Pxbxu - ax^2P - bxuxP + xaxP + xbxc' \in Z(R)$$

*for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1, \ldots, x_n)$ in $R$, then either $b$ or $u$ or $P$ is central.*

*Proof.* By our assumption, $a'f(r)^2 + b'f(r)uf(r) - Pf(r)af(r) - Pf(r)bf(r)u - af(r)^2P - bf(r)uf(r)P + f(r)af(r)P + f(r)bf(r)c' \in Z(R)$ for all $r = (r_1, \ldots, r_n)$, where $r_1, \cdots, r_n \in R$. Hence it commutes with $f(r)$ for all $r = (r_1, \ldots, r_n)$, where $r_1, \cdots, r_n \in R$. Thus $R$ satisfies the generalized polynomial identity

$$\begin{aligned}
&\Big[a'f(r_1, \ldots, r_n)^2 + b'f(r_1, \ldots, r_n)uf(r_1, \ldots, r_n) - Pf(r_1, \ldots, r_n)af(r_1, \ldots, r_n) \\
&-Pf(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)u - af(r_1, \ldots, r_n)^2P - bf(r_1, \ldots, r_n)uf(r_1, \ldots, r_n)P \\
&+f(r_1, \ldots, r_n)af(r_1, \ldots, r_n)P + f(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)c', f(r_1, \ldots, r_n)\Big] = 0. \qquad (1)
\end{aligned}$$

We shall prove this result by contradiction. Suppose that $b \notin Z(R)$, $u \notin C$ and $P \notin Z(R)$. Then by Lemma 4.2 there exists a $C$-automorphism $\phi$ of $M_m(C)$ such that $\phi(b)$, $\phi(u)$ and $\phi(P)$ have all non zero entries. Clearly $\phi(b)$, $\phi(u)$, $\phi(P)$, $\phi(a)$, $\phi(a')$, $\phi(b')$ and $\phi(c')$ must satisfy the condition (1).

Here $e_{ij}$ denotes the matrix whose $(i, j)$-entry is 1 and rest entries are zero. Since $f(x_1, \ldots, x_n)$ is not central, by [20] (see also [22]), there exist $s_1, \ldots, s_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(s_1, \ldots, s_n) = \gamma e_{ij}$, with $i \neq j$. Moreover, since the set $\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in M_m(C)\}$ is invariant under the action of all $C$-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \ldots, r_n \in M_m(C)$ such that $f(r_1, \ldots, r_n) = e_{ij}$. Hence by (1) we have

$$\begin{aligned}
&\Big[\phi(a')e_{ij}^2 + \phi(b')e_{ij}\phi(u)e_{ij} - \phi(P)e_{ij}\phi(a)e_{ij} - \phi(P)e_{ij}\phi(b)e_{ij}\phi(u) - \phi(a)e_{ij}^2\phi(P) \\
&\quad -\phi(b)e_{ij}\phi(u)e_{ij}\phi(P) + e_{ij}\phi(a)e_{ij}\phi(P) + e_{ij}\phi(b)e_{ij}\phi(c'), e_{ij}\Big] = 0.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\Big[\phi(b')e_{ij}\phi(u)e_{ij} - \phi(P)e_{ij}\phi(a)e_{ij} - \phi(P)e_{ij}\phi(b)e_{ij}\phi(u) \\
&-\phi(b)e_{ij}\phi(u)e_{ij}\phi(P) + e_{ij}\phi(a)e_{ij}\phi(P) + e_{ij}\phi(b)e_{ij}\phi(c'), e_{ij}\Big] = 0. \qquad (2)
\end{aligned}$$

Left multiplying by $e_{ij}$ in (2), we obtain

$$-e_{ij}\phi(P)e_{ij}\phi(b)e_{ij}\phi(u)e_{ij} - e_{ij}\phi(b)e_{ij}\phi(u)e_{ij}\phi(P)e_{ij} = 0.$$

Thus we have $2\phi(P)_{ji}\phi(b)_{ji}\phi(u)_{ji}e_{ij} = 0$. Since char$(R) \neq 2$, it implies that $\phi(P)_{ji}\phi(b)_{ji}\phi(u)_{ji}e_{ij} = 0$. It gives that either $\phi(P)_{ji} = 0$ or $\phi(u)_{ji} = 0$ or $\phi(b)_{ji} = 0$, a contradiction, since $\phi(b)$, $\phi(u)$ and $\phi(P)$ have all non zero entries. Thus we conclude that either $\phi(b)$ or $\phi(u)$ or $\phi(P)$ is central. Since $\phi$ is an automorphism, hence it gives that either $b$ or $u$ or $P$ is central. $\square$

**Proposition 4.4.** *Let $R = M_m(C)$, $m \geq 2$, be the ring of all matrices over the field $C$ with characteristic different from 2 and $f(x_1, \ldots, x_n)$ a non central multilinear polynomial over $C$. Let $a, b, u, P, a', b', c' \in R$ such that*

$$a'x^2 + b'xux - Pxax - Pxbxu - ax^2P - bxuxP + xaxP + xbxc' \in Z(R)$$

*for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1, \ldots, x_n)$ in $R$, then either $b$ or $u$ or $P$ is central.*

*Proof.* The conclusions follow from Proposition 4.3 in the case of infinite field $C$. Now we assume that $C$ is a finite field. Suppose that $K$ is an infinite extension of the field of $C$. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central valued on $R$ if and only if it is central valued on $\overline{R}$. Suppose that the generalized polynomial $Q(r_1, \ldots, r_{n+1})$ such that

$$\begin{aligned} Q(r_1, \ldots, r_{n+1}) = \Big[&a'f(r_1, \ldots, r_n)^2 + b'f(r_1, \ldots, r_n)uf(r_1, \ldots, r_n) - Pf(r_1, \ldots, r_n)af(r_1, \ldots, r_n) \\ &-Pf(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)u - af(r_1, \ldots, r_n)^2P - bf(r_1, \ldots, r_n)uf(r_1, \ldots, r_n)P \\ &+f(r_1, \ldots, r_n)af(r_1, \ldots, r_n)P + f(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)c', r_{n+1}\Big] \end{aligned} \tag{3}$$

is a generalized polynomial identity for $R$.

Moreover, it is a multihomogeneous of multidegree $(2, \ldots, 2)$ in the indeterminates $r_1, \ldots, r_{n+1}$. Hence the complete linearization of $Q(r_1, \ldots, r_{n+1})$ is a multilinear generalized polynomial $\Theta(r_1, \ldots, r_{n+1}, x_1, \ldots, x_{n+1})$ in $2n + 2$ indeterminates, moreover

$$\Theta(r_1, \ldots, r_{n+1}, r_1, \ldots, r_{n+1}) = 2^{2n+2}Q(r_1, \ldots, r_{n+1}).$$

It is clear that the multilinear polynomial $\Theta(r_1, \ldots, r_{n+1}, x_1, \ldots, x_{n+1})$ is a generalized polynomial identity for both $R$ and $\overline{R}$. By assumption $char(R) \neq 2$ we obtain $Q(r_1, \ldots, r_{n+1}) = 0$ for all $r_1, \ldots, r_{n+1} \in \overline{R}$ and then conclusion follows from Proposition 4.3. □

**Lemma 4.5.** *Let $R$ be a prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$ and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Let $a, b, u, P, a', b', c' \in R$ such that*

$$a'x^2 + b'xux - Pxax - Pxbxu - ax^2P - bxuxP + xaxP + xbxc' \in Z(R)$$

*for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1, \ldots, x_n)$ in $R$, then either $b$ or $u$ or $P$ is central.*

*Proof.* We shall prove this by contradiction. Suppose that none of $b$, $u$ and $P$ is not in $C$, that is $b \notin C$, $u \notin C$ and $P \notin C$. By hypothesis, we have

$$\begin{aligned} h(x_1, \ldots, x_n) \quad = \quad & \Big[a'f(x_1, \ldots, x_n)^2 + b'f(x_1, \ldots, x_n)uf(x_1, \ldots, x_n) - Pf(x_1, \ldots, x_n)af(x_1, \ldots, x_n) \\ - \quad & Pf(x_1, \ldots, x_n)bf(x_1, \ldots, x_n)u - af(x_1, \ldots, x_n)^2P - bf(x_1, \ldots, x_n)uf(x_1, \ldots, x_n)P \\ + \quad & f(x_1, \ldots, x_n)af(x_1, \ldots, x_n)P + f(x_1, \ldots, x_n)bf(x_1, \ldots, x_n)c', f(x_1, \ldots, x_n)\Big] \end{aligned} \tag{4}$$

for all $x_1, \ldots, x_n \in R$. Since $R$ and $U$ satisfy same generalized polynomial identity (GPI) (see [5]), $U$ satisfies $h(x_1, \ldots, x_n) = 0_T$. Suppose that $h(x_1, \ldots, x_n)$ is a trivial GPI for $U$. Let $T = U *_C C\{x_1, \ldots, x_n\}$, the free product of $U$ and $C\{x_1, \ldots, x_n\}$, the free $C$-algebra in non commuting indeterminates $x_1, \ldots, x_n$. Then, $h(x_1, \ldots, x_n)$ is zero element in $T = U *_C C\{x_1, \ldots, x_n\}$. Since neither $b$ nor $u$ nor $P$ is central, hence the term

$$\Big[ - Pf(x_1, \ldots, x_n)bf(x_1, \ldots, x_n)u - bf(x_1, \ldots, x_n)uf(x_1, \ldots, x_n)P, f(x_1, \ldots, x_n)\Big]$$

appears non trivially in $h(x_1, \ldots, x_n)$. Thus $U$ satisfies

$$\begin{aligned} &Pf(x_1, \ldots, x_n)bf(x_1, \ldots, x_n)uf(x_1, \ldots, x_n) + bf(x_1, \ldots, x_n)uf(x_1, \ldots, x_n)Pf(x_1, \ldots, x_n) \\ &-f(x_1, \ldots, x_n)Pf(x_1, \ldots, x_n)bf(x_1, \ldots, x_n)u - f(x_1, \ldots, x_n)bf(x_1, \ldots, x_n)uf(x_1, \ldots, x_n)P = 0_T. \end{aligned}$$

Since $P \notin C$, hence it implies that

$$Pf(x_1,\dots,x_n)bf(x_1,\dots,x_n)uf(x_1,\dots,x_n) = 0.$$

This gives a contradiction that is we have either $P \in C$ or $u \in C$ or $b \in C$.

Next, suppose that $h(x_1,\dots,x_n)$ is a non trivial GPI for $U$. In case $C$ is infinite, we have $h(x_1,\dots,x_n) = 0$ for all $x_1,\dots,x_n \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of $C$. Since both $U$ and $U \otimes_C \overline{C}$ are prime and centrally closed [19, Theorems 2.5 and 3.5], we may replace $R$ by $U$ or $U \otimes_C \overline{C}$ according to $C$ finite or infinite. Then $R$ is centrally closed over $C$ and $h(x_1,\dots,x_n) = 0$ for all $x_1,\dots,x_n \in R$. By Martindale's theorem [26], $R$ is then a primitive ring with non zero socle and with $C$ as its associated division ring. Then, by Jacobson's theorem [15, p.75], $R$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$.

Assume first that $V$ is finite dimensional over $C$, that is, $\dim_C V = m$. By density of $R$, we have $R \cong M_m(C)$. Since $f(r_1,\dots,r_n)$ is not central valued on $R$, $R$ must be non commutative and so $m \geq 2$. In this case, by Proposition 4.4, we get that either $P \in C$ or $b \in C$ or $u \in C$, a contradiction.

Next we suppose that $V$ is infinite dimensional over $C$. By Martindale's theorem [26, Theorem 3], for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since we have assumed that neither $P$ nor $b$ nor $u$ in the center. Then there exist $h_1, h_2, h_3 \in soc(R)$ such that $[P, h_1] \neq 0$, $[b, h_2] \neq 0$ and $[u, h_3] \neq 0$. By Litoff's Theorem [8], there exists an idempotent $e \in soc(R)$ such that $Ph_1, h_1P, bh_2, h_2b, uh_3, h_3u, h_1, h_2, h_3 \in eRe$. Since $R$ satisfies generalized identity

$$\begin{aligned} e\Big\{\Big[&a'f(ex_1e,\dots,ex_ne)^2 + b'f(ex_1e,\dots,ex_ne)uf(ex_1e,\dots,ex_ne) - Pf(ex_1e,\dots,ex_ne)af(ex_1e,\dots,ex_ne)\\ &-Pf(ex_1e,\dots,ex_ne)bf(ex_1e,\dots,ex_ne)u - af(ex_1e,\dots,ex_ne)^2P - bf(ex_1e,\dots,ex_ne)uf(ex_1e,\dots,ex_ne)P\\ &+f(ex_1e,\dots,ex_ne)af(ex_1e,\dots,ex_ne)P + f(ex_1e,\dots,ex_ne)bf(ex_1e,\dots,ex_ne)c', f(ex_1e,\dots,ex_ne)\Big]\Big\}e. \end{aligned}$$

That is

$$\begin{aligned} e\Big\{&a'f(ex_1e,\dots,ex_ne)^3 + b'f(ex_1e,\dots,ex_ne)uf(ex_1e,\dots,ex_ne)^2 - Pf(ex_1e,\dots,ex_ne)af(ex_1e,\dots,ex_ne)^2\\ &-Pf(ex_1e,\dots,ex_ne)bf(ex_1e,\dots,ex_ne)uf(ex_1e,\dots,ex_ne) - af(ex_1e,\dots,ex_ne)^2Pf(ex_1e,\dots,ex_ne)\\ &-bf(ex_1e,\dots,ex_ne)uf(ex_1e,\dots,ex_ne)Pf(ex_1e,\dots,ex_ne) + f(ex_1e,\dots,ex_ne)af(ex_1e,\dots,ex_ne)Pf(ex_1e,\dots,ex_ne)\\ &+f(ex_1e,\dots,ex_ne)bf(ex_1e,\dots,ex_ne)c'f(ex_1e,\dots,ex_ne) - f(ex_1e,\dots,ex_ne)a'f(ex_1e,\dots,ex_ne)^2\\ &-f(ex_1e,\dots,ex_ne)b'f(ex_1e,\dots,ex_ne)uf(ex_1e,\dots,ex_ne) + f(ex_1e,\dots,ex_ne)Pf(ex_1e,\dots,ex_ne)af(ex_1e,\dots,ex_ne)\\ &+f(ex_1e,\dots,ex_ne)Pf(ex_1e,\dots,ex_ne)bf(ex_1e,\dots,ex_ne)u + f(ex_1e,\dots,ex_ne)af(ex_1e,\dots,ex_ne)^2P\\ &+f(ex_1e,\dots,ex_ne)bf(ex_1e,\dots,ex_ne)uf(ex_1e,\dots,ex_ne)P - f(ex_1e,\dots,ex_ne)^2af(ex_1e,\dots,ex_ne)P\\ &-f(ex_1e,\dots,ex_ne)^2bf(ex_1e,\dots,ex_ne)c'\Big\}e. \end{aligned}$$

The subring $eRe$ satisfies

$$\begin{aligned} &ea'ef(x_1,\dots,x_n)^3 + eb'ef(x_1,\dots,x_n)euef(x_1,\dots,x_n)^2 - ePef(x_1,\dots,x_n)eaef(x_1,\dots,x_n)^2\\ &-ePef(x_1,\dots,x_n)ebef(x_1,\dots,x_n)euef(x_1,\dots,x_n) - eaef(x_1,\dots,x_n)^2ePef(x_1,\dots,x_n)\\ &-ebef(x_1,\dots,x_n)euef(x_1,\dots,x_n)ePef(x_1,\dots,x_n) + f(x_1,\dots,x_n)eaef(x_1,\dots,x_n)ePef(x_1,\dots,x_n)\\ &+f(x_1,\dots,x_n)ebef(x_1,\dots,x_n)ec'ef(x_1,\dots,x_n) - f(x_1,\dots,x_n)ea'ef(x_1,\dots,x_n)^2\\ &-f(x_1,\dots,x_n)eb'ef(x_1,\dots,x_n)euef(x_1,\dots,x_n) + f(x_1,\dots,x_n)ePef(x_1,\dots,x_n)eaef(x_1,\dots,x_n)\\ &+f(x_1,\dots,x_n)ePef(x_1,\dots,x_n)ebef(x_1,\dots,x_n)eue + f(x_1,\dots,x_n)eaef(x_1,\dots,x_n)^2ePe\\ &+f(x_1,\dots,x_n)ebef(x_1,\dots,x_n)euef(x_1,\dots,x_n)ePe - f(x_1,\dots,x_n)^2eaef(x_1,\dots,x_n)ePe\\ &-f(x_1,\dots,x_n)^2ebef(x_1,\dots,x_n)ec'e. \end{aligned}$$

This can be re-written as

$$\Big[ea'ef(x_1,\ldots,x_n)^2 + eb'ef(x_1,\ldots,x_n)euef(x_1,\ldots,x_n) - ePef(x_1,\ldots,x_n)eaef(x_1,\ldots,x_n)$$
$$-ePef(x_1,\ldots,x_n)ebef(x_1,\ldots,x_n)eue - eaef(x_1,\ldots,x_n)^2ePe - ebef(x_1,\ldots,x_n)euef(x_1,\ldots,x_n)ePe$$
$$+f(x_1,\ldots,x_n)eaef(x_1,\ldots,x_n)ePe + f(x_1,\ldots,x_n)ebef(x_1,\ldots,x_n)ec'e, f(x_1,\ldots,x_n)\Big].$$

Then by the above finite dimensional case, either $ePe$ or $ebe$ or $eue$ is central element of $eRe$. Thus either $Ph_1 = (ePe)h_1 = h_1ePe = h_1P$ or $bh_2 = (ebe)h_2 = h_2(ebe) = h_2b$ or $uh_3 = (eue)h_3 = h_3(eue) = h_3u$, a contradiction. □

**Lemma 4.6.** *Let $R$ be a prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$ and $f(x_1,\ldots,x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that for some $a, b, u, c, P, a' \in R$*

$$a'x^2 + Pxux - Px^2c - ax^2P - xuxP + x^2b \in Z(R)$$

*for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1,\ldots,x_n)$ in $R$, then either $P$ or $u$ is central.*

*Proof.* Suppose that $P \notin C$ and $u \notin C$. By hypothesis, we have

$$\begin{aligned} h(x_1,\ldots,x_n) &= \Big[a'f(x_1,\ldots,x_n)^2 + Pf(x_1,\ldots,x_n)uf(x_1,\ldots,x_n) \\ &- Pf(x_1,\ldots,x_n)^2c - af(x_1,\ldots,x_n)^2P \\ &- f(x_1,\ldots,x_n)uf(x_1,\ldots,x_n)P + f(x_1,\ldots,x_n)^2b, f(x_1,\ldots,x_n)\Big] \end{aligned} \tag{5}$$

for all $x_1,\ldots,x_n \in R$. Since $R$ and $U$ satisfy same generalized polynomial identity (GPI) (see [5]), $U$ satisfies $h(x_1,\ldots,x_n) = 0_T$. Suppose that $h(x_1,\ldots,x_n)$ is a trivial GPI for $U$. Let $T = U *_C C\{x_1,\ldots,x_n\}$, the free product of $U$ and $C\{x_1,\ldots,x_n\}$, the free $C$-algebra in non commuting indeterminates $x_1,\ldots,x_n$. Then, $h(x_1,\ldots,x_n)$ is zero element in $T = U *_C C\{x_1,\ldots,x_n\}$. Since neither $P$ nor $u$ is central, hence the term

$$\Big[Pf(x_1,\ldots,x_n)uf(x_1,\ldots,x_n) - f(x_1,\ldots,x_n)uf(x_1,\ldots,x_n)P, f(x_1,\ldots,x_n)\Big]$$

appears non trivially in $h(x_1,\ldots,x_n)$. Thus $U$ satisfies

$$Pf(x_1,\ldots,x_n)uf(x_1,\ldots,x_n)^2 - f(x_1,\ldots,x_n)uf(x_1,\ldots,x_n)Pf(x_1,\ldots,x_n)$$
$$-f(x_1,\ldots,x_n)Pf(x_1,\ldots,x_n)uf(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)^2uf(x_1,\ldots,x_n)P = 0_T.$$

Since $P \notin C$, $u \notin C$ hence it implies that $Pf(x_1,\ldots,x_n)uf(x_1,\ldots,x_n)^2 = 0$ is a trivial GPI for $U$. This gives $P = 0$ or $u = 0$, a contradiction.

Next, suppose that $h(x_1,\ldots,x_n)$ is a non trivial GPI for $U$. In case $C$ is infinite, we have $h(x_1,\ldots,x_n) = 0$ for all $x_1,\ldots,x_n \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of $C$. Since both $U$ and $U \otimes_C \overline{C}$ are prime and centrally closed [19, Theorems 2.5 and 3.5], we may replace $R$ by $U$ or $U \otimes_C \overline{C}$ according to $C$ finite or infinite. Then $R$ is centrally closed over $C$ and $h(x_1,\ldots,x_n) = 0$ for all $x_1,\ldots,x_n \in R$. By Martindale's theorem [26], $R$ is then a primitive ring with non zero socle and with $C$ as its associated division ring. Then, by Jacobson's theorem [15, p.75], $R$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$. Then we the following cases.

**Case-I:** If $V$ is finite dimensional over $C$, that is, $\dim_C V = m$. By density of $R$, we have $R \cong M_m(C)$. Since $f(r_1,\ldots,r_n)$ is not central valued on $R$, $R$ must be non commutative and so $m \geq 2$. We have the following subcases.

**Subcase-I:** Let $C$ be an infinite field. By Lemma 4.2 there exists a $C$-automorphism $\phi$ of $M_m(C)$ such that $\phi(P)$ and $\phi(u)$ have all non zero entries. Clearly $\phi(P)$, $\phi(u)$, $\phi(a')$, $\phi(c)$, $\phi(a)$ and $\phi(b)$ must satisfy the condition (5).

Here $e_{ij}$ denotes the matrix whose $(i, j)$-entry is 1 and rest entries are zero. Since $f(x_1,\ldots,x_n)$ is not central, by [20] (see also [22]), there exist $s_1,\ldots,s_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(s_1,\ldots,s_n) = \gamma e_{ij}$, with $i \neq j$. Moreover, since the set $\{f(r_1,\ldots,r_n) : r_1,\ldots,r_n \in M_m(C)\}$ is invariant under the action of all

$C$-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \ldots, r_n \in M_m(C)$ such that $f(r_1, \ldots, r_n) = e_{ij}$. Hence by (5) we have

$$\left[\phi(a')e_{ij}^2 + \phi(P)e_{ij}\phi(u)e_{ij} - \phi(P)e_{ij}^2\phi(c) - \phi(a)e_{ij}^2\phi(P) - e_{ij}\phi(u)e_{ij}\phi(P) + e_{ij}^2\phi(b), e_{ij}\right] = 0.$$

It implies that $\left[\phi(P)e_{ij}\phi(u)e_{ij} - e_{ij}\phi(u)e_{ij}\phi(P), e_{ij}\right] = 0$. This gives that $-e_{ij}\phi(u)e_{ij}\phi(P)e_{ij} - e_{ij}\phi(P)e_{ij}\phi(u)e_{ij} = 0$. Thus we have $2\phi(P)_{ji}\phi(u)_{ji}e_{ij} = 0$. Since char$(R) \neq 2$, it implies that $\phi(P)_{ji}\phi(u)_{ji}e_{ij} = 0$. It gives that either $\phi(P)_{ji} = 0$ or $\phi(u)_{ji} = 0$, a contradiction, since $\phi(u)$ and $\phi(P)$ have all non zero entries. Thus we conclude that either $\phi(u)$ or $\phi(P)$ is central. Since $\phi$ is an automorphism, hence it gives that either $u$ or $P$ is central, a contradiction.

**Subcase-II:** Let $C$ be a finite field. Suppose that $K$ is an infinite extension of the field of $C$. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central valued on $R$ if and only if it is central valued on $\overline{R}$. Suppose that the generalized polynomial $Q(r_1, \ldots, r_{n+1})$ such that

$$\begin{aligned}Q(r_1, \ldots, r_{n+1}) = \Big[&a'f(r_1, \ldots, r_n)^2 + Pf(r_1, \ldots, r_n)uf(r_1, \ldots, r_n) - Pf(r_1, \ldots, r_n)^2c \\ &-af(r_1, \ldots, r_n)^2P - f(r_1, \ldots, r_n)uf(r_1, \ldots, r_n)P + f(r_1, \ldots, r_n)^2b, r_{n+1}\Big] \end{aligned} \tag{6}$$

is a generalized polynomial identity for $R$.

Moreover, it is a multihomogeneous of multidegree $(2, \ldots, 2)$ in the indeterminates $r_1, \ldots, r_{n+1}$. Hence the complete linearization of $Q(r_1, \ldots, r_{n+1})$ is a multilinear generalized polynomial $\Theta(r_1, \ldots, r_{n+1}, x_1, \ldots, x_{n+1})$ in $2n + 2$ indeterminates, moreover

$$\Theta(r_1, \ldots, r_{n+1}, r_1, \ldots, r_{n+1}) = 2^{2n+2}Q(r_1, \ldots, r_{n+1}).$$

It is clear that the multilinear polynomial $\Theta(r_1, \ldots, r_{n+1}, x_1, \ldots, x_{n+1})$ is a generalized polynomial identity for both $R$ and $\overline{R}$. By assumption $char(R) \neq 2$ we obtain $Q(r_1, \ldots, r_{n+1}) = 0$ for all $r_1, \ldots, r_{n+1} \in \overline{R}$ and then conclusion follows from Subcase-*I*.

**Case-II:** Next we suppose that $V$ is infinite dimensional over $C$. By Martindale's theorem [26, Theorem 3], for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since we have assumed that neither $P$ nor $u$ in the center. Then there exist $h_1, h_2 \in soc(R)$ such that $[P, h_1] \neq 0$ and $[u, h_2] \neq 0$. By Litoff's Theorem [8], there exists an idempotent $e \in soc(R)$ such that $Ph_1, h_1P, uh_2, h_2u, h_1, h_2 \in eRe$. Since $R$ satisfies generalized identity

$$\begin{aligned}e\Big\{\Big[&a'f(ex_1e, \ldots, ex_ne)^2 + Pf(ex_1e, \ldots, ex_ne)uf(ex_1e, \ldots, ex_ne) - Pf(ex_1e, \ldots, ex_ne)^2c - af(ex_1e, \ldots, ex_ne)^2P \\ &-f(ex_1e, \ldots, ex_ne)uf(ex_1e, \ldots, ex_ne)P + f(ex_1e, \ldots, ex_ne)^2b, f(ex_1e, \ldots, ex_ne)\Big]\Big\}e.\end{aligned}$$

That is

$$\begin{aligned}e\Big\{&a'f(ex_1e, \ldots, ex_ne)^3 + Pf(ex_1e, \ldots, ex_ne)uf(ex_1e, \ldots, ex_ne)^2 - Pf(ex_1e, \ldots, ex_ne)^2cf(ex_1e, \ldots, ex_ne) \\ &-af(ex_1e, \ldots, ex_ne)^2Pf(ex_1e, \ldots, ex_ne) - f(ex_1e, \ldots, ex_ne)uf(ex_1e, \ldots, ex_ne)Pf(ex_1e, \ldots, ex_ne) \\ &+f(ex_1e, \ldots, ex_ne)^2bf(ex_1e, \ldots, ex_ne) - f(ex_1e, \ldots, ex_ne)a'f(ex_1e, \ldots, ex_ne)^2 \\ &-f(ex_1e, \ldots, ex_ne)Pf(ex_1e, \ldots, ex_ne)uf(ex_1e, \ldots, ex_ne) + f(ex_1e, \ldots, ex_ne)Pf(ex_1e, \ldots, ex_ne)^2c \\ &+f(ex_1e, \ldots, ex_ne)af(ex_1e, \ldots, ex_ne)^2P + f(ex_1e, \ldots, ex_ne)^2uf(ex_1e, \ldots, ex_ne)P - f(ex_1e, \ldots, ex_ne)^3b\Big\}e.\end{aligned}$$

the subring $eRe$ satisfies

$$\begin{aligned}&ea'ef(x_1, \ldots, x_n)^3 + ePef(x_1, \ldots, x_n)euef(x_1, \ldots, x_n)^2 - ePef(x_1, \ldots, x_n)^2ecef(x_1, \ldots, x_n) \\ &-eaef(x_1, \ldots, x_n)^2ePef(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)euef(x_1, \ldots, x_n)ePef(x_1, \ldots, x_n) \\ &+f(x_1, \ldots, x_n)^2ebef(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)ea'ef(x_1, \ldots, x_n)^2 \\ &-f(x_1, \ldots, x_n)ePef(x_1, \ldots, x_n)euef(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)ePef(x_1, \ldots, x_n)^2ece \\ &+f(x_1, \ldots, x_n)eaef(x_1, \ldots, x_n)^2ePe + f(x_1, \ldots, x_n)^2euef(x_1, \ldots, x_n)ePe - f(x_1, \ldots, x_n)^3ebe.\end{aligned}$$

This can be re-written as

$$\Big[ea'ef(x_1,\ldots,x_n)^2 + ePef(x_1,\ldots,x_n)euef(x_1,\ldots,x_n) - ePef(x_1,\ldots,x_n)^2ece - eaef(x_1,\ldots,x_n)^2ePe$$
$$-f(x_1,\ldots,x_n)euef(x_1,\ldots,x_n)ePe + f(x_1,\ldots,x_n)^2ebe, f(x_1,\ldots,x_n)\Big].$$

Then by the above finite dimensional case, either $ePe$ or $eue$ is central element of $eRe$. Thus either $Ph_1 = (ePe)h_1 = h_1ePe = h_1P$ or $uh_2 = (eue)h_2 = h_2(eue) = h_2u$, a contradiction. □

**Lemma 4.7.** *Let $R$ be a prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$ and $f(x_1,\ldots,x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that for some $a, P \in R$*

$$\Big[P, af(r)^2 - f(r)af(r)\Big] \in Z(R)$$

*that is*

$$Paf(r)^2 - Pf(r)af(r) - af(r)^2P + f(r)af(r)P \in Z(R),$$

*where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1,\ldots,x_n)$ in $R$, then either $P$ or $a$ is central.*

*Proof.* Substitute $a' = Pa$, $u = -a$, $c = 0$ and $b = 0$ in Lemma 4.6, we get the result. □

Now we are in a position to prove Proposition 4.1.

**Proof of Proposition 4.1:** From Lemma 4.5, we get either $P \in C$ or $b \in C$ or $u \in C$. Since $d \neq 0$ so $P$ can not be central. Hence, we have either $b \in C$ or $u \in C$. Now we shall study following two cases.

**Case-I:** If $b \in C$, then $F(x) = ax + xbu$ for all $x \in R$. Thus our hypothesis reduces to

$$\Big[P, af(r)^2 + f(r)(bu-a)f(r) - f(r)^2bu\Big] \in Z(R) \tag{7}$$

for all $r = (r_1,\ldots,r_n) \in R^n$. Since $[P, af(r)^2 + f(r)(bu-a)f(r) - f(r)^2bu] \in Z(R)$, hence it commutes with $f(r)$ for all $r = (r_1,\ldots,r_n) \in R^n$. Thus it implies that

$$\Big[Paf(r)^2 + Pf(r)(bu-a)f(r) - Pf(r)^2bu - af(r)^2P - f(r)(bu-a)f(r)P + f(r)^2buP, f(r)\Big] = 0 \tag{8}$$

for all $r = (r_1,\ldots,r_n) \in R^n$. By Lemma 4.6, we have either $P \in C$ or $bu - a \in C$. Since $d \neq 0$, hence $P$ can not be central. Thus we have $bu - a \in C$. In this case, we have $F(x) = ax + xbu = ax + x(\lambda + a) = ax + \lambda x + xa$ for some $\lambda \in C$, for all $x \in R$. Thus our hypothesis gives that

$$\Big[P, buf(r)^2 - f(r)^2bu\Big] \in Z(R) \tag{9}$$

for all $r = (r_1,\ldots,r_n) \in R^n$. If $f(x_1,\ldots,x_n)^2$ is central valued on $R$, then we find our conclusion (1). Now assume that $f(x_1,\ldots,x_n)^2$ is not central valued on $R$. Then $R$ must be non commutative. Let $S$ be the additive subgroup of $R$ generated by the set $S_1 = \{f(x_1,\ldots,x_n)^2 | x_1,\ldots,x_n \in R\}$. Then $S_1 \neq \{0\}$, since $f(x_1,\ldots,x_n)^2$ is non central valued on $R$. Thus we have

$$\Big[P, bux - xbu\Big] \in Z(R) \tag{10}$$

for all $x \in S$. By [6], either $S \subseteq Z(R)$ or char $(R) = 2$ and $R$ satisfies $s_4$, except when $S$ contains a non central Lie ideal $L$ of $R$. Since $f(x_1,\ldots,x_n)^2$ is not central valued on $R$, the first case can not occur. Since char $(R) \neq 2$, second case also can not occur. Thus $S$ contains a non central Lie ideal $L$ of $R$. By [10, Lemma 1], there exists a non central two sided ideal $I$ of $R$ such that $[I, R] \subseteq L$. In particular, (10) reduces to

$$\Big[P, bu[x,y] - [x,y]bu\Big] \in Z(R) \tag{11}$$

for all $x, y \in I$, non zero ideal of $R$. That is

$$\Big[P, \big[bu, [x, y]\big]\Big] \in Z(R) \tag{12}$$

for all $x, y \in I$. If we assume $d(x) = [P, x]$ and $\delta(x) = [bu, x]$ for all $x \in I$, then equation (12) can be rewritten as

$$d\Big(\delta([x, y])\Big) \in Z(R)$$

for all $x, y \in I$. By Lemma 3.6, we have $\delta = 0$. That is $bu \in C$. Since $bu - a \in C$ and $bu \in C$, it implies that $a \in C$. Hence $F(x) = ax + \lambda x + xa = (2a + \lambda)x$, where $2a + \lambda \in C$, which is our conclusion (2).

Case-II: If $u \in C$, then $F(x) = (a + bu)x$ for all $x \in R$. Thus by our hypothesis, we have

$$\Big[P, (a + bu)f(r)^2 - f(r)(a + bu)f(r)\Big] \in Z(R) \tag{13}$$

for all $r = (r_1, \ldots, r_n) \in R^n$. By Lemma 4.7, we get $a + bu \in C$, which is our conclusion (2).

## 5. Proof of the Main Theorem

We can write $F(x) = ax + b\delta(x)$ for all $x \in R$ and for some $a, b \in U$, where $\delta$ is a derivation on $R$. If $d$ and $\delta$ both are an inner derivations on $R$, then by Proposition 4.1, we get our conclusions. Now we assume that both are not an inner derivations. Now we shall study the following cases.

**Case-I:** Let $d$ is an inner and $\delta$ is outer. From given hypothesis we get

$$\Big[P, \Big[af(x_1, \ldots, x_n) + b\delta(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)\Big]\Big] \in C. \tag{14}$$

Substituting the value of $\delta(f(x_1, \ldots, x_n))$ with $f^{\delta}(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, \delta(x_i), \ldots, x_n)$ in equation (14), we get

$$\Big[P, \Big[af(x_1, \ldots, x_n) + bf^{\delta}(x_1, \ldots, x_n) + b\sum_i f(x_1, \ldots, \delta(x_i), \ldots, x_n), f(x_1, \ldots, x_n)\Big]\Big] \in C. \tag{15}$$

Since $\delta$ is outer, by using Kharchenko's theorem (see remark 3.3) in above expression, we get

$$\Big[P, \Big[af(x_1, \ldots, x_n) + bf^{\delta}(x_1, \ldots, x_n) + b\sum_i f(x_1, \ldots, y_i, \ldots, x_n), f(x_1, \ldots, x_n)\Big]\Big] \in C. \tag{16}$$

In particular $U$ satisfies the blended component

$$\Big[P, \Big[b\sum_i f(x_1, \ldots, y_i, \ldots, x_n), f(x_1, \ldots, x_n)\Big]\Big] \in C. \tag{17}$$

In (17) replace $y_1 = x_1$ and $y_i = 0$ for all $i > 1$, we get

$$\Big[P, \Big[bf(x_1, \ldots, x_n), f(x_1, \ldots, x_n)\Big]\Big] \in C.$$

This expression is same as equation (13), hence we get our conclusion (2).

**Case-II:** Now we assume that $d$ is outer derivation and $\delta$ is an inner derivation, say $\delta(x) = qx - xq$, for some $q \in U$. Then our hypothesis becomes

$$d\Big(\Big[(a + bq)f(x_1, \ldots, x_n) - bf(x_1, \ldots, x_n)q, f(x_1, \ldots, x_n)\Big]\Big) \in C.$$

We can replace $d(f(x_1,\ldots,x_n))$ with $f^d(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,d(x_i),\ldots,x_n)$ in above equation, we get that $U$ satisfies

$$\Big[d(a+bq)f(x_1,\ldots,x_n)+(a+bq)f^d(x_1,\ldots,x_n)+(a+bq)\sum_i f(x_1,\ldots,d(x_i),\ldots,x_n)-d(b)f(x_1,\ldots,x_n)q$$
$$-bf^d(x_1,\ldots,x_n)q-b\sum_i f(x_1,\ldots,d(x_i),\ldots,x_n)q-bf(x_1,\ldots,x_n)d(q),f(x_1,\ldots,x_n)\Big]$$
$$+\Big[(a+bq)f(x_1,\ldots,x_n)-bf(x_1,\ldots,x_n)q,f^d(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,d(x_i),\ldots,x_n)\Big]\in C.$$

Since $d$ is outer derivation, by using Kharchenko's theorem (see remark 3.3) in above expression, we get

$$\Big[d(a+bq)f(x_1,\ldots,x_n)+(a+bq)f^d(x_1,\ldots,x_n)+(a+bq)\sum_i f(x_1,\ldots,y_i,\ldots,x_n)-d(b)f(x_1,\ldots,x_n)q$$
$$-bf^d(x_1,\ldots,x_n)q-b\sum_i f(x_1,\ldots,y_i,\ldots,x_n)q-bf(x_1,\ldots,x_n)d(q),f(x_1,\ldots,x_n)\Big]$$
$$+\Big[(a+bq)f(x_1,\ldots,x_n)-bf(x_1,\ldots,x_n)q,f^d(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n)\Big]\in C,$$

where $d(x_i)=y_i$ and $x_1,\cdots,x_n\in R$. Since $R$ and $U$ satisfies same GPI (see remark-3.2), hence $U$ satisfies the blended component

$$\Big[(a+bq)\sum_i f(x_1,\ldots,y_i,\ldots,x_n)-b\sum_i f(x_1,\ldots,y_i,\ldots,x_n)q,f(x_1,\ldots,x_n)\Big]$$
$$+\Big[(a+bq)f(x_1,\ldots,x_n)-bf(x_1,\ldots,x_n)q,\sum_i f(x_1,\ldots,y_i,\ldots,x_n)\Big]\in C.$$

Replace $y_1=x_1$ and $y_i=0$ for all $i\geq 2$ in (18) we get

$$2\Big[(a+bq)f(x_1,\ldots,x_n)-bf(x_1,\ldots,x_n)q,f(x_1,\ldots,x_n)\Big]\in C.$$

Since char$(R)\neq 2$, it gives that

$$\Big[(a+bq)f(x_1,\ldots,x_n)-bf(x_1,\ldots,x_n)q,f(x_1,\ldots,x_n)\Big]\in C.$$

This is a particular case of Proposition 4.1 and thus result follows from Proposition 4.1.

**Case-III:** Now we suppose that none of $d$ and $\delta$ are an inner derivations. That is we consider, in this case $d$ and $\delta$ both are outer derivations. Now we have the following two subcases.

### $d$ and $\delta$ be $C$-linearly independent modulo $D_{in}$

In this case from our hypothesis, $U$ satisfies

$$d\Big(\Big[af(x_1,\ldots,x_n)+b\delta(f(x_1,\ldots,x_n)),f(x_1,\ldots,x_n)\Big]\Big)\in C. \tag{18}$$

By applying Kharchenko's theorem (see remark-3.2) to (18), we can replace $\delta(f(x_1,\ldots,x_n))$ with $f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n)$, where $y_i=\delta(x_i)$ and then $U$ satisfies

$$d\Big(\Big[af(x_1,\ldots,x_n)+bf^\delta(x_1,\ldots,x_n)+b\sum_i f(x_1,\ldots,y_i,\ldots,x_n),f(x_1,\ldots,x_n)\Big]\Big)\in C.$$

Then $U$ satisfies the blended components of the above expression as

$$d\Big(\Big[b\sum_i f(x_1,\dots,y_i,\dots,x_n), f(x_1,\dots,x_n)\Big]\Big)\in C.$$

Substitute $y_1=x_1$ and $y_i=0$ for all $i\geq 2$ we get

$$d\Big(\Big[bf(x_1,\dots,x_n), f(x_1,\dots,x_n)\Big]\Big)\in C. \tag{19}$$

Again applying Kharchenko's theorem (see remark-3.2) to (19), we can replace $d(f(x_1,\dots,x_n))$ with $f^d(x_1,\dots,x_n)+\sum_i f(x_1,\dots,z_i,\dots,x_n)$, where $z_i=d(x_i)$ and then $U$ satisfies

$$\Big[d(b)f(x_1,\dots,x_n)+bf^d(x_1,\dots,x_n)+b\sum_i f(x_1,\dots,z_i,\dots,x_n), f(x_1,\dots,x_n)\Big]$$
$$+\Big[bf(x_1,\dots,x_n), f^d(x_1,\dots,x_n)+\sum_i f(x_1,\dots,z_i,\dots,x_n)\Big]\in C.$$

In particular $U$ satisfies the blended component

$$\Big[b\sum_i f(x_1,\dots,z_i,\dots,x_n), f(x_1,\dots,x_n)\Big]+\Big[bf(x_1,\dots,x_n),\sum_i f(x_1,\dots,z_i,\dots,x_n)\Big]\in C.$$

For $z_1=x_1$ and $z_i=0$ for all $i\geq 2$ and using char$(R)\neq 2$ in above expression we get

$$\Big[bf(x_1,\dots,x_n), f(x_1,\dots,x_n)\Big]\in C. \tag{20}$$

This equation (20) is a particular case of Proposition 4.1 and thus result follows from Proposition 4.1.

### $d$ and $\delta$ be $C$-linearly dependent modulo $D_{in}$

Since $d$ and $\delta$ be $C$-linearly dependent modulo $D_{in}$, hence for some $\lambda,\mu\in C$, $p\in U$ such that $\lambda d(x)+\mu\delta(x)=px-xp$ for all $x\in R$.

If $\lambda=0$, then $\mu$ can not be zero. Thus it implies that $d$ is an inner derivation, a contradiction.

If $\mu=0$, then $\lambda$ can not be zero. Thus it implies that $\delta$ is an inner derivation, a contradiction.

Suppose $\lambda\neq 0$ and $\mu\neq 0$ and we can write $d(x)=\beta\delta(x)+qx-xq$, where $\beta=-\lambda^{-1}\mu$, $q=\lambda^{-1}p$. Now from our hypothesis we have

$$\beta\delta\Big(\Big[af(x_1,\dots,x_n)+b\delta(f(x_1,\dots,x_n)), f(x_1,\dots,x_n)\Big]\Big)+q\Big[af(x_1,\dots,x_n)+b\delta(f(x_1,\dots,x_n)), f(x_1,\dots,x_n)\Big]$$
$$-\Big[af(x_1,\dots,x_n)+b\delta(f(x_1,\dots,x_n)), f(x_1,\dots,x_n)\Big]q\in C,$$

which can be written as

$$\beta\Big(\Big[\delta(a)f(x_1,\dots,x_n)+a\delta(f(x_1,\dots,x_n))+\delta(b)\delta(f(x_1,\dots,x_n))+b\delta^2(f(x_1,\dots,x_n)), f(x_1,\dots,x_n)\Big]\Big)$$
$$+\beta\Big(\Big[af(x_1,\dots,x_n)+b\delta(f(x_1,\dots,x_n)),\delta(f(x_1,\dots,x_n))\Big]\Big)+q\Big[af(x_1,\dots,x_n)+b\delta(f(x_1,\dots,x_n)), f(x_1,\dots,x_n)\Big]$$
$$-\Big[af(x_1,\dots,x_n)+b\delta(f(x_1,\dots,x_n)), f(x_1,\dots,x_n)\Big]q\in C.$$

Substituting the value of $\delta(f(x_1,\ldots,x_n))$ with $f^{\delta}(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n)$ and $\delta^2(f(x_1,\ldots,x_n))$ with $f^{\delta^2}(x_1,\ldots,x_n)+2\sum_i f^{\delta}(x_1,\ldots,y_i,\ldots,x_n)+\sum_i f(x_1,\ldots,z_i,\ldots,x_n)+\sum_{i\neq j} f(x_1,\ldots,y_i,\ldots,y_j,\ldots,x_n)$, where $y_i=\delta(x_i)$ and $\delta^2(x_i)=z_i$ then $U$ satisfies

$$\begin{aligned}
&\beta\Big[\delta(a)f(x_1,\ldots,x_n)+af^{\delta}(x_1,\ldots,x_n)+a\sum_i f(x_1,\ldots,y_i,\ldots,x_n)+\delta(b)f^{\delta}(x_1,\ldots,x_n)\\
&+\delta(b)\sum_i f(x_1,\ldots,y_i,\ldots,x_n)+bf^{\delta^2}(x_1,\ldots,x_n)+2b\sum_i f^{\delta}(x_1,\ldots,y_i,\ldots,x_n)\\
&+b\sum_i f(x_1,\ldots,z_i,\ldots,x_n)+b\sum_{i\neq j} f(x_1,\ldots,y_i,\ldots,y_j,\ldots,x_n),f(x_1,\ldots,x_n)\Big]+\beta\Big[af(x_1,\ldots,x_n)\\
&+bf^{\delta}(x_1,\ldots,x_n)+b\sum_i f(x_1,\ldots,y_i,\ldots,x_n),f^{\delta}(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n)\Big]\\
&+q\Big[af(x_1,\ldots,x_n)+bf^{\delta}(x_1,\ldots,x_n)+b\sum_i f(x_1,\ldots,y_i,\ldots,x_n),f(x_1,\ldots,x_n)\Big]-\Big[af(x_1,\ldots,x_n)\\
&+bf^{\delta}(x_1,\ldots,x_n)+b\sum_i f(x_1,\ldots,y_i,\ldots,x_n),f(x_1,\ldots,x_n)\Big]q\in C.
\end{aligned}$$

Then $U$ satisfies the blended component

$$\beta\Big[b\sum_i f(x_1,\ldots,z_i,\ldots,x_n),f(x_1,\ldots,x_n)\Big]\in C.$$

Since $0\neq\beta\in C$, from above we get

$$\Big[b\sum_i f(x_1,\ldots,z_i,\ldots,x_n),f(x_1,\ldots,x_n)\Big]\in C. \tag{21}$$

In particular for $z_1=x_1$ and $z_i=0$ for all $i\geq 2$ and then $U$ satisfies

$$\Big[bf(x_1,\ldots,x_n),f(x_1,\ldots,x_n)\Big]\in C \tag{22}$$

for all $x_1,\cdots,x_n\in R$. Equation (22) is same as equation (20). Hence we get our conclusions.


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