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Counting Visible Levels in Bargraphs and Set Partitions

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Abstract. In this paper, we study the generating functions for the number of visible levels in compositions of n and set partitions of $[n]$.

1. Introduction

A *composition* of the positive integer n is a word $\sigma = \sigma_1 \cdots \sigma_m$ over the alphabet of positive integers such that $\sigma_1 + \cdots + \sigma_m = n$. The letters of σ are called *parts*. Let the set of compositions of n be denoted by $C(n)$. Each composition can be represented as a *bargraph* which is a column convex polyomino, where the lower edge lies on the x-axis. Specifically, the bargraph is made up of square cells on a regular planar lattice grid such that the size and the number of parts of σ, namely n and m, are the total number of cells and number of columns of the bargraph, respectively. The part σ_i is the height of the i-th column in the corresponding bargraph. For example, the bargraph of the composition 1223434124 of 26 is represented on Figure 1. The most recent results regarding bargraphs are due to Blecher, Brennan, Knopfmacher, and

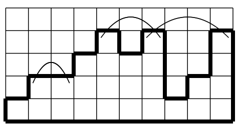

Figure 1: The bargraph 1223434124 where the visible levels are shown with arcs.

in the fourth listed paper also Prodinger in [1–4], where the authors study several statistics on bargraphs including height, width, number of peaks, number of levels, etc. Motivated by work of Gutin, Severini, and the second author [8] as well as work of Shattuck and the second author [10], we define the following statistic on bargraphs. Given the bargraph $\sigma = \sigma_1 \cdots \sigma_m$, we say the ith column is *visible* and *level* to the jth column, $i < j$, if $\sigma_k < \sigma_i = \sigma_j$ for all $k = i + 1, i + 2, \ldots, j - 1$. In this context, the pair (i, j) is called a *visible*

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level. For example, Figure 1 presents the bargraph $\sigma = 1223434124$ with three visible levels, namely (σ_2, σ_3), (σ_5, σ_7), and (σ_7, σ_{10}), which are shown by the three arcs.

A *set partition* of $[n] = \{1, 2, \ldots, n\}$ is a collection of nonempty, pairwise disjoint subsets, called *blocks*, whose union is $[n]$. If $n = 0$, there is a single empty set partition of $[0]$ which has no blocks. We will show in Theorem 2.2 that the ordinary generating function for the total number of visible levels in bargraphs representing compositions of $[n]$ is

$$\frac{(1-x)^3}{(1-2x)^2} \sum_{j\geq 1} \frac{x^{2j}}{1-2x+x^j}.$$

Further, in Corollary 3.2 we show that the generating function for the total number of visible levels in all words over the alphabet $[k]$ of length n, that is the total number of visible levels in all bargraphs with exactly n columns each of height at most k, is given by

$$J_k(x) = \frac{x^2}{(1-kx)^2} \sum_{j=0}^{k-1} \frac{1}{1-jx}.$$

Let $P(n, k)$ denote the set of all set partitions of $[n]$ with exactly k blocks. Further denote set of all set partitions of $[n]$ by $P(n)$. The *standard form* of a set partition $\pi \in P(n, k)$ is $B_1, B_2, \ldots, B_k$, where $\min(B_1) < \cdots < \min(B_k)$. Equivalently, one may also represent a set partition by the *canonical sequential form* $\pi = \pi_1\pi_2\cdots\pi_n$, wherein $i \in B_{\pi_i}$ for $i \in [n]$. That is, $\pi_i = j$ if $i \in B_j$. For example, the set partition $\pi = \{1, 6, 8\}, \{2, 3, 9\}, \{4, 7, 10\}, \{5\}$ has the canonical sequential form $\pi = 1223413123$. Recently, statistics on set partitions have been considered by several authors including Shattuck and the second author (for example, see [9, 11–13]).

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be any set partition of $P(n, k)$. As before, the pair (π_i, π_j) with $i < j$ is said to be a *visible level* (respectively, *visible rise, visible descent*) if $\pi_i = \pi_j$ (respectively, $\pi_i < \pi_j$, $\pi_i > \pi_j$) and $\pi_m < \min\{\pi_i, \pi_j\}$ for all $m = i+1, i+2, \ldots, j-1$. For instance, the set partition $\pi = \pi_1\pi_2\cdots\pi_{19} = 1121213232212421412$ of $P(19, 4)$ has six visible levels, namely $(\pi_1, \pi_2) = (1, 1)$, $(\pi_3, \pi_5) = (2, 2)$, $(\pi_7, \pi_9) = (3, 3)$, $(\pi_{10}, \pi_{11}) = (2, 2)$, $(\pi_{11}, \pi_{13}) = (2, 2)$, $(\pi_{14}, \pi_{17}) = (4, 4)$, and $(\pi_{18}, \pi_{19}) = (1, 1)$.

This notion of visibility in bargraphs is the defining property of horizontal visibility graphs. A *horizontal visibility graph* is a simple graph such that if the vertices $v_1, v_2, \ldots, v_k$ are totally ordered, then there is an edge v_iv_j exactly if $v_i, v_j > v_m$ and $i < m < j$. The formal definition of horizontal visibility graphs was given by Luque, Lacasa, Ballesteros, and Luque [7]. These graphs can be used to describe diagrams that represent the floorplan and routing in circuits (see Ho, Suzuki, and Sarrafzadeh [6]), show up in computational geometry (see de Berg, van Kreveld, Overmars, and Schwarzkopf [5]), and discrete dynamical systems (see [7]). These horizontal visibility graphs can be viewed as bargraphs where there is an edge for each visible level, rise, or descent. In this work, we consider bargraphs that represent partitions of n and set partitions of $[n]$ focusing on the special case of visible levels.

In Section 3, we study the generating function for the number of set partitions of $[n]$ with k blocks according to the number of visible levels. In particular, we show in Theorem 3.5 that the exponential generating function for the total number of visible levels in all set partitions of $[n]$ is given by

$$e^{e^x-1} \int_0^x \left(re^r - e^{1-e^r} \int_0^r te^{e^t+2t-1} dt \right) dr.$$

2. Visible levels in bargraphs

We begin our study by utilizing generating functions that track the number of bargraphs by both the composition that they represent and the number of visible levels present.

Let $C(x, q) = \sum_{n\geq 0} \sum_{\pi \in C(n)} q^{visl(\pi)} x^n$ be the generating function for the number of bargraphs (compositions) according to the number of visible levels. In order to study the generating function $C(x, q)$, we refine it as follows. Let $C_d(n)$ be the set of all bargraphs π of n where the height of each column (value of each part) of π is at most d. Define $C_d(x, q) = \sum_{n\geq d} \sum_{\pi \in C_d(n)} q^{visl(\pi)} x^n$.

Proposition 2.1. *We have*

$$C_d(x,q) = C_{d-1}(x,q) + \frac{x^d(C_{d-1}(x,q))^2}{1 - x^d q C_{d-1}(x,q)}$$

with $C_0(x,q) = 1$.

Proof. Note that each bargraph π in $C_d(n)$ can be decomposed as a bargraph in $C_{d-1}(n)$ or $\pi = \pi' d\pi''$ with $\pi' \in C_{d-1}(n')$ and $\pi'' \in C_d(n'')$ such that $n' + n'' + d = n$. The recurrence follows. $\square$

Note that $C_d(x,1) = \frac{C_{d-1}(x,1)}{1-x^dC_{d-1}(x,1)}$ with $C_0(x,1) = 1$. So $C_d(x,1) = \frac{1}{1-x-x^2-\cdots-x^d}$, for all $d \geq 1$.

Now let us consider the total number of visible levels in bargraphs of $C_d(n)$. To do so, we define $D_d(x) = \frac{d}{dq}C_d(x,q)\mid_{q=1}$.

Theorem 2.2. *The generating function for the total number of visible levels over all bargraphs of n is given by*

$$\frac{(1-x)^3}{(1-2x)^2}\sum_{j\geq 1}\frac{x^{2j}}{1-2x+x^j}.$$

Proof. By Proposition 2.1, we have

$$D_d(x) = D_{d-1}(x) + \frac{2x^dC_{d-1}(x,1)D_{d-1}(x)}{1-x^dC_{d-1}(x,1)} + \frac{x^d(C_{d-1}(x,1))^2(x^dC_{d-1}(x,1)+x^dD_{d-1}(x))}{(1-x^dC_{d-1}(x,1))^2}$$

with $D_0(x) = 0$. Thus,

$$D_d(x) = (1 + x^dC_d(x,1))^2 D_{d-1}(x) + x^{2d}(C_d(x,1))^2 C_{d-1}(x,1)$$

with $D_0(x) = 0$. Hence, by induction on d and the fact that $C_d(x,1) = \frac{1}{1-x-x^2-\cdots-x^d}$, we have

$$D_d(x) = \sum_{j=1}^{d}\frac{x^{2j}}{(1-x-\cdots-x^j)^2(1-x-\cdots-x^{j-1})}\prod_{i=j+1}^{d}\left(1+\frac{x^i}{1-x-\cdots-x^i}\right)^2,$$

which is equivalent to

$$D_d(x) = \frac{1}{(1-x-\cdots-x^d)^2}\sum_{j=1}^{d}\frac{x^{2j}}{1-\frac{x-x^j}{1-x}}.$$

By taking $d \to \infty$, we obtain the desired result. $\square$

Theorem 2.2 and analytic generating function techniques (see Sedgewick and Flajolet [14] for more information) allow us to obtain the following corollary.

Corollary 2.3. *The total number of visible levels over all bargraphs of n asymptotically approaches* $(n+1)2^{n-3}$.

Proof. By Theorem 2.2, we have the generating function for the total number of visible levels over all bargraphs of n is

$$f(x) = \frac{(1-x)^3}{(1-2x)^2}\sum_{j\geq 1}\frac{x^{2j}}{1-2x+x^j}.$$

Let

$$f_N(x) = \frac{(1-x)^3}{(1-2x)^2}\sum_{j=1}^{N}\frac{x^{2j}}{1-2x+x^j}.$$

Note that the generating function $f_N(x)$ is meromorphic in the interval $|x| \le \frac{3}{5}$ with unique pole at 1/2 of multiplicity 2. Then (applying Theorem IV.9 in [14]), the coefficient of x^n in $f_N(x)$ asymptotically approaches

$$(n+1)2^{n-3}\sum_{j=1}^{N}\left(\frac{1}{2}\right)^j = (n+1)2^{n-3}\left(1-\left(\frac{1}{2}\right)^N\right),$$

for all $N \ge n$. Hence the coefficient of x^n in $f(x)$ is given by $(n+1)2^{n-3}$ as $n \to \infty$. □

3. Visible levels in set partitions

For the remainder of the paper, we consider bargraphs as set partitions. Let

$$P_k(x,q) = \sum_{n\ge 0}\sum_{\pi\in P(n,k)} q^{visl(\pi)}x^n$$

be the generating function for the number of set partitions in $P(n,k)$ according to the number visible levels.

Theorem 3.1. *The generating function $P_k(x,q)$ is given by*

$$P_k(x,q) = x^k\prod_{j=0}^{k-1}\frac{L_j(x,q)}{1-xqL_j(x,q)},$$

where $L_k(x,q) = L_{k-1}(x,q) + \frac{x(L_{k-1}(x,q))^2}{1-xqL_{k-1}(x,q)}$ with $L_0(x,q) = 1$.

Proof. Note that each set partition π in $P(n,k)$ can be decomposed as $\pi = \pi' k\pi''$, where $\pi' \in P(n', k-1)$ and π'' is a word over the alphabet $[k]$ of length n'' such that $n' + n'' = n - 1$. Thus,

$$P_k(x,q) = P_{k-1}(x,q)L'_k(x,q),$$

where $L'_k(x,q) = \sum_{n\ge 0}\sum_{\pi=k\pi'\in[k]^n} q^{visl(\pi)}x^n$ is the generating function for the number of words $\pi = k\pi'$ over the alphabet $[k]$ of length n according to the number visible levels. Therefore,

$$P_k(x,q) = L'_k(x,q)L'_{k-1}(x,q)\cdots L'_1(x,q). \tag{1}$$

To study the generating function $L'_k(x,q)$, we focus on $L_k(x,q) = \sum_{n\ge 0}\sum_{\pi\in[k]^n} q^{visl(\pi)}x^n$, which is the generating function for the number of words over the alphabet $[k]$ of length n according to the number visible levels. Note that each word π over the alphabet $[k]$ can be written as $\pi = \pi^{(0)}k\pi^{(1)}\cdots k\pi^{(s)}$ with $s \ge 0$, where $\pi^{(j)}$ is a word over the alphabet $[k-1]$. Thus (similar to Proposition 2.1),

$$L_k(x,q) = L_{k-1}(x,q) + \frac{x(L_{k-1}(x,q))^2}{1-xqL_{k-1}(x,q)} \tag{2}$$

with $L_0(x,q) = 1$.

Note that each word $\pi = k\pi'$ over the alphabet $[k]$ is such that either π' is a word over the alphabet $[k-1]$ or π' contains the letter k at least once. So $L'_k(x,q) = xL_{k-1}(x,q) + xqL_{k-1}(x,q)L'_k(x,q)$, which leads to

$$L'_k(x,q) = \frac{xL_{k-1}(x,q)}{1-xqL_{k-1}(x,q)}. \tag{3}$$

From (1)-(3), we have the desired result. □

For instance, Theorem 3.1 gives $P_0(x,q)=1$, $P_1(x,q)=\frac{x}{1-xq}$ and

$$P_2(x,q)=\frac{(1+x-xq)x^2}{(1-2xq-x^2q+x^2q^2)(1-xq)}.$$

We can also use Theorem 3.1 to obtain the generating function for the total number of the visible levels in bargraphs representing words of length n on the alphabet $[k]$.

Corollary 3.2. *The generating function for the total number of visible levels in all words over the alphabet* $[k]$ *of length n is given by*

$$J_k(x)=\frac{x^2}{(1-kx)^2}\sum_{j=0}^{k-1}\frac{1}{1-jx}.$$

Proof. Define $J_k(x)=\frac{\partial}{\partial q}L_k(x,q)\mid_{q=1}$, so by (2), we have

$$J_k(x)=\frac{J_{k-1}(x)+x^2(L_{k-1}(x,1))^3}{(1-xL_{k-1}(x,1))^2}$$

with $J_0(x)=0$. Note that $L_{k-1}(x,1)=\frac{1}{1-(k-1)x}$, so

$$J_k(x)=\frac{(1-(k-1)x)^2}{(1-kx)^2}J_{k-1}(x)+\frac{x^2}{(1-kx)^2(1-(k-1)x)}.$$

Hence, by induction on k, we have the desired result. □

Next, we examine the generating function for the total number of visible levels in bargraphs representing set partitions of $[n]$.

Define, $Q_k(x)=\frac{\partial}{\partial q}P_k(x,q)\mid_{q=1}$, so Theorem 3.1 gives

$$Q_k(x)=x^k\prod_{j=0}^{k-1}\frac{L_j(x,1)}{1-xL_j(x,1)}\sum_{j=0}^{k-1}\frac{J_j(x)+x(L_j(x,1))^2}{(1-xL_j(x,1))L_j(x,1)},$$

which, by $L_j(x,1)=\frac{1}{1-jx}$, implies

$$Q_k(x)=\frac{x^k}{\prod_{j=1}^{k}(1-jx)}\sum_{j=0}^{k-1}\frac{x+x^2\sum_{i=0}^{j-1}1/(1-ix)}{1-(j+1)x}.$$

Therefore, by Corollary 3.2, we have

$$Q_k(x)=\frac{x^{k+1}}{\prod_{j=1}^{k}(1-jx)}\sum_{j=1}^{k}\frac{1+x\sum_{i=0}^{j-2}\frac{1}{1-ix}}{1-jx}. \tag{4}$$

Given the ordinary generating function $A(x)=\sum_{n\geq 0}a_nx^n$, we define the corresponding exponential generating function as $\tilde{A}(x)=\sum_{n\geq 0}a_n\frac{x^n}{n!}$ (for example, see [14] to see basic and advanced properties of ordinary and exponential generating functions). In order to study further the coefficients of the generating function $Q_k(x)$, we denote the corresponding exponential generating function of $Q_k(x)$ as $\widetilde{Q}_k(x)$.

Theorem 3.3. *The generating function* $\widetilde{Q}(x,y)$ *is given by*

$$\widetilde{Q}(x,y)=\int_0^x e^{y(e^{x-t}-1)}\frac{\partial}{\partial x}\widetilde{Q'}(x,y)\mid_{(x,y)=(t,ye^{x-t})}dt,$$

where

$$\tilde{Q}'(x,y) = \frac{y}{2}\int_0^x t^2 e^{y(e^{x-t}-1)+x-t}dt + \int_0^x \frac{\partial}{\partial x}\widetilde{Q''}(x,y)\,|_{(x,y)=(t,ye^{x-t})}\; e^{y(e^{x-t}-1)}dt$$
$$- y\int_0^x \frac{\partial}{\partial y}\widetilde{Q''}(x,y)\,|_{(x,y)=(t,ye^{x-t})}\; e^{y(e^{x-t}-1)+x-t}dt$$

and

$$\frac{\partial}{\partial x}\widetilde{Q''}(x,y) = (x+1)y - \frac{1+ye^x}{2} - \frac{y^2}{2}e^{ye^x}(Ei(1,ye^x)-Ei(1,y)) + \frac{1-y}{2}e^{y(e^x-1)},$$

with

$$Ei(1,ye^x) - Ei(1,y) = x + \sum_{j\ge 1}(-1)^j\frac{e^{jx}-1}{j\cdot j!}y^j.$$

Proof. By (4), we have that $Q_0(x) = 0$, $Q_1(x) = \frac{x^2}{(1-x)^2}$, and $Q_2(x) = \frac{x^3(2-2x-x^2)}{(1-x)^2(1-2x)^2}$. For all $k \ge 1$, we also have

$$Q'_k(x) := (1-kx)Q_k(x) - xQ_{k-1}(x) = \frac{x^{k+1}}{\prod_{j=1}^k(1-jx)}(1 + x\sum_{i=0}^{k-2}\frac{1}{1-ix}) \tag{5}$$

with $Q'_0(x) = 0$. Moreover, for all $k \ge 2$, we have

$$Q''_k(x) := Q'_k(x) - \frac{x}{1-kx}Q'_{k-1}(x) = \frac{x^{k+2}}{\prod_{j=1}^k(1-jx)}\frac{1}{1-(k-2)x} \tag{6}$$

with $Q''_0(x) = 0$ and $Q''_1(x) = x^2$.

Rewriting (6) as $\frac{1}{x^2}Q''_k(x) - \frac{k-2}{x}Q''_k(x) = \frac{x^k}{\prod_{j=1}^k(1-jx)}$, we obtain

$$\frac{d^2}{dx^2}\widetilde{Q''}_k(x) - (k-2)\frac{d}{dx}\widetilde{Q''}_k(x) = \frac{(e^x-1)^k}{k!} \tag{7}$$

with $\widetilde{Q''}_0(x) = 0$ and $\widetilde{Q''}_1(x) = x^2/2!$, where the corresponding exponential generating function for the ordinary generating function $\frac{x^k}{\prod_{j=1}^k(1-jx)}$ is given by $\frac{(e^x-1)^k}{k!}$ (for instance, see [9]).

For given a sequence of generating functions $A_k(x)$, we define $A(x,y) = \sum_{k\ge 0}A_k(x)y^k$. Thus, by (7), we obtain

$$\frac{\partial^2}{\partial x^2}\widetilde{Q''}(x,y) - y\frac{\partial^2}{\partial x\partial y}\widetilde{Q''}(x,y) + 2\frac{\partial}{\partial x}\widetilde{Q''}(x,y) = e^{y(e^x-1)} - 1 - (e^x-1)y + y + xy$$

with $\widetilde{Q''}(0,y) = 0$ and $\frac{\partial}{\partial x}\widetilde{Q''}(0,y) = 0$. Thus,

$$\frac{\partial}{\partial x}\widetilde{Q''}(x,y) = (x+1)y - \frac{1+ye^x}{2} - \frac{y^2}{2}e^{ye^x}(Ei(1,ye^x)-Ei(1,y)) + \frac{1-y}{2}e^{y(e^x-1)},$$

where

$$Ei(1,y) - Ei(1,ye^x) = -x - \sum_{j\ge 1}(-1)^j\frac{e^{jx}-1}{j\cdot j!}y^j.$$

By (6), we have

$$\frac{d}{dx}\widetilde{Q''}_k(x) - k\widetilde{Q''}_k(x) = \frac{d}{dx}\widetilde{Q'}_k(x) - k\widetilde{Q'}_k(x) - \widetilde{Q'}_{k-1}(x)$$

with $\widetilde{Q''}_0(x) = \widetilde{Q'}_0(x) = 0$, $\widetilde{Q''}_1(x) = x^2/2$ and $\widetilde{Q'}_1(x) = e^x - 1 - x$. Thus, by multiplying by y^k and summing over $k \geq 2$, we obtain

$$\frac{\partial}{\partial x}\widetilde{Q''}(x,y) - y\frac{\partial}{\partial y}\widetilde{Q''}(x,y) = \frac{\partial}{\partial x}\widetilde{Q'}(x,y) - y\frac{\partial}{\partial y}\widetilde{Q'}(x,y) - y\widetilde{Q'}(x,y) - \frac{1}{2}x^2y.$$

Hence,

$$\tilde{Q}'(x,y) = \int_0^x ((\frac{ye^xt^2}{2} + e^t\frac{\partial}{\partial x}\widetilde{Q''}(x,y)\mid_{(x,y)=(t,ye^{x-t})} - ye^x\frac{\partial}{\partial y}\widetilde{Q''}(x,y)\mid_{(x,y)=(t,ye^{x-t})})e^{y(e^{x-t}-1)-t}dt.$$

By (5), we have

$$\frac{d}{dx}\widetilde{Q'}_k(x) = \frac{d}{dx}\widetilde{Q}_k(x) - k\widetilde{Q}_k(x) - \widetilde{Q}_{k-1}(x)$$

with $\widetilde{Q'}_0(x) = \widetilde{Q}_0(x) = 0$. Thus, by multiplying by y^k and summing over $k \geq 2$, we obtain

$$\frac{\partial}{\partial x}\widetilde{Q'}(x,y) = \frac{\partial}{\partial x}\widetilde{Q}(x,y) - y\frac{\partial}{\partial y}\widetilde{Q}(x,y) - y\widetilde{Q}(x,y).$$

Hence,

$$\widetilde{Q}(x,y) = \int_0^x e^{y(e^{x-t}-1)}\frac{\partial}{\partial x}\widetilde{Q'}(x,y)\mid_{(x,y)=(t,ye^{x-t})} dt,$$

which leads to the desired result. □

3.1. *Another recurrence for the number of visible levels in set partitions*

A different combinatorial approach can be taken to obtain the following result. First recall that the n-th Bell Number, denoted $B(n)$, is the number of ways to partition the set $[n]$ into nonempty subsets.

Proposition 3.4. *Let $V(n)$ be the number of visible levels found in all set partitions of $[n]$. Then*

$$V(n) = \sum_{k=0}^{n-1}\binom{n-1}{k}V(k) + \sum_{k=0}^{n-1}B(k)\sum_{j=0}^{n-1-k}\binom{k}{j}\binom{n-k-1}{j}(n-k-j-1)$$

where $B(k)$ is the k-th Bell number.

Proof. Consider the word representing the set partition π of $[n]$. The visible levels involving integers other than 1 can be found by considering the word obtained when all of the 1s are removed from π. This resulting word π' also represents a set partition. Further, as the 1s do not interfere with visible levels of greater entries, π' has exactly the same visible levels involving entries greater than 1 as π does. Thus there are $\sum_{k=0}^{n-1}\binom{n-1}{k}V(k)$ visible levels in set partitions of $[n]$ that do not involve the 1 entries.

Note that each visible level $(1,1)$ of π must have the 1s in consecutive positions in the word representation. Consider set partitions of $[n]$ with exactly k entries larger than 1. There are $B(k)$ such partitions. Let j be the number of entries of value 1 that have an entry other than 1 to their left. There are $\binom{k}{j}$ ways to intersperse these entries with the larger entries. Finally the $(n-k-j-1)$ 1s that do have a 1 to their left can be placed in $\binom{n-k-1}{j}$ ways. Such a word representation will have exactly $n-k-j-1$ visible levels of 1s. Hence there are $\sum_{k=0}^{n-1}B(k)\sum_{j=0}^{n-1-k}\binom{k}{j}\binom{n-k-1}{j}(n-k-j-1)$ visible levels $(1,1)$ in all set partitions of $[n]$. □

Now define the exponential generating function $E(x) = \sum_{n\geq 0} V(n)\frac{x^n}{n!}$.

Theorem 3.5. *The exponential generating function $E(x) = \widetilde{Q}(x,1)$ is given by*

$$E(x) = e^{e^x-1}\int_0^x\left(re^r - e^{1-e^r}\int_0^r te^{e^t+2t-1}dt\right)dr.$$

Proof. By Proposition 3.4 and the identity

$$\sum_{j=0}^{n-k-1}\binom{k}{j}\binom{n-k-1}{j}(n-k-j-1) = (n-k-1)\binom{n-2}{k},$$

we have that

$$V(n) = \sum_{k=0}^{n-1}\binom{n-1}{k}V(k) + \sum_{k=0}^{n-2}(n-k-1)\binom{n-2}{k}B(k).$$

Hence,

$$\frac{V(n)}{(n-1)!} = \sum_{k=0}^{n-1}\frac{1}{(n-1-k)!}\frac{V(k)}{k!} + \sum_{k=0}^{n-2}\frac{1}{(n-2-k)!}\frac{B(k)}{k!} - \frac{1}{n-1}\sum_{k=0}^{n-2}\frac{k}{(n-2-k)!}\frac{B(k)}{k!}.$$

By multiplying the last recurrence by x^{n-1} and summing over all $n \geq 1$, we obtain

$$\frac{d}{dx}E(x) = e^xE(x) + xe^{e^x+x-1} - \int_0^x te^{e^t+2t-1}dt.$$

Hence, we have

$$E(x) = e^{e^x-1}\int_0^x\left(re^r - e^{1-e^r}\int_0^r te^{e^t+2t-1}dt\right)dr,$$

as required. □

4. Conclusion

In this paper we provided the ordinary generating function for the number of visible levels over compositions of n and the exponential generating function for the number of visible levels over set partitions of $[n]$. As natural next step, we plan to extend the study of visible levels to visible descents as follows: Given the bargraph $\sigma = \sigma_1\cdots\sigma_m$, we say (i,j) is a *visible descent* if $i < j$ and $\sigma_k < \sigma_j < \sigma_i$ for all $k = i+1, i+2, \ldots, j-1$.

For example, Figure 1 presents the bargraph $\sigma = 1223434124$ with three visible descents, namely (σ_5,σ_6), (σ_7,σ_8), and (σ_7,σ_9), which are shown by the three arcs. Thus, as next step, we plan to study the ordinary generating function for the number of visible descents over compositions of n and the exponential generating function for the number of visible descents over set partitions of $[n]$.

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