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On Nearly Menger and Nearly Star-Menger Spaces

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Abstract. In 1999, Kočinac defined and characterized the almost Menger property. Following this concept, we define and investigate nearly Menger and nearly star-Menger spaces. Every Menger space is nearly Menger, and every nearly Menger space is almost Menger. It is demonstrated that a nearly Menger space may not necessarily be a Menger space. In the similar way, we consider nearly $\gamma-$sets.

1. Introduction

The theory of Selection Principles (SP_s) is an area of Mathematics that possesses a rich history dating back to papers published in 1920-1930's by Borel, Hurewicz, Menger, Rothberger, Sierpiński. After Scheeper's paper [27], research in this particular field expanded immensely and attracted many researchers. The theory of SP_s has extraordinary connections with numerous subareas of mathematics, for example, Set theory and General topology, Game theory, Ramsey theory, Function spaces, Hyperspaces, Cardinal invariants, Dimension theory, Uniform structures, Topological groups and relatives, Karamata theory, and Ditopological texture spaces. Various survey papers exist in the field of selection principles theory (see, for example, [15, 16, 26] and the paper [29] for open problems).

In mathematics, there are three classical SP_s [12, 13, 25] that can provide a base for the theory. Menger's property is a special type of the Lindelöf property and is defined as follows:

Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of an infinite set X. The selection hypothesis denoted by $\mathbf{S}_{fin}(\mathcal{A},\mathcal{B})$ states that:

for every sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of elements of $\mathcal{A}$, there is a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for each n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\bigcup_{n\in\mathbf{N}} \mathcal{B}_n$ is a member of $\mathcal{B}$. For more information, see the paper [26].

If a space X satisfies the selection hypothesis $\mathbf{S}_{fin}(\mathcal{O},\mathcal{O})$, where $\mathcal{O}$ is the collection of all open covers of X, then it is said to possess the *Menger property* and is expressed as $X \models \mathbf{P}_{\mathbf{S}_{fin}}$.

In [11, 24] the authors studied topological properties by using the method of stars. Kočinac in [14] initiated the study of star selection principles by using the operator $\mathbf{St}$ which is defined as follows: For a subset A of X and a collection $\mathcal{P}$ of subsets of X, we denote the set $\cup\{P \in \mathcal{P} : A \cap P \neq \emptyset\}$ by $\mathbf{St}(A,\mathcal{P})$.

Kočinac in [14] stated that $\mathbf{S}^*_{fin}(\mathcal{A},\mathcal{B})$ (resp. $\mathbf{SS}^*_{fin}(\mathcal{A},\mathcal{B})$) represents the selection hypothesis that for each sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ (resp. $(F_n : n \in \mathbf{N})$) such that for each n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$, (resp. F_n is a finite subset of X) and $\bigcup_{n\in\mathbf{N}}\{\mathbf{St}(B,\mathcal{A}_n) : B \in \mathcal{B}_n\}$

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(resp. $\{\mathbf{St}(F_n, \mathcal{A}_n) : n \in \mathbf{N}\}$) is an element of $\mathcal{B}$. The space X possesses the *star-Menger property* (resp. *strongly star-Menger property*) if it satisfies the selection hypothesis $\mathbf{S}^*_{fin}(\mathcal{O},\mathcal{O})$ (resp. $\mathbf{SS}^*_{fin}(\mathcal{O},\mathcal{O})$).

For simplicity, we will use the following notations:

$X \models \mathbf{P}$ $\left(X \models \mathbf{P}_{\mathbf{S}_{fin}},\ X \models \mathbf{P}_{\mathbf{S}^*_{fin}}\ \textit{and}\ X \models \mathbf{P}_{\mathbf{SS}^*_{fin}}\right)$; X satisfies the property $\mathbf{P}$ (resp. the Menger property $\mathbf{P}_{\mathbf{S}_{fin}}$, the star-Menger property $\mathbf{P}_{\mathbf{S}^*_{fin}}$, the strongly star-Menger property $\mathbf{P}_{\mathbf{SS}^*_{fin}}$).

Recall that in a space X:

1) a *γ-cover* is an open cover $\mathcal{A}$ of X that is infinite and for every $x \in X$, the collection $\{A \in \mathcal{A}\colon x \notin A\}$ is finite.

2) an *ω-cover* $\mathcal{A}$ is an open cover of X such that $X \notin \mathcal{A}$ and for every finite subset F of X, $F \subset A$ for some $A \in \mathcal{A}$.

The Menger property is a familiar topological idea presented by Karl Menger in 1924 and deliberately examined by Scheepers [27]. Di Maio and Kočinac defined the almost Menger property in hyperspaces in [8]. In this paper, we define and study nearly Menger spaces, nearly star Menger spaces, and nearly γ-sets. For this, we utilized the semi closure of an open set, and the idea is not totally new. Let us mention that nearly Menger spaces have been defined in a different, but equivalent way, in [17]; see also [18].

Normann Levine [23] in 1963, gave the definition of semi open set in a space X. From that point forward, a number of mathematicians generalized several concepts and investigated their properties in the new setting. A set $S \subset X$ is semi open in X if and only if there is an open set O such that $O \subset S \subset cl(O)$, where $cl(O)$ is the closure of the open set O. The complement of a semi open set is known as a semi closed set [5]. An open set is always semi open but a semi open set may not be an open set. $SO(X)$ denotes the collection of all semi open subsets of X. According to Crossley [5], the semi interior and semi closure were defined analogously to the interior and closure. A set S is semi open if and only if $sInt(S) = S$, where $sInt(S)$ is the semi interior in a space X and is the largest semi open set contained in S. A set T is semi closed if and only if $scl(T) = T$, where $scl(T)$ is the semi closure of T in a space X and is the smallest semi closed set containing T. A point x belongs to X is semi limit point of subset A of a space X if $U \cap A \neq \emptyset$ for each semi open set U containing x. For any subset S of X, $Int(S) \subseteq sInt(S) \subseteq S \subseteq scl(S) \subseteq cl(S)$. For more explanation on semi open sets and semi closed sets, see [5–7, 10, 21].

In Section 2 of this paper, we define nearly Menger spaces and give certain results. We also give counterexamples in this section. Further, in Sections 3 and 4, we define and study nearly γ-sets and nearly star-Menger spaces.

2. Nearly Menger Spaces

In 1924, Menger presented the Menger basis property [25]. In 1925, Hurewicz [12] gave the proof of the statement that a metric space X has the Menger basis property, if and only if, $X \models \mathbf{P}_{\mathbf{S}_{fin}}$. In 1999, Kočinac [14] characterized and considered almost Menger spaces. A space X has the almost Menger property if for every sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of open covers of a space X there is a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for every n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\bigcup\{\mathcal{B}'_n : n \in \mathbf{N}\}$ is a cover of X, where $\mathcal{B}'_n = \{cl(B) : B \in \mathcal{B}_n\}$. In this contrast, here we define a new class of Menger-type spaces by utilizing the semi closure of an open set. It is demonstrated that newly defined Menger-type property is different from the almost Menger property and the Menger property. Kočinac [17] defined nearly Menger spaces by utilizing the interior of the closure of an open set. We notice that both notions of nearly Menger spaces coincide in the presence of open covers.

Lemma 2.1. ([9]) *For an open set O in a space X, $int\,(cl(O)) = scl(O)$.*

It is worth mentioning that for an open set O, the set $scl(O)$ is open as well as semi closed.

Lemma 2.2. ([1]) *For any subset A of a space X, $A \cup int\,(cl(A)) = scl(A)$.*

Definition 2.3. A space X is *nearly Menger* if for every sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of open covers of X there exists a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for every n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\bigcup\{\mathcal{B}'_n : n \in \mathbf{N}\}$ is a cover of X, where $\mathcal{B}'_n = \{scl(B) = int(cl(B)) : B \in \mathcal{B}_n\}$.

We immediately note that:

$$\text{Menger space} \Rightarrow \text{nearly Menger space} \Rightarrow \text{almost Menger space}.$$

A space X satisfying the nearly Menger (resp. almost Menger) property is denoted by $X \models \mathbf{P}_{\mathbf{nS}_{fin}}$ $\left(\text{resp. } X \models \mathbf{P}_{\mathbf{aS}_{fin}}\right)$.

However, the nearly Menger property does not imply the Menger property in general as the following example shows.

Example 2.4. Let X be an Euclidean plane and define the deleted diameter topology on it. X does not satisfy $\mathbf{P}_{\mathbf{S}_{fin}}$, because the space X with the deleted diameter topology $\mathcal{T}_D$ is not Lindelöf [28]. To prove $X \models \mathbf{P}_{\mathbf{nS}_{fin}}$, we will use the fact that points on the diameter of an open disc are always its semi limit points, therefore the semi closure of an open set in the deleted diameter topology is the same set which we obtain by taking semi closure of an open set in the Euclidean topology. Euclidean plane with the usual Euclidean topology $\mathcal{T}_E$ satisfies $\mathbf{P}_{\mathbf{S}_{fin}}$, because it is $\sigma-$compact, so also satisfies $\mathbf{P}_{\mathbf{nS}_{fin}}$ and this implies $\cup\{scl(O) : O \in \mathcal{T}_D\} = X = \cup\{scl(O) : O \in \mathcal{T}_E\}$. This shows that the Euclidean plane with the deleted diameter topology satisfies $\mathbf{P}_{\mathbf{nS}_{fin}}$ but not $\mathbf{P}_{\mathbf{S}_{fin}}$.

Example 2.5. (Uncountable particular point topology) Let X be an uncountable set and $p \in X$. Then $\mathcal{T}_p = \{O \subseteq X; p \in O \text{ or } O = \emptyset\}$ is the uncountable particular point topology on X. The uncountable particular point topology is not Lindelöf [28] so it does not satisfy $\mathbf{P}_{\mathbf{S}_{fin}}$. To show that $X \models \mathbf{P}_{\mathbf{nS}_{fin}}$, we will show that $\cup\{scl(O) : O \in \mathcal{T}_p\} = X$. As for $A \subseteq X$ and $A \in \mathcal{T}_p$, implies $p \in A$. No closed set other than X contains p. Thus, the closure of any open set other than $\emptyset$ is X. This implies $cl(A) = X$. And in the particular point topology $\mathcal{T}_p \supseteq SO(X)$, because if $A \in SO(X)$, then $p \in A$ so $A \in \mathcal{T}_p$ whereas $\mathcal{T}_p \subseteq SO(X)$ is always true. Therefore, the collection of semi closures of open sets is $\{\emptyset , X\}$ which obviously covers X.

Following is an example of an almost Menger space which is not nearly Menger.

Example 2.6. Let Ω be the smallest uncountable ordinal number and $A = [0, \Omega)$. The set A has the property that for each $\alpha \in A$ the set $[0, \alpha)$ is countable, while A is not. Let $X = \{a_{ij}, b_{ij}, c_i, a, b\}$ where $i \in A$ and $j \in N$. We define in X a topology such that the points $\{a_{ij}\}$ and $\{b_{ij}\}$ are isolated and the fundamental system of neighborhoods of the points $\{c_i\}$, $\{a\}$ and $\{b\}$ are $B^n_{c_i} = \{c_i, a_{ij}, b_{ij}\}_{j\geq n}$, $B^\alpha_a = \{a, a_{ij}\}_{i\geq\alpha, j\in N}$ and $B^\alpha_b = \{b, b_{ij}\}_{i\geq\alpha, j\in N}$ respectively. X is not nearly Lindelöf [4, Example 3.5], so it can not be nearly Menger. As the collection of fundamental neighbourhoods is uncountable, therefore bases is uncountable and is the intersection of finite number of elements of subbases. Hence this space is a P-space, and an almost Lindelöf P-space is almost Menger ([20, Proposition 2.4]).

Theorem 2.7. *If a regular space $X \models \mathbf{P}_{\mathbf{nS}_{fin}}$, then $X \models \mathbf{P}_{\mathbf{S}_{fin}}$.*

Proof. Let $(\mathcal{A}_n : n \in \mathbf{N})$ be a sequence of open covers of a regular space X. Then for every n in $\mathbf{N}$, there exists an open cover $\mathcal{B}_n$ of X such that $\mathcal{B}'_n = \{scl(B) : B \in \mathcal{B}_n\}$ forms an open refinement of $\mathcal{A}_n$. Now using the fact that $X \models \mathbf{P}_{\mathbf{nS}_{fin}}$, there exists a sequence $(\mathcal{G}_n : n \in \mathbf{N})$ such that for every n in $\mathbf{N}$, $\mathcal{G}_n$ is a finite subset of $\mathcal{B}'_n$ and $\cup(\mathcal{G}_n : n \in \mathbf{N})$ covers X. For every n in $\mathbf{N}$ and every G in $\mathcal{G}_n$, we have $A_G \in \mathcal{A}_n$ such that $G \subset A_G$. Let $\mathcal{A}'_n = \{A_G : G \in \mathcal{G}_n\}$. We will prove that $\cup(\mathcal{A}'_n : n \in \mathbf{N})$ covers X. Let $x \in X$. There is an n in $\mathbf{N}$ and G in $\mathcal{G}_n$ such that $x \in G$. By regularity of X, there will be $A_G \in \mathcal{A}'_n$ such that $G \subset A_G$. Therefore, $x \in A_G$. □

Definition 2.8. A subset A of a space X is *s-regular open* (*s-regular closed*) if $A = int(scl(A))(A = cl(sInt(A)))$.

We note that an s-regular open set is regular open, open as well as semi closed, and if A is open, then $cl(A)$ is an s-regular closed set.

Lemma 2.9. *For every subset A of a space X, $int\,(scl(A))$ is s-regular open.*

Theorem 2.10. *$X \models \mathbf{P}_{\mathbf{nS}_{fin}}$ if and only if for every sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of covers of X by s-regular open sets, there is a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for every n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\cup(\mathcal{B}_n : n \in \mathbf{N})$ is a cover of X.*

Proof. The direct part is trivial. Conversely, let a sequence of open covers of X be denoted by $(\mathcal{A}_n : n \in \mathbf{N})$. Also, consider a sequence $(\mathcal{A}'_n : n \in \mathbf{N})$ such that $\mathcal{A}'_n = \{int(scl(A)) : A \in \mathcal{A}_n\}$. Then for every $n \in \mathbf{N}$, $\mathcal{A}'_n$ is a cover of X by s-regular open sets.

By hypothesis, there exists a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for every n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}'_n$ and $\cup(\mathcal{B}_n : n \in \mathbf{N})$ covers X. By construction, for every n in $\mathbf{N}$ and B in $\mathcal{B}_n$ there is an $A_B \in \mathcal{A}_n$ such that $B = int(scl(A_B))$. $int(scl(A_B)) \subseteq scl(A_B)$. Therefore, $\bigcup_{n\in\mathbf{N}}\{scl(A_B) : B \in \mathcal{B}_n\}$ covers X. This implies that $X \models \mathbf{P}_{\mathbf{nS}_{fin}}$. □

Corollary 2.11. *$X \models \mathbf{P}_{\mathbf{aS}_{fin}}$ if and only if for every sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of covers of X by regular open sets, there exists a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for every n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\cup(\mathcal{B}_n : n \in \mathbf{N})$ is a cover of X.*

Theorem 2.12. *If $X \models \mathbf{P}_{\mathbf{nS}_{fin}}$ and $int(cl(A))$ is finite for any $A \subset X$, then $X \models \mathbf{P}_{\mathbf{S}_{fin}}$.*

Proof. Let the sequence of open covers of X be denoted by $(\mathcal{A}_n : n \in \mathbf{N})$. Hence, there is a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for each n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\cup(\mathcal{B}'_n : n \in \mathbf{N})$ covers X, where $\mathcal{B}'_n = \{scl(B) : B \in \mathcal{B}_n\}$. By Lemma 2.2, for any $A \subset X$, $scl(A) = A \cup int(cl(A))$ and $scl(A) = int(cl(A))$ if A is open. Thus, $X = \cup_{n\in\mathbf{N}} \cup \{int(cl(B)) : B \in \mathcal{B}_n\}$. For every n, let $\mathcal{G}_n$ be a set of a finite number of members of $\mathcal{A}_n$ whose union is $int(cl(A))$. Then the sequence $(\mathcal{G}_n : n \in \mathbf{N})$ covers X. This proves that $X \models \mathbf{P}_{\mathbf{S}_{fin}}$. □

Theorem 2.13. *If $X \models \mathbf{P}_{\mathbf{nS}_{fin}}$ and C is clopen subset of X, then $C \models \mathbf{P}_{\mathbf{nS}_{fin}}$.*

Proof. Let $(\mathcal{A}_n : n \in \mathbf{N})$ be a sequence of open covers of C. Then, for each $n \in \mathbf{N}$, $\mathcal{B}_n = \{\mathcal{A}_n\} \cup \{X \backslash C\}$ is an open cover of X. Since $X \models \mathbf{P}_{\mathbf{nS}_{fin}}$, there exists a finite subset $\mathcal{B}'_n$ of $\mathcal{B}_n$ for every n in $\mathbf{N}$ such that $\cup_{n\in N}\{scl_X(B) : B \in \mathcal{B}'_n\} = X$. But $X \backslash C$ is clopen, so $scl(X \setminus C) = X \setminus C$, and $\cup_{n\in N}\{scl_X(B) : B \in \mathcal{B}'_n, B \neq X \setminus C\}$ covers C. □

Remark 2.14. If $X \models \mathbf{P}_{\mathbf{nS}_{fin}}$, then X^2 may not satisfy $\mathbf{P}_{\mathbf{nS}_{fin}}$.

Example 2.15. Let $\mathbf{S}$ be the Sorgenfrey line, the set of real numbers be denoted by $\mathbf{R}$ and $i : \mathbf{S} \longrightarrow \mathbf{R}$ be the identity mapping. If A is a subset of $\mathbf{R}$, then denote $i^{-1}(A) = A_{\mathbf{S}}$. Lelek proved in [22] that for every Lusin set L in $\mathbf{R}$, $L_{\mathbf{S}}$ satisfies $\mathbf{P}_{\mathbf{S}_{fin}}$ so $L_{\mathbf{S}} \models \mathbf{P}_{\mathbf{nS}_{fin}}$, but he stated that if $(L \times L) \cap \{(a, b) : a + b = 0\}$ is an uncountable set, then $L_{\mathbf{S}} \times L_{\mathbf{S}}$ does not satisfy $\mathbf{P}_{\mathbf{S}_{fin}}$. Now since $\mathbf{S} \times \mathbf{S}$ is a regular space and $L_{\mathbf{S}} \times L_{\mathbf{S}}$ is a subspace of $\mathbf{S} \times \mathbf{S}$ is regular but not satisfying $\mathbf{P}_{\mathbf{S}_{fin}}$, therefore by Theorem 2.7, $L^2_{\mathbf{S}}$ does not satisfy $\mathbf{P}_{\mathbf{nS}_{fin}}$.

Theorem 2.16. *If $X_1 \models \mathbf{P}_{\mathbf{nS}_{fin}}$ and X_2 is nearly compact, then $X_1 \times X_2 \models \mathbf{P}_{\mathbf{nS}_{fin}}$.*

Proof. Let the sequence of open covers of $X_1 \times X_2$ be $(\mathcal{A}_n : n \in \mathbf{N})$. Then for every n in $\mathbf{N}$ there are open covers $\mathcal{B}_n$ and C_n of X_1 and X_2 respectively such that $\mathcal{A}_n = \mathcal{B}_n \times C_n$. As $X_1 \models \mathbf{P}_{\mathbf{nS}_{fin}}$ and X_2 is nearly compact, there are sequences $(\mathcal{B}'_n : n \in \mathbf{N})$ of finite subsets of $\mathcal{B}_n$ for every n in $\mathbf{N}$ and $(C'_n : n \in \mathbf{N})$ of finite subsets of C_n such that $\cup_{n\in N}\{scl_{X_1}(B) : B \in \mathcal{B}'_n\} = X_1$ and $\cup\{scl_{X_2}(C) : C \in C'_n\} = X_2$ for all n in $\mathbf{N}$. Let $\mathcal{R}_n = \mathcal{B}'_n \times C'_n$. Then for every n in $\mathbf{N}$, $\mathcal{R}_n$ is a finite subset of $\mathcal{A}_n$, and we show that $\cup_{n\in N}\{scl_{X_1\times X_2}(R) : R \in \mathcal{R}_n\}$ covers $X_1 \times X_2$.

Let $(x, y) \in X_1 \times X_2$. Then there exists n in $\mathbf{N}$ and B in $\mathcal{B}'_n$ such that $x \in scl_{X_1}(B)$. There is also $C \in C'_n$ such that $y \in scl_{X_2}(C)$. This shows that $(x, y) \in scl_{X_1}(B) \times scl_{X_2}(C) = scl_{X_1\times X_2}(B \times C)$. This completes the proof. □

Theorem 2.17. *If $X^n \models \mathbf{P}_{\mathbf{nS}_{fin}}$ for every n in $\mathbf{N}$, then X satisfies the following selection hypothesis:*

For every sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of ω−covers of X, there is a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for each n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and for each finite subset F of X there is an n in $\mathbf{N}$ and B in $\mathcal{B}_n$ such that $F \subset scl(B)$.

Proof. Let a sequence of ω−covers of X be denoted by $(\mathcal{A}_n : n \in \mathbf{N})$ and consider a partition of $\mathbf{N}$ into countably many pair wise disjoint infinite subsets such that $\mathbf{N} = N_1 \cup N_2 \cup ... \cup N_n \cup ....$ For all i in $\mathbf{N}$ and each $j \in N_i$, consider $\mathcal{V}_j = \{A^i : A \in \mathcal{A}_j\}$. Then $\{\mathcal{V}_j : j \in N_i\}$ is a sequence of open covers of X^i. As $X^i \models \mathbf{P}_{\mathbf{nS}_{fin}}$, for each i in $\mathbf{N}$, we have a sequence $(C_j : j \in N_i)$ such that for every j, $C_j = \{A^i_{j1}, A^i_{j2}, ..., A^i_{jk(j)}\}$ is a finite subset of $\mathcal{V}_j$ and $\cup_{j\in N_i}\{scl(C) : C \in C_j\}$ is an open cover of X^i. Now, we show that $\{scl(C_{jp}) : 1 \leq p \leq k(j), j \in \mathbf{N}\}$ is an $\omega-$cover of X. Let $F = \{x_1, x_2, ...x_t\}$ be a finite subset of X. Then $(x_1, x_2, ...x_t) \in X^t$, so there is an $l \in N_t$ such that $(x_1, x_2, ...x_t) \in C_l$. Thus, we can find $1 \leq r \leq k(l)$ such that $(x_1, x_2, ...x_t) \in scl(A^t_{l_r}) = (sclA_{l_r})^t$. It is clear that $F \subset scl(A_{l_r}) = scl(B)$. □

Definition 2.18. A mapping $f : Y \longrightarrow Z$ is *nearly continuous* if for every s-regular open set $A \subset Z$, $f^{-1}(A)$ is an open set in Y.

Every continuous mapping is almost continuous and every almost continuous mapping is nearly continuous.

Lemma 2.19. *If $f : Y \longrightarrow Z$ is nearly continuous and open mapping, then for every s-regular open set A, $scl(f^{-1}(A)) \subseteq f^{-1}(scl(A))$.*

Theorem 2.20. *Let $Y \models \mathbf{P}_{\mathbf{nS}_{fin}}$ and $f : Y \longrightarrow Z$ be a nearly continuous and open surjection. Then $Z \models \mathbf{P}_{\mathbf{nS}_{fin}}$.*

Proof. Let $(\mathcal{A}_n : n \in \mathbf{N})$ be a sequence of covers of Z by s-regular open sets and $\mathcal{A}'_n = \{f^{-1}(A) : A \in \mathcal{A}_n\}, n \in \mathbf{N}$. Then $(\mathcal{A}'_n : n \in \mathbf{N})$ is a sequence of open covers of Y. As f is a nearly continuous surjection and $Y \models \mathbf{P}_{\mathbf{nS}_{fin}}$, there exists a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for each n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}'_n$ and $\cup(\mathcal{B}'_n : n \in \mathbf{N})$ covers Y, where $\mathcal{B}'_n = \{scl(B) : B \in \mathcal{B}_n\}$. For every n in $\mathbf{N}$ and B in $\mathcal{B}_n$ we can choose a member A_B in $\mathcal{A}_n$ such that $B = f^{-1}(A_B)$. Let $C_n = \{A_B : B \in \mathcal{B}_n\}$. Now we show that $\cup(C_n : n \in \mathbf{N})$ covers Z. If $z = f(y) \in Z$, then there is an n in $\mathbf{N}$ and B in $\mathcal{B}_n$ such that $y \in scl(B)$. As $B = f^{-1}(A_B)$, $y \in scl(f^{-1}(A_B)) \subset f^{-1}(scl(A_B)) = f^{-1}(A_B)$. Hence, $z = f(y) \in A_B \in C_n$. □

Corollary 2.21. *A continuous open surjective image Z of a space $Y \models \mathbf{P}_{\mathbf{nS}_{fin}}$ satisfies $\mathbf{P}_{\mathbf{nS}_{fin}}$.*

Corollary 2.22. *An almost continuous open surjective image Z of a space $Y \models \mathbf{P}_{\mathbf{nS}_{fin}}$ satisfies $\mathbf{P}_{\mathbf{aS}_{fin}}$.*

Definition 2.23. A mapping $f : Y \longrightarrow Z$ is *nearly open* if the image of every open set is s-regular open.

Lemma 2.24. *If a mapping $f : Y \longrightarrow Z$ is nearly open and f^{-1} is open, then $f^{-1}(scl(A)) \subseteq scl(f^{-1}(A))$ for every open subset A of Z.*

Theorem 2.25. *If $f : Y \to Z$ is nearly open and perfect continuous mapping and $Z \models \mathbf{P}_{\mathbf{nS}_{fin}}$, then $Y \models \mathbf{P}_{\mathbf{nS}_{fin}}$.*

Proof. Let a sequence of open covers of Z be denoted by $(\mathcal{A}_n : n \in \mathbf{N})$. Then due to perfect continuity, there is a finite subcollection $\mathcal{A}_{n_z}$ of $\mathcal{A}_n$ such that $f^{-1}(z) \subset \cup\mathcal{A}_{n_z}$, for all z in Z and every n in $\mathbf{N}$. Let $A_{n_z} = \cup\mathcal{A}_{n_z}$. Then $B_{n_z} = Z - f(Y \setminus A_{n_z})$ is an open neighborhood of z, since f is closed. For all n in $\mathbf{N}$, let $\mathcal{B}_n = \{B_{n_z} : z \in Z\}$. Then $(\mathcal{B}_n : n \in \mathbf{N})$ is a sequence of open covers of Z. Since $Z \models \mathbf{P}_{\mathbf{nS}_{fin}}$, there is a sequence $(\mathcal{B}'_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $\mathcal{B}'_n$ is a finite subset of $\mathcal{B}_n$ and $\cup\{scl(B) : B \in \mathcal{B}'_n\}$ covers Z. We may assume $\mathcal{B}'_n = \{B_{n_{z_i}} : i \leq n'\}$ for all n in $\mathbf{N}$. For every n in $\mathbf{N}$, let $\mathcal{A}'_n = \bigcup_{i\leq n'} \mathcal{A}_{n_{z_i}}$. Then $\mathcal{A}'_n$ is a finite subset of $\mathcal{A}_n$. Since f is nearly open,

$$\begin{aligned} Y &= f^{-1}(\cup_{n\in\mathbf{N}} \cup \{scl(B_{n_{z_i}}) : i \leq n'\}) = \cup_{n\in\mathbf{N}} \cup \{f^{-1}(scl(B_{n_{z_i}})) : i \leq n'\} \\ &\subset \cup_{n\in\mathbf{N}} \cup \{scl(f^{-1}(B_{n_{z_i}})) : i \leq n'\} \subset \cup_{n\in\mathbf{N}} \cup \{scl(A_{n_{z_i}}) : i \leq n'\} \\ &= \cup_{n\in\mathbf{N}} \cup \{scl(\cup\mathcal{A}_{n_{z_i}}) : i \leq n'\} = \cup_{n\in\mathbf{N}} \cup \{scl(A) : A \in \mathcal{A}'_n\}. \end{aligned}$$

Hence, $Y \models \mathbf{P}_{\mathbf{nS}_{fin}}$. □

Definition 2.26. A mapping $f : Y \to Z$ is *n-continuous* if the inverse image of each open set is s-regular open.

Theorem 2.27. *An n-continuous surjective image of a space satisfying $\mathbf{P}_{\mathbf{nS}_{fin}}$ satisfies $\mathbf{P}_{\mathbf{S}_{fin}}$.*

Proof. Let $f : Y \to Z$ be an n-continuous surjective mapping and sequence of open covers of Z be denoted by $(\mathcal{A}_n : n \in \mathbf{N})$. As f is n-continuous, for all n in N and every A in $\mathcal{A}_n$, $f^{-1}(A)$ is s-regular open and $\acute{\mathcal{A}}_n = \{f^{-1}(A) : A \in \mathcal{A}_n\}$ is a cover of Y by s-regular open sets. Since $Y \models \mathbf{P}_{\mathbf{nS}_{fin}}$, there is a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for every n, $\mathcal{B}_n$ is a finite subset of $\acute{\mathcal{A}}_n$ and $\cup_{n\in\mathbf{N}}\{B : B \in \mathcal{B}_n\}$ is a cover of Y. Then $C_n = \{f(B) : B \in \mathcal{B}_n\}$ is a finite subset of $\mathcal{A}_n$ for every n in $\mathbf{N}$ and $\cup_{n\in\mathbf{N}} C_n$ is an open cover of Z. This shows that $Z \models \mathbf{P}_{\mathbf{S}_{fin}}$. □

3. Nearly γ-Sets

A cover $\mathcal{A}$ of X is a *nearly $\gamma-$cover* (resp. *almost $\gamma-$cover* [19]) if it is an infinite cover and for all $x \in X$, $\{A \in \mathcal{A}\colon x \notin scl(A)\}$ (resp. $\{A \in \mathcal{A}\colon x \notin cl(A)\}$) is a finite collection. An almost $\gamma-$cover is a nearly $\gamma-$cover and a nearly $\gamma-$cover is a $\gamma-$cover.

Definition 3.1. A space X is a *nearly $\gamma-$set* if for every sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of $\omega-$covers of X, there is a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for every n in $\mathbf{N}$, $\mathcal{B}_n$ in $\mathcal{A}_n$, $\{\mathcal{B}_n : n \in \mathbf{N}\}$ is a nearly $\gamma-$cover of X.

If a space X satisfies the property of nearly $\gamma-$set (resp. almost $\gamma-$set), then we denote it by $X \models \mathbf{P}_{\mathbf{n}\gamma}$ (resp. $X \models \mathbf{P}_{\mathbf{a}\gamma}$).

Remark 3.2. If $X \models \mathbf{P}_{\gamma}$, then $X \models \mathbf{P}_{\mathbf{n}\gamma}$ but the converse is not true generally.

Example 3.3. Let X be an uncountable set with the uncountable particular point topology. Then $X \models \mathbf{P}_{\mathbf{n}\gamma}$ but it fails to satisfy $\mathbf{P}_{\gamma}$.

Remark 3.4. If a space $X \models \mathbf{P}_{\mathbf{n}\gamma}$ then $X \models \mathbf{P}_{\mathbf{a}\gamma}$ but if $X \models \mathbf{P}_{\mathbf{a}\gamma}$, then it may not satisfy $\mathbf{P}_{\mathbf{n}\gamma}$.

Problem 3.5. *Can we find a space X with $X \models \mathbf{P}_{\mathbf{a}\gamma}$ that does not satisfy $\mathbf{P}_{\mathbf{n}\gamma}$?*

Theorem 3.6. *$X \models \mathbf{P}_{\mathbf{n}\gamma}$ if and only if for every sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of $\omega-$covers of X by s-regular open sets, there is a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for every n in $\mathbf{N}$, $\mathcal{B}_n$ in $\mathcal{A}_n$, $\{\mathcal{B}_n : n \in \mathbf{N}\}$ is a nearly $\gamma-$cover of X.*

Proof. The direct part is obvious from the definition of $\mathbf{P}_{\mathbf{n}\gamma}$ set.

Converse: We have to show that $X \models \mathbf{P}_{\mathbf{n}\gamma}$. Let the sequence of $\omega-$ covers of X be denoted by $(\mathcal{A}_n : n \in \mathbf{N})$ and $(\mathcal{A}'_n : n \in \mathbf{N})$ be a sequence where $\mathcal{A}'_n = \{int(scl(A))\colon A \in \mathcal{A}_n\}$. Then every $\mathcal{A}'_n$ is an $\omega-$ cover of X by $s-$regular open sets. By assumption, there exists a sequence $\{\mathcal{B}_n : n \in \mathbf{N}\}$ such that for every n in $\mathbf{N}$, $\mathcal{B}_n$ in $\mathcal{A}'_n$, $\{\mathcal{B}_n : n \in \mathbf{N}\}$ is a nearly $\gamma-$cover of X. By our construction, for every n in $\mathbf{N}$ and B in $\mathcal{B}_n$ there is A_B in $\mathcal{A}_n$ such that $B = int(scl(A_B))$. Since, $int(scl(A_B)) \subseteq scl(A_B)$, $x \in B = int(scl(A_B)) \subseteq scl\,(A_B)$ for $n > n_0$ and $\bigcup_{n\in\mathbf{N}}\{scl(A_B) : B \in \mathcal{B}_n\}$ covers X. This implies that $X \models \mathbf{P}_{\mathbf{n}\gamma}$. □

Theorem 3.7. *Let $Y \models \mathbf{P}_{\mathbf{n}\gamma}$ and Z be a space. If $f : Y \longrightarrow Z$ is a nearly continuous open surjection, then $Z \models \mathbf{P}_{\mathbf{n}\gamma}$.*

Proof. Consider a sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of $\omega-$covers of Z by s-regular open sets. Let $\mathcal{A}'_n = \{f^{-1}(A) : A \in \mathcal{A}_n\}$. If F is a finite set in Y, then $f(F)$ is a finite set in Z. There exists an $A \in \mathcal{A}_n$ such that $f(F) \subset A$. Then $f^{-1}(A)$ is an open set containing F. Thus, $\mathcal{A}'_n$ is really an $\omega-$cover of Y.

As $Y \models \mathbf{P}_{\mathbf{n}\gamma}$, there exists a sequence $\{B'_n : n \in \mathbf{N}\}$ such that for every n in $\mathbf{N}$ there exists $A_n \in \mathcal{A}_n$ such that $B'_n = f^{-1}(A_n)$ and $\{B'_n : n \in \mathbf{N}\}$ is a nearly $\gamma-$cover of Y. For each n in $\mathbf{N}$, let $B_n = A_n$ so that $f^{-1}(A_n) = B'_n$. If $z = f(y) \in Y$, then there is $n_\circ$ in $\mathbf{N}$ such that for every $n > n_\circ$, $y \in scl(B'_n)$. As $y \in scl(B'_n) = scl\ f^{-1}(B_n) \subseteq f^{-1}(scl(B_n)) = f^{-1}(B_n)$, we have that for every $n > n_\circ$, $z \in B_n$. Hence, $Z \models \mathbf{P}_{\mathbf{n}\gamma}$. □

4. Nearly Star-Menger Spaces

Definition 4.1. If for every sequence $\{\mathcal{A}_n : n \in \mathbf{N}\}$ of open covers of a space X there is a sequence $\{\mathcal{B}_n : n \in \mathbf{N}\}$ such that for every n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\{scl\,(\mathbf{St}(\cup\mathcal{B}_n, \mathcal{A}_n)) : n \in \mathbf{N}\}$ is a cover of X, then X is said to be a *nearly star-Menger space*.

If a space X satisfies the nearly star-Menger property, then we will write it as $X \models \mathbf{P}_{\mathbf{nS}^*_{fin}}$.

Theorem 4.2. *$X \models \mathbf{P}_{\mathbf{nS}^*_{fin}}$ if and only if for every sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of covers of X by s-regular open sets, there is a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for every n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\{scl\,(\mathbf{St}(\cup\mathcal{B}_n, \mathcal{A}_n)) : n \in \mathbf{N}\}$ is a cover of X.*

Proof. An s-regular open set is always open therefore the direct part is obvious.

Converse: We prove that $X \models \mathbf{P}_{\mathbf{nS}^*_{fin}}$. Let $(\mathcal{A}_n : n \in \mathbf{N})$ be a sequence of open covers of X and $(\mathcal{A}'_n : n \in \mathbf{N})$ be a sequence such that for every n in $\mathbf{N}$, $\mathcal{A}'_n = \{int(scl(A)) : A \in \mathcal{A}_n\}$. Then $\mathcal{A}'_n$ is a cover of X by s-regular open sets. By hypothesis, there exists a sequence $(\mathcal{B}_n : n \in \mathbf{N})$ such that for every n in $\mathbf{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}'_n$ and $\{scl(\mathbf{St}(\cup\mathcal{B}_n, \mathcal{A}'_n)) : n \in \mathbf{N}\}$ is a cover of X.

Claim: $\mathbf{St}(A, \mathcal{A}_n) = \mathbf{St}(int(scl(A)), \mathcal{A}_n)$, for each A in $\mathcal{A}_n$.

Since A is open, $A \subset int(scl(A))$ and $\mathbf{St}(A, \mathcal{A}_n) \subset \mathbf{St}(int(scl(A)), \mathcal{A}_n)$. Now let $x \in \mathbf{St}(int(scl(A)), \mathcal{A}_n)$. Then by definition there exists B in $\mathcal{A}_n$ such that x is in B and $B \cap int(scl(A)) \neq \emptyset$. This implies that $B \cap A \neq \emptyset$ and therefore $x \in \mathbf{St}(A, \mathcal{A}_n)$. Hence, $\mathbf{St}(int(scl(A)), \mathcal{A}_n) \subset \mathbf{St}(A, \mathcal{A}_n)$.

Now for each B in $\mathcal{B}_n$, we can choose A_B in $\mathcal{A}_n$ such that $B = int(scl(A_B))$ by our construction. Consider $\mathcal{G}_n = \{A_B : B \in \mathcal{B}_n\}$. We show that $\{\mathbf{St}(\cup\mathcal{G}_n, \mathcal{A}_n) : n \in \mathbf{N}\}$ is a cover of X.

Let $x \in X$. Then there is an n in $\mathbf{N}$ such that $x \in scl(\mathbf{St}(\cup\mathcal{B}_n, \mathcal{A}'_n))$. For each semi neighbourhood B of X, we have $B \cap \mathbf{St}(\cup\mathcal{B}_n, \mathcal{A}'_n) \neq \emptyset$. Then there will be $A \in \mathcal{A}_n$ such that $(B \cap int(scl(A)) \neq \emptyset)$ and $(\cup\mathcal{B}_n \cap int(scl(A)) \neq \emptyset)$ this shows $(B \cap A \neq \emptyset)$ and $(\cup\mathcal{B}_n \cap A \neq \emptyset)$. By claim, we have $\cup\mathcal{G}_n \cap A \neq \emptyset$, thus $x \in scl(\mathbf{St}(\cup\mathcal{G}_n, \mathcal{A}_n))$. □

Lemma 4.3. *If $f : Y \longrightarrow Z$ is nearly continuous and open mapping, then $f(scl(A)) \subseteq scl(f(A))$, where A is an open set in Z.*

Theorem 4.4. *A nearly continuous, open and surjective image Z of a space $Y \models \mathbf{P}_{\mathbf{nS}^*_{fin}}$ satisfies $\mathbf{P}_{\mathbf{nS}^*_{fin}}$.*

Proof. Let $f : Y \longrightarrow Z$ be a nearly continuous, open surjection and $(\mathcal{A}_n : n \in \mathbf{N})$ be a sequence of covers of Z by s-regular open sets. Consider $\mathcal{A}'_n = \{f^{-1}(A) : A \in \mathcal{A}_n\}$ for every n in $\mathbf{N}$. Then due to nearly continuity of f, $(\mathcal{A}'_n : n \in \mathbf{N})$ is a sequence of open covers of Y. Since $Y \models \mathbf{P}_{\mathbf{nS}^*_{fin}}$, there exists a sequence $(\mathcal{B}'_n : n \in \mathbf{N})$ such that for every n in $\mathbf{N}$, $\mathcal{B}'_n$ is a finite subset of $\mathcal{A}'_n$ and $\{scl(\mathbf{St}(\cup\mathcal{B}'_n, \mathcal{A}'_n)) : n \in \mathbf{N}\}$ is a cover of the space Y. Let $\mathcal{B}_n = \{A : f^{-1}(A) \in \mathcal{B}'_n\}$ and $y \in Y$. This implies that $f^{-1}(\cup\mathcal{B}_n) = \cup\mathcal{B}'_n$ and there is an n in $\mathbf{N}$ such that $y \in scl(\mathbf{St}(f^{-1}(\cup\mathcal{B}_n), \mathcal{A}'_n))$. If $z = f(y) \in Z$, then $z \in f(scl(\mathbf{St}(f^{-1}(\cup\mathcal{B}_n), \mathcal{A}'_n)) \subseteq scl(f(\mathbf{St}(f^{-1}(\cup\mathcal{B}_n)), \mathcal{A}'_n)) \subseteq scl(\mathbf{St}(\cup\mathcal{B}_n, \mathcal{A}_n)) = \mathbf{St}(\cup\mathcal{B}_n, \mathcal{A}_n)$. To prove the last inclusion, let $f^{-1}(\cup\mathcal{B}_n) \cap f^{-1}(A) \neq \emptyset$. Then $f(f^{-1}(\cup\mathcal{B}_n)) \cap f(f^{-1}(A)) \neq \emptyset$, therefore $\cup\mathcal{B}_n \cap A \neq \emptyset$.

Hence, the sequence $(\mathcal{B}_n : n \in \mathbf{N})$ guarantees that $Z \models \mathbf{P}_{\mathbf{nS}^*_{fin}}$. □

Theorem 4.5. *If all finite powers of a space X satisfy $\mathbf{P}_{\mathbf{nS}^*_{fin}}$, then $X \models \mathbf{P}_{\mathbf{nS}^*_{fin}}$.*

Proof. Let the sequence of open covers of X be denoted by $(\mathcal{A}_n : n \in \mathbf{N})$. Consider a partitioning of $\mathbf{N}$ into infinitely many pairwise disjoint sets such as $\mathbf{N} = N_1 \cup N_2 \cup ....$ For each k in $\mathbf{N}$ and each j in N_k, let $C_j = \{A_1 \times A_2 \times ... \times A_k : A_1, A_2, ..., A_k \in \mathcal{A}_j\} = \mathcal{A}^k_j$. Then $(C_j : j \in N_k)$ is a sequence of covers of X^k by open sets. As $X^k \models \mathbf{P}_{\mathbf{nS}^*_{fin}}$, we have a sequence $(\mathcal{G}_j : j \in N_k)$ such that for every j, $\mathcal{G}_j$ is a finite subset of C_j and $\bigcup_{j \in N_k}\{scl\left(\mathbf{St}(G, C_j)\right) : H \in \mathcal{G}_j\}$ is a cover of X^k. For each j in N_k and each G in $\mathcal{G}_j$, we have $A_i(G) \in \mathcal{A}_j$ such that $G = A_1(G) \times A_2(G) \times ... \times A_k(G)$, for each $i \leq k$. Now let $\mathcal{V}_j = \{A_i(G) : i \leq k, G \in \mathcal{G}_j\}$. Then for every j in N_k, $\mathcal{V}_j$ is a finite subset of $\mathcal{A}_j$.

We claim that $\{scl\,(\mathbf{St}(\cup\mathcal{B}_n, \mathcal{A}_n)) : n \in \mathbf{N}\}$ is an $\omega-$ cover of X. Let $F = \{x_1, ..., x_t\}$ be a finite subset of X. Then $y = (x_1, ..., x_t) \in X^t$ such that there exists an n in $\mathbf{N}_p$ such that $y \in scl\,(\mathbf{St}(G, \mathcal{A}_n))$, $G \in \mathcal{G}_n$. But $G = A_1(G) \times A_2(G) \times ... \times A_t(G)$, where $A_1(G), A_2(G), ..., A_t(G) \in \mathcal{B}_n$. The point y belongs to some C in C_n of the form $V_1 \times V_2 \times ... \times V_t$, $\mathcal{V}_i \in \mathcal{A}_n$ for every $i \leq t$, which is of the form $A_1(G) \times A_2(G) \times ... \times A_t(G)$. Therefore, for every $i \leq t$, we have $x_i \in scl\,(\mathbf{St}(A_i(G), \mathcal{A}_n)) \subset scl\,(\mathbf{St}(\cup\mathcal{B}_n, \mathcal{A}_n))$, that is, $F \subset scl\,(\mathbf{St}(\cup\mathcal{B}_n, \mathcal{A}_n))$. Therefore, $X \models \mathbf{P}_{\mathbf{nS}^*_{fin}}$. □

Definition 4.6. If for every sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of open covers of X there is a sequence $(F_n : n \in \mathbf{N})$ of finite subsets of X such that $\{scl\,(\mathbf{St}(F_n, \mathcal{U}_n)) : n \in \mathbf{N}\}$ is a cover of X, then X is said to have the *nearly strongly star-Menger property* and is denoted by $X \models \mathbf{P}_{\mathbf{nSS}^*_{fin}}$.

Remark 4.7. A space that satisfies $\mathbf{P}_{\mathbf{nSS}^*_{fin}}$ may not satisfy $\mathbf{P}_{\mathbf{S}_{fin}}$.

Example 4.8. Consider the particular point topology τ_p on the real line $\mathbf{R}$ and consider the open cover $\mathcal{A} = \{\{p, x\} : x \in \mathbf{R}\}$ of $\mathbf{R}$ that does not have a countable subcover. Then $(\mathbf{R}, \tau_p)$ is not Lindelöf and does not satisfy the $\mathbf{P}_{\mathbf{S}_{fin}}$. But if $\mathcal{A}$ is any open cover and $F = \{p\}$ a finite subset of X, $\mathbf{St}(F, \mathcal{A})$ covers $\mathbf{R}$ that is $(\mathbf{R}, \tau_p)$ is strongly star compact so $X \models \mathbf{P}_{\mathbf{SS}^*_{fin}}$ and hence, $X \models \mathbf{P}_{\mathbf{nSS}^*_{fin}}$.

Definition 4.9. ([3]) A space X is *metacompact* if each open cover $\mathcal{A}$ of X has a point-finite open refinement $\mathcal{V}$.

Theorem 4.10. *If $X \models \mathbf{P}_{\mathbf{nSS}^*_{fin}}$ and is metacompact space then $X \models \mathbf{P}_{\mathbf{S}_{fin}}$.*

Proof. Let $X \models \mathbf{P}_{\mathbf{nSS}^*_{fin}}$ and a metacompact space and $(\mathcal{A}_n : n \in \mathbf{N})$ be a sequence of covers of X by open sets. For every n in $\mathbf{N}$, let $\mathcal{B}_n$ be a point-finite open refinement of $\mathcal{A}_n$. As $X \models \mathbf{P}_{\mathbf{nSS}^*_{fin}}$, there exists a sequence $(F_n : n \in \mathbf{N})$ of finite subsets of X such that $\bigcup_{n\in\mathbf{N}} scl\,(\mathbf{St}(F_n, \mathcal{B}_n))$ covers X.

As $\mathcal{B}_n$ is point-finite open refinement and F_n is finite for each n, elements of every F_n belongs to finitely many members of $\mathcal{B}_n$ say $B_{n1}, B_{n2}, B_{n3}, ...B_{nk}$. Consider $\mathcal{B}'_n = \{B_{n1}, B_{n2}, B_{n3}, ...B_{nk}\}$. Therefore, $scl\,(\mathbf{St}(F_n, \mathcal{B}_n)) = \cup\mathcal{B}'_n$ for every n in $\mathbf{N}$, so we have $\bigcup_{n\in\mathbf{N}}(\cup\mathcal{B}'_n)$ covers X. For each $B \in \mathcal{B}'_n$, choose A_B in $\mathcal{A}_n$ such that $B \subset A_B$. Then for each n, $\mathcal{G}_n = \{A_B : B \in \mathcal{B}'_n\}$ is a finite subcollection of $\mathcal{A}_n$ and $\bigcup_{n\in\mathbf{N}}(\cup\mathcal{G}_n)$ covers X. Hence, $X \models \mathbf{P}_{\mathbf{S}_{fin}}$. □

Definition 4.11. ([2]) A space X is said to be *meta-Lindelöf* if every open cover $\mathcal{A}$ of X has a point-countable open refinement $\mathcal{V}$.

Theorem 4.12. *If $X \models \mathbf{P}_{\mathbf{nSS}^*_{fin}}$ and is meta-Lindelöf space, then X is a Lindelöf space.*

Proof. Let $X \models \mathbf{P}_{\mathbf{nSS}^*_{fin}}$ and a meta-Lindelöf space. Let $\mathcal{A}$ be an open cover of X and $\mathcal{B}$ a point-countable open refinement of $\mathcal{A}$. So by our hypothesis, there is $(F_n : n \in \mathbf{N})$ a sequence of finite subsets of X such that $\bigcup_{n\in\mathbf{N}} scl\,(\mathbf{St}(F_n, \mathcal{B}_n))$ covers X.

For every n in $\mathbf{N}$, denote by $\mathcal{G}_n$, the collection of all members of $\mathcal{V}$ which intersect F_n. As $\mathcal{B}$ is point-countable and F_n is finite, $\mathcal{G}_n$ is countable. Hence, the collection $\mathcal{G} = \bigcup_{n\in\mathbf{N}} \mathcal{G}_n$ is a countable subfamily of $\mathcal{B}$ and covers X. For every $G \in \mathcal{G}$, pick a member $A_G \in \mathcal{A}$ such that $G \in A_G$. Then $\{A_G : G \in \mathcal{G}\}$ is a countable subcover of $\mathcal{A}$. Therefore X is a Lindelöf space. □

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