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A Note on the FIP Property for Extensions of Commutative Rings

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Abstract. A ring extension $R \subset S$ is said to be FIP if it has only finitely many intermediate rings between R and S. The main purpose of this paper is to characterize the FIP property for a ring extension, where R is not (necessarily) an integral domain and S may not be an integral domain. Precisely, we establish a generalization of the classical Primitive Element Theorem for an arbitrary ring extension. Also, various sufficient and necessary conditions are given for a ring extension to have or not to have FIP, where $S = R[\alpha]$ with α a nilpotent element of S.

1. Introduction

All rings considered below are commutative and unital; all inclusions of rings are unital. For a ring R, we frequently use Spec(R) (respectively, Max(R)) to denote the set of all prime (respectively, maximal) ideals of R. If $R \subset S$ is an extension of rings, we will denote by $[R, S]$ the set of all R-subalebras of S (that is, the set of rings T such that $R \subseteq T \subseteq S$), by $(R : S) = \{x \in R : xS \subseteq R\}$ the conductor of R in S. In particular, if $[R, S] = \{R, S\}$, we say that $R \subset S$ is a minimal extension [6,9]. Recall from [1] that a ring extension $R \subset S$ is said to have (or to satisfy) FIP (for the "finitely many intermediate algebras property") if $[R, S]$ is finite. The initial work on the FIP property in [1] was motivated in part by a desire to generalize the Primitive Element Theorem, a classical result in field theory: If $K \subset L$ is a finite-dimensional field extension, $L = K[\alpha]$ for some element $\alpha \in L$ if and only if $[K, L]$ is finite. One example of a FIP extension would be any minimal ring extension , and whenever that condition holds, then $S = R[x]$ for each $x \in S \setminus R$. The key connection between the above ideas is that if a ring extension $R \subset S$ has FIP, then any maximal chain $R = R_0 \subset R_1 \subset \ldots \subset R_n = S$ is finite and results from juxtaposing n minimal extensions $R_i \subset R_{i+1}$, $0 \leq i \leq n-1$. The FIP property was introduced and studied in [1] and, along with various related properties, has been treated in many other papers [2–5, 8–11]. In particular, Section 3 of [1] was devoted to the study of ring extension $R \subset S$ satisfying FIP when R is a field. That work culminated in [1, Theorem 3.8] which gave a generalization of the Primitive Element Theorem. Later, Dobbs et al. in [2] completed this study in the case where R is replaced by an arbitrary Artinian reduced ring (cf. [2, Theorem III.2] and [2, Theorem III.5]). The present paper heavily relies on [1] and [2]; we will freely use the characterizations of the FIP extensions that were given there. The plan of this article is as follows: Section 2 was central to the work in [1, Section 3] and that led to the above-mentioned generalizations of the classical Primitive Element. The main result is the following: Let R be an infinite ring all of whose residue class fields are infinite and let $R \subset S$ be an extension such that S/C

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is a reduced ring, where $C = (R : S)$. Then $R \subset S$ has FIP if and only if R/C is an Artinian ring and $S = R[\alpha]$ for some $\alpha \in S$ where α is algebraic over R. (Recall that a ring is said to be reduced if it has no nonzero nilpotent elements). As a consequence, we recover the result obtained by Anderson et al. in [1, Lemma 3.5].

Section 3 studies when FIP holds for ring extensions $R \subset S$ such that $S = R[\alpha]$, where α is a nilpotent element. We establish some necessary and sufficient conditions for which a ring extension of this form has FIP. The first of these appears in Theorem 3.4 which states: Let R be a reduced ring and assume that $S = R[\alpha]$ where α is a nilpotent element of S. Suppose that $R/(R : S)$ is an infinite ring. Then $R \subset S$ is a minimal extension if and only if $(R : S) \in \mathrm{Max}(R)$ and $\alpha^2 \in (R : S)$. Also, we obtain a characterization of $[R, S]$ which satisfies FIP, in term of finite maximal chains. We present the following result in Theorem 3.5: If $S = R[\alpha]$ where $\alpha \in S$ satisfies $\alpha^2 = 0$, then $R \subset S$ has FIP if and only if there exists a finite maximal chain from R to S. As consequence of this result, we establish that if $S = R[\alpha]$ where $\alpha^2 = 0$ and $(R : S)$ is a maximal ideal of R or R has only finitely many ideals, then $R \subset S$ has FIP. Another context for which we find a complete answer is given in Theorem 3.9: If R is a infinite domain and $S = R[\alpha, \beta]$, where $\alpha^2 = \beta^2 = 0$. Then $R \subset S$ has FIP if and only if there exists a finite maximal chain from R to S and either $S = R[\alpha]$ or $S = R[\beta]$.
Finally, any unexplained terminology is standard as in [12] and [13].

2. A generalized Primitive Element Theorem

Consider a ring extension $R \subset S$ that has FIP. Recall from [1, Proposition 2.2 (a), (b)] that S must be a finite-type R-algebra and algebraic over R. Moreover, in case R contains an infinite field, we have that $S = R[\alpha]$ for some $\alpha \in S$ that is algebraic over R (cf. [1, Corollary 3.2] and [1, Lemma 3.5]). Our primary interest in this section is to complete this study, we generalize the last cited results.

Proposition 2.1. *Let $R \subset S$ be an extension of rings such that:*

(i) R/C is a finite ring, where $C = (R : S)$;

(ii) $S = R[\alpha]$ for some $\alpha \in S$.

Then $R \subset S$ has FIP if and only if α is integral over R.

Proof. For the "only if" part, since R/C is a finite ring, we have $\dim(R/C) = 0$ (the Krull dimension of R/C). Moreover, as $R \subset S$ has FIP, then so is $R/C \subset S/C$ [2, Proposition II.4]. It follows from [1,Proposition 3.4 (b)] that S/C is integral over R/C. Whence, S is integral over R, in particular α is integral over R. Conversely, we assume that α is integral over R, then $S/C = (R/C)[\overline{\alpha}]$ where $\overline{\alpha} = \alpha + C \in S/C$ is integral over R/C. Thus, S/C is a finitely generated R/C-module and since R/C is a finite ring, hence S/C is also finite. Then, $R/C \subset S/C$ has FIP. This prove that $R \subset S$ has FIP.
□

Corollary 2.2. *If $S = \mathbb{Z}[\alpha]$ where $\alpha \in S$ is integral over $\mathbb{Z}$, then $\mathbb{Z} \subset S$ has FIP if and only if $(\mathbb{Z} : S) \neq 0$.*

Proof. Suppose that $\mathbb{Z} \subset S$ has FIP and assume, by way of contradiction, that $(\mathbb{Z} : S) = 0$. Since S is a finitely generated $\mathbb{Z}$-module and each non unit of $\mathbb{Z}$ is a non-zero-divisor of $\mathbb{Z}$, then [3, Theorem 2.1] ensures that there exists a infinite chain of intermediate rings between $\mathbb{Z}$ and S. This contradicts the fact that $\mathbb{Z} \subset S$ has FIP. Conversely, it suffice to notice that since $(\mathbb{Z} : S) \neq 0$, then $\mathbb{Z}/(\mathbb{Z} : S)$ is finite. Hence, the result follows from Proposition 2.1. □

To prove our main result, Theorem 2.4, we need the following lemma.

Lemma 2.3. *Let $R \subset S$ be an extension of rings. Denote $C = (R : S)$. If $R \subset S$ has FIP, then R/C is a reduced ring if and only if C is the intersection of finitely many maximal ideals of R.*

Proof. It is clear that if C is the intersection of finitely many maximal ideals of R, then R/C is a finite direct sum of fields. Thus R/C is a reduced ring. Conversely, because $R \subset S$ has FIP, hence $R \subset S$ has FCP (in the sense of [4]). It follows from [4, Theorem 4.2] that R/C is a Artinian ring. Since R/C is a reduced Artinian ring, Wedderburn-Artin Theory (cf. [13, Theorem 3, page 209]) expresses R/C uniquely as the internal direct product of finitely many fields K_i, that is, $R/C = K_1 \times \ldots \times K_n$. Let $\mathrm{Max}(R/C) = \{N_1, \ldots, N_n\} = \{M_1/C, \ldots, M_n/C\}$, where $M_i \in \mathrm{Max}(R)$ and $C \subseteq M_i$ for each $i = 1, \ldots, n$. As $N_1 \cap \ldots \cap N_n = 0$, then $(M_1/C) \cap \ldots \cap (M_n/C) = (M_1 \cap \ldots \cap M_n)/C = 0$. Thus $C = M_1 \cap \ldots \cap M_n$. □

Theorem 2.4 below provides a generalization of the Primitive Element Theorem.

Theorem 2.4. *Let R be an infinite ring all of whose residue class fields are infinite. Let $R \subset S$ be an extension such that S/C is a reduced ring, where $C = (R : S)$. Then $R \subset S$ has FIP if and only if R/C is an Artinian ring and $S = R[\alpha]$ for some $\alpha \in S$ where α is algebraic over R.*

Proof. Notice by [2, Proposition II.4] that $R \subset S$ has FIP if and only if $R/C \subset S/C$ has FIP. For the "only if" part, since S/C is a reduced ring, then R/C is also a reduced ring. It follows from Lemma 2.3 that $C = \bigcap_{i=1}^{n} M_i$, where $M_i \in \mathrm{Max}(R)$ for each i. By the Chinese Remainder Theorem, $R/C = K_1 \times \ldots \times K_n$ such that K_i is a infinite field for each i, and hence R/C is an Artinian ring. It remains to prove that $S = R[\alpha]$ for some $\alpha \in S$. By virtue of [4, Proposition 3.7 (d)], we can identify S/C with $S_1 \times \ldots \times S_n$ such that $K_i \subseteq S_i$ and $R/C \subset S/C$ satisfies FIP if and only if $K_i \subset S_i$ satisfies FIP for each i. Notice that since S/C is a reduced ring, then so is S_i. Then, we conclude form [1, Lemma 3.5] that $R/C \subset S/C$ satisfies FIP if and only if $S_i = K_i[\beta_i]$ where $\beta_i \in S_i$ for each i. Denote $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, then it is easy to verify that $K_1[\beta_1] \times \ldots \times K_n[\beta_n] \cong (K_1 \times \ldots \times K_n)[(\beta_1, \ldots, \beta_n)] = R/C[\beta]$. Therefore, $R/C \subset S/C$ satisfies FIP if and only if $S/C = R/C[\beta]$, where β is algebraic over R/C. This implies that $R \subset S$ satisfies FIP if and only if $S = R[\alpha]$ for some $\alpha \in S$ which is algebraic over R and satisfies $\overline{\alpha} = \alpha + C = \beta$.

For the "if" part, assume that $S = R[\alpha]$ for some $\alpha \in S$ where α is algebraic over R and R/C is an Artinian ring. Since, in addition, R/C is reduced, hence Wedderburn-Artin Theory (cf. [13, Theorem 3, page 209]) expresses R/C uniquely as the internal direct product of finitely many fields K_i, that is, $R/C = K_1 \times \ldots \times K_n$. Again [4, Proposition 3.7 (d)], the ring S/C can be uniquely expressed as a product of rings $S_1 \times \ldots S_n$ where $K_i \subseteq S_i$ for each $i \in \{1, \ldots, n\}$. Moreover, since $S/C = R/C[\overline{\alpha}]$ where $\overline{\alpha} = \alpha + C$, hence reasoning as in the proof of the "only if" part, we deduce that $S_i = K_i[\beta_i]$ where $\overline{\alpha} = (\beta_1, \ldots, \beta_n)$ and β_i is algebraic over K_i. Hence, if K_i is a finite field, then S_i is a finite K_i-vector space. Then, S_i is finite and so $K_i \subseteq S_i$ has FIP. Now, if K_i is infinite field, then [1, Lemma 3.5] ensures that $K_i \subseteq S_i$ has FIP. By globalization, we deduce that $K_i \subseteq S_i$ has FIP for each $i \in \{1, \ldots, n\}$. Then, $R/C \subseteq S/C$ has FIP [4, Proposition 3.7 (d)]. Finally, according to [2, Proposition II.4], we conclude that $R \subset S$ has FIP, which completes the proof. □

In view of Theorem 2.4, the "if" implication is valid, for if R/C is an Artinian ring. The following example will show that the hypothesis "R/C is an Artinian ring" cannot be omitted in the above theorem .

Example 2.5. *Let R be an infinite-dimensional valuation domain with a height 1 prime ideal P. Pick $\alpha \in P$ where $\alpha \neq 0$ and set $S = qf(R)$ the quotient field of R. It is clear that $C = (R : S) = 0$, and hence $R/C \cong R$ is not Artinian. Also $S/C \cong S$ is a reduced ring. On the other hand, [12, Theorem 19] ensures that $S = R[\alpha^{-1}]$. But $R \subset S$ does not have FIP since $\{R_p, p \in \mathrm{Spec}(R)\}$ is an infinite set of intermediate rings between R and $qf(R)$.*

Corollary 2.6. *([1, Lemma 3.5]) Let R be an infinite field, and let $R \subset S$ be an extension such that S is a reduced ring. Then $R \subset S$ has FIP if and only if $S = R[\alpha]$ for some $\alpha \in S$ such that α is algebraic over R.*

Proof. Since R is quasi-local with maximal ideal 0, then $R/0 \cong R$ is infinite. Moreover, as $(R : S) = 0$, hence $S/(R : S) \cong S$ is a reduced ring. Therefore, the conclusion follows readily from Theorem 2.4. □

3. When the generator is a nilpotent element

Consider a ring extension $R \subset S$. In view of the central role that nilpotent elements have played in the study of the FIP property for a ring extension (cf. [1, Theorem 3.8] and Section IV of [2]), we devote this section to completing this study and to investigating when $R \subset S$ has FIP where $S = R[\alpha]$ with α is a nilpotent element of S. We begin with two results giving useful sufficient conditions for FIP to fail.

Proposition 3.1. *Let $R \subset S$ be a ring extension such that $S = R[\alpha]$ where α is a nilpotent element of S. If $(R : S) \in \text{Spec}(R) \setminus \text{Max}(R)$, then $R \subset S$ does not have FIP.*

Proof. Since $(R : S) \in \text{Spec}(R) \setminus \text{Max}(R)$, then $R/(R : S)$ is a integral domain (not a field), and we have $S/(R : S) = (R/(R : S))[\overline{\alpha}]$ where $\overline{\alpha} = \alpha + (R : S)$. We prove that $(0 : \overline{\alpha}) = \{\overline{r} \in R/(R : S) | \overline{r}.\overline{\alpha} = 0\} = 0$. Let $\overline{r} \in R/(R : S)$ such that $\overline{r}.\overline{\alpha} = 0$, hence $\overline{r\alpha} = 0$. It follows that $r\alpha \in (R : S)$. As $(R : S)$ is a prime ideal of R and $\alpha \notin (R : S)$, we conclude that $r \in (R : S)$. This implies that $\overline{r} = 0$, and so $(0 : \overline{\alpha}) = 0$. According to [2, Proposition IV.1], we have that $R/(R : S) \subset S/(R : S)$ does not have FIP, and so is $R \subset S$. □

The following result is a generalization of [2, Proposition IV.1].

Corollary 3.2. *Let R be an integral domain that is not a field, and $R \subset S$ such that $S = R[\alpha]$ where α is a nilpotent element of S. If $(R : S) = 0$, then $R \subset S$ does not have FIP.*

Proposition 3.3. *Let $R \subset S$ be an extension such that $S = R[\alpha]$ where α is a nilpotent element of S. Denote $C = (R : S)$. If $C \in \text{Max}(R)$, then $R \subset S$ has FIP if and only if R/C is finite or R/C is an infinite field and $\alpha^3 \in C$.*

Proof. Notice by [2, Proposition II.4] that $R \subset S$ has FIP if and only if $R/C \subset S/C$ has FIP. We have $S/C = R/C[\overline{\alpha}]$ where $\overline{\alpha} = \alpha + C$. If R/C is finite, then S/C is also finite since S/C is a R/C-vector space. Thus $R/C \subset S/C$ has FIP, and so is $R \subset S$. Now, if R/C is a infinite field, then [1, Lemma 3.6 (b)] ensures that $R/C \subset S/C$ has FIP if and only if $\overline{\alpha}^3 = 0$, that is, $R \subset S$ has FIP if and only if $\alpha^3 \in C$. □

The following result is a characterization of minimal extensions where S is the form $R[\alpha]$ for some nilpotent element $\alpha \in S$.

Theorem 3.4. *Let R be a reduced ring and let $S = R[\alpha]$ where α is a nilpotent element of S. Suppose that $R/(R : S)$ is a infinite ring. Then $R \subset S$ is a minimal extension if and only if $(R : S) \in \text{Max}(R)$ and $\alpha^2 \in (R : S)$.*

Proof. If $R \subset S$ is a minimal (integral) extension, then $C = (R : S) \in \text{Max}(R)$ and from Proposition 3.3 we have $\alpha^3 \in C$. It follows that R/C is a infinite field and $S/C = R/C[\overline{\alpha}]$ where $\overline{\alpha} = \alpha + C$, and so $\overline{\alpha}^3 = 0$. Hence, the proof of [1, Lemma 3.6 (b)] shows that $[R/C, S/C] = \{R/C, R/C[\overline{\alpha}^2], S/C = R/C[\overline{\alpha}]\}$. Moreover, $R/C \subset S/C$ is a minimal extension since $R \subset S$ is a minimal extension, we conclude that either $R/C = R/C[\overline{\alpha}^2]$ or $R/C[\overline{\alpha}^2] = R/C[\overline{\alpha}]$. Then, either $R = R[\alpha^2]$ or $R[\alpha^2] = R[\alpha]$. Suppose that $R[\alpha^2] = R[\alpha]$ and let $n(\geq 2)$ be the index of nilpotency for α. Hence, $\alpha = r_0 + r_1\alpha^2 + r_2\alpha^4 + \ldots + r_{n-1}\alpha^{2(n-1)}$, for some $r_0, r_1 \ldots, r_{n-1} \in R$. Thus, $r_0 = \alpha - (r_1\alpha^2 + r_2\alpha^4 + \ldots + r_{n-1}\alpha^{2(n-1)})$ is a nilpotent element, and so $r_0 = 0$ since R is reduced. This implies that $\alpha = \alpha(r_1\alpha + r_2\alpha^3 + \ldots + r_{n-1}\alpha^{2n-3})$, hence $(r_1\alpha + r_2\alpha^3 + \ldots + r_n\alpha^{2n-3}) = 1$, a contradiction since $(r_1\alpha + r_2\alpha^3 + \ldots + r_n\alpha^{2n-3})$ is a nilpotent element. Therefore, $R = R[\alpha^2]$, and hence $\alpha^2 \in R$. Now, we prove that $\alpha^2 \in C$. Let $x \in S$, then $x = a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_{n-1}\alpha^{n-1}$ for some $a_0, a_1, \ldots, a_{n-1} \in R$. Hence, $\alpha^2 x = a_0\alpha^2 + a_1\alpha^3 + a_2\alpha^5 + \ldots + a_{n-1}\alpha^{n+1}$. Notice that any power of α is a product of a power of α^2 and a power of α^3. As $\alpha^2, \alpha^3 \in R$, it follows that $\alpha^2 x \in R$, and hence $\alpha^2 \in C$. Conversely, since $\alpha^2 \in C$, then $S/C = R/C[\overline{\alpha}]$ where $\overline{\alpha}^2 = 0$. As, in addition, R/C is a infinite field since C is a maximal ideal of R, then the end of the proof of [1, Lemma 3.6 (b)] ensures that $R/C \subset S/C$ is a minimal extension, this implies that $R \subset S$ is also a minimal extension [9, Corollary 1.4]. □

We are now in position to give a characterization of $[R, S]$ which satisfies FIP, in term of finite maximal chains.

Theorem 3.5. *If $R \subset S$ is an extension of rings such that $S = R[\alpha]$ where $\alpha^2 = 0$, then the following conditions are equivalent:*

(i) $R \subset S$ has FIP;

(ii) There exists a finite maximal chain from R to S.

Proof. $(i) \Rightarrow (ii)$ The result is clear since the condition "$R \subset S$ has FIP", implies that any maximal chain from R to S is finite.

$(ii) \Rightarrow (i)$ Since $S = R + R\alpha$, therefore [7, Proposition 4.12] gives a bijection between $[R, S]$ and the set of ideals of R containing $C = (R : S)$. On the other hand, by assumption, there is a finite maximal chain $R = R_0 \subset R_1 \subset \ldots \subset R_n = S$ in $[R, S]$. For each $i = 0, \ldots, n-1$, denote $C_i = (R_i : R_{i+1})$ and $m_i = C_i \cap R$. Since $R_i \subset R_{i+1}$ is both minimal and integral, hence $C_i \in \mathrm{Max}(R_i)$ and so $m_i \in \mathrm{Max}(R)$ [6, Thorme 2.2]. Moreover, it is clear that $C \subseteq C_i$ for each i, thus $C \subseteq \bigcap_{i=0}^{n-1} m_i$. By iteration, we get

$$(\prod_{i=0}^{n-1} m_i)R_n \subseteq (\prod_{i=0}^{n-2} m_i)R_{n-1} \subseteq \ldots \subseteq m_0R_1 \subseteq R.$$

Then, $\prod_{i=0}^{n-1} m_i \subseteq C \subseteq \bigcap_{i=0}^{n-1} m_i$. Hence, the m_i are precisely the uniquely ideals of R containing C. Therefore, $|[R, S]| = |\{m_i \mid i = 0, \ldots, n-1\}|$, this prove that $R \subset S$ has FIP. □

The proof of Theorem 3.5 established the following result.

Proposition 3.6. *Let $R \subset S$ be a ring extension such that $S = R[\alpha]$ where $\alpha^2 = 0$. If $(R : S)$ is a maximal ideal of R or R has only finitely many ideals, then $R \subset S$ has FIP. Moreover, $R \subset S$ is a minimal extension if and only if $(R : S) \in \mathrm{Max}(R)$.*

Remark 3.7. *If $S = R[\alpha]$ where α is a nilpotent element of S of index $n \neq 2$, then Theorem 3.5 does not follow in general. For instance, let R be any infinite field K of characteristic 2 and take $S = K[X]/(X^4) = K[x]$ where $x = X + (X^4)$ and $x^4 = 0$. Then, $\{1, x, x^2, x^3\}$ is a K-vector space basis of S. As $\dim_K(S) < \infty$, then any maximal chain of intermediate rings between K and S is finite, while the failure to satisfy FIP can be seen by applying [1, Lemma 3.6(a)].*

We next give the following lemma which be used often later. Lemma 3.8 provides a generalization of [1, Lemma 2.6 (c)].

Lemma 3.8. *Let $R \subset S$ be an extension. If R is infinite domain and $R \subset S$ has FIP, then S does not contain two nilpotent elements of index 2 which are algebraically independent over R.*

Proof. If the assertion fails, S contains two nilpotent elements α and β of index 2 which are algebraically independent over R. We consider two cases:

Case.1. $\alpha\beta = 0$, then $\{1, \alpha, \beta\}$ is a basis of $R[\alpha, \beta]$ as a finitely generated R-module. For each $r \in R$, consider $T_r = \{a + b\alpha + rb\beta : a, b \in R\}$. It is clear that $R \subseteq T_r \subseteq S$ for each r. Moreover, since α and β are nilpotent elements of index 2, on easy verifies that each T_r is a ring. Also, $T_r \neq T_{r'}$ for each $r \neq r'$. Indeed, if $T_r = T_{r'}$ then $\alpha + r\beta = a_0 + b_0\alpha + r'b_0\beta$ for some $a_0, b_0 \in R$. Since $\{1, \alpha, \beta\}$ is a basis of $R[\alpha, \beta]$, it follows that $a_0 = 0$, $b_0 = 1$ and $r = b_0r'$. This yields that $r = r'$. Since R is infinite, $\{T_r, r \in R\}$ is an infinite collection of intermediate rings between R and S, contradicting that $R \subset S$ has FIP.

Case.2. $\alpha\beta \neq 0$. First, suppose that $\alpha\beta$ is algebraically independent with α and β over R, then $\{1, \alpha, \beta, \alpha\beta\}$ is a basis of $R[\alpha, \beta]$ as a finitely generated R-module. For each $r \in R$, consider $T_r = \{a + b\alpha + rb\alpha\beta : a, b \in R\}$. Reasoning as in the first case, we show that $\{T_r, r \in R\}$ describes an infinite family of rings, contradicting that $R \subset S$ has FIP. In the remaining case, $\alpha\beta = r_0\alpha + r_1\beta$ where $r_0, r_1 \in R$. Let $r \in R$, consider $T_r = \{a + rb\alpha + rc\beta : a, b, c \in R$ such that $b \neq c\}$. Then, T_r is intermediate ring between R and S. Moreover, $T_r \neq T_{r'}$ for each $r \neq r'$. Indeed, if $r\alpha + r\beta = a_0 + r'b_0\alpha + r'c_0\beta$ for some $a_0, b_0, c_0 \in R$ where $b_0 \neq c_0$. Since $\{1, \alpha, \beta\}$ is a basis of $R[\alpha, \beta]$ as a finitely generated R-module, then $a_0 = 0$ and $r = r'b_0 = r'c_0$. Because R is integral domain, it follows that $b_0 = c_0$, the desired contradiction. Therefore, $\{T_r, r \in R\}$ is an infinite collection of intermediate rings between R and S, contradicting that $R \subset S$ has FIP. □

Again, by combining Lemma 3.8 and Theorem 3.5, we obtain directly another characterization of $[R, S]$ which satisfies FIP where $S = R[\alpha, \beta]$ and $\alpha^2 = \beta^2 = 0$:

Theorem 3.9. *Let $R \subset S$ be an extension such that R is infinite domain and $S = R[\alpha, \beta]$, where $\alpha^2 = \beta^2 = 0$. Then $R \subset S$ has FIP if and only if there exists a finite maximal chain from R to S and either $S = R[\alpha]$ or $S = R[\beta]$.*

We close this section by the following proposition.

Proposition 3.10. *Let $R = R_1 \times \ldots \times R_n$ be a finite product of rings and let $R \subset S$ be a ring extension. Using [2, Lemma III.3], identify S with $S_1 \times \ldots \times S_n$. For each $i \in \{1 \ldots, n\}$, consider the following three conditions (which depend on i):*

1. *R_i is finite and S_i is a finitely generated R_i-module;*
2. *R_i is infinite ring all of whose residue class fields are infinite, S_i/C_i is a reduced ring where $C_i = (R_i : S_i)$, R_i/C_i is Artinian and $S_i = R_i[\alpha_i]$ for some $\alpha_i \in S_i$ which is algebraic over R_i.*
3. *R_i is infinite, $(R_i : S_i) \in Max(R_i)$ and $S_i = R_i[\alpha_i]$ for some $\alpha_i \in S_i$ which satisfies $\alpha_i^3 \in (R_i : S_i)$.*

If for each $i \in \{1, \ldots, n\}$, at least one of the conditions (1), (2), (3) holds, then $R \subset S$ has FIP.

Proof. Combine [2, Proposition III.4 (a)] with [4, Proposition 5.1], Theorem 2.4 and Proposition 3.3 . □

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