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Sheffer Type Degenerate Euler and Bernoulli Polynomials

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Abstract. In this paper, we study some special polynomials which are related to sheffer sequence. In addition, we give some new identities for these numbers and polynomials.

1. Introduction

The Sheffer polynomials $S_n(x), (n \geq 0)$ are defined by the generating function to be (see [13])

$$f(t)e^{xg(t)} = \sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!}, \tag{1}$$

where $f(t) = \sum_{i=0}^{n} a_i t^i, (a_0 \neq 0)$ and $g(t) = \sum_{i=0}^{\infty} a_i t^i, (a_1 \neq 0)$.

It is well known that the most famous Sheffer polynomials are the Bernoulli polynomials and the Euler polynomials: the Bernoulli polynomials are defined by the generating function to be (see [1–6])

$$\frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{2}$$

When $x = 0, B_n = B_n(0)$ are called the Bernoulli numbers. The Euler polynomials are given by the generating function to be (see [7, 8, 11, 13, 14])

$$\frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{3}$$

When $x = 0, E_n = E_n(0)$ are called the Euler numbers.

For $\lambda \in \mathbb{R}\backslash\{0\}$, the degenerate Bernoulli and Euler polynomials are defined by Carlitz which are given by the generating functions to be (see [3, 9, 10, 12])

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!} \tag{4}$$

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and

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\varepsilon_{n,\lambda}(x)\frac{t^n}{n!}. \tag{5}$$

Note that $\lim_{\lambda\to\infty}\beta_{n,\lambda}(x)=B_n(x)$ and $\lim_{\lambda\to\infty}\varepsilon_{n,\lambda}(x)=E_n(x)$. When $x=0, \beta_{n,\lambda}=\beta_{n,\lambda}(0)$ and $\varepsilon_{n,\lambda}=\varepsilon_{n,\lambda}(0)$ are called degenerate Bernoulli and Euler numbers, respectively.

Now, we define the λ-analogue of the falling factorial sequences as follows: $(x)_{0,\lambda}=1, (x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), (n\geq 1)$. Note that $\lim_{\lambda\to 1}(x)_{n,\lambda}=x(x-1)(x-2)\cdots(x-(n-1))=(x)_n, (n\geq 1)$.

In this paper, we study some special polynomials which are related to the Sheffer polynomials. Also, we give some identities for these polynomials and numbers. Although some of results in this paper are already presented in [9, 10, 12], the main goal of this paper is to support numerical results for the theoretical identities and investigate new interesting pattern of the zeros of the Sheffer type degenerate Bernoulli and Euler polynomials. To do this, we display the shapes of the polynomials and investigate their zeros.

2. Sheffer type degenerate Euler and Bernoulli polynomials

For $p,q\in\mathbb{R}$, we define the following two degenerate polynomials which are derived from the Taylor expansions of $(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt)$ and $(1+\lambda t)^{\frac{p}{\lambda}}\sin(qt)$:

$$(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt)=\sum_{k=0}^{\infty}C_{k,\lambda}(p,q)\frac{t^k}{k!} \tag{6}$$

and

$$(1+\lambda t)^{\frac{p}{\lambda}}\sin(qt)=\sum_{k=0}^{\infty}S_{k,\lambda}(p,q)\frac{t^k}{k!}. \tag{7}$$

From (6) and (7), we easily derive the following equations:

$$(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt)=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k}{2}]}(p)_{k-2m,\lambda}(-1)^m\binom{k}{2m}q^{2m}\right)\frac{t^k}{k!} \tag{8}$$

and

$$(1+\lambda t)^{\frac{p}{\lambda}}\sin(qt)=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k-1}{2}]}(p)_{k-2m-1,\lambda}(-1)^m\binom{k}{2m+1}q^{2m+1}\right)\frac{t^k}{k!}. \tag{9}$$

By (6), (7), (8) and (9), we get

$$C_{k,\lambda}(p,q)=\sum_{m=0}^{[\frac{k}{2}]}(p)_{k-2m,\lambda}(-1)^m\binom{k}{2m}q^{2m}, \tag{10}$$

and

$$S_{k,\lambda}(p,q)=\sum_{m=0}^{[\frac{k-1}{2}]}(p)_{k-2m-1,\lambda}(-1)^m\binom{k}{2m+1}q^{2m+1}, (k\geq 0). \tag{11}$$

The two degenerate polynomials can be determined explicitly. A few of them are

$$\begin{aligned}
&C_{0,\lambda}(p,q)=1, \quad C_{1,\lambda}(p,q)=p,\\
&C_{2,\lambda}(p,q)=-\lambda p+p^2-q^2,\\
&C_{3,\lambda}(p,q)=2\lambda^2p-3\lambda p^2+p^3-3pq^2,\\
&C_{4,\lambda}(p,q)=-6\lambda^3p+11\lambda^2p^2-6\lambda p^3+p^4+6\lambda pq^2-6p^2q^2+q^4,
\end{aligned}$$

and

$$\begin{aligned}
&S_{0,\lambda}(p,q)=0, \quad S_{1,\lambda}(p,q)=q,\\
&S_{2,\lambda}(p,q)=2pq,\\
&S_{3,\lambda}(p,q)=-3\lambda pq+3p^2q-q^3,\\
&S_{4,\lambda}(p,q)=8\lambda^2pq-12\lambda p^2q+4p^3q-4pq^3.
\end{aligned}$$

Now, we define the Sheffer type degenerate Euler polynomials which are given by the generating function to be

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt)=\sum_{n=0}^{\infty}\varepsilon_{n,\lambda}^{(C)}(p,q)\frac{t^n}{n!} \tag{12}$$

and

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{p}{\lambda}}\sin(qt)=\sum_{n=0}^{\infty}\varepsilon_{n,\lambda}^{(S)}(p,q)\frac{t^n}{n!}. \tag{13}$$

Note that $\varepsilon_{n,\lambda}^{(C)}(p,0)=\varepsilon_{n,\lambda}(p), \varepsilon_{n,\lambda}^{(S)}(p,0)=0, (n\geq 0)$. The Sheffer type degenerate Euler polynomials can be determined explicitly. A few of them are

$$\begin{aligned}
&\varepsilon_{0,\lambda}^{(C)}(p,q)=1, \quad \varepsilon_{1,\lambda}^{(C)}(p,q)=-\frac{1}{2}+p,\\
&\varepsilon_{2,\lambda}^{(C)}(p,q)=\frac{\lambda}{2}-p-\lambda p+p^2-q^2,\\
&\varepsilon_{3,\lambda}^{(C)}(p,q)=\frac{1}{4}-\lambda^2+3\lambda p+2\lambda^2p-\frac{3p^2}{2}-3\lambda p^2+p^3+\frac{3q^2}{2}-3pq^2,\\
&\varepsilon_{4,\lambda}^{(C)}(p,q)=-\frac{3\lambda}{2}+3\lambda^3+p-11\lambda^2p-6\lambda^3p+9\lambda p^2+11\lambda^2p^2-2p^3-6\lambda p^3\\
&\qquad +p^4-3\lambda q^2+6pq^2+6\lambda pq^2-6p^2q^2+q^4,
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon_{0,\lambda}^{(S)}(p,q)=0, \quad \varepsilon_{1,\lambda}^{(S)}(p,q)=q,\\
&\varepsilon_{2,\lambda}^{(S)}(p,q)=-q+2pq,\\
&\varepsilon_{3,\lambda}^{(S)}(p,q)=\frac{3\lambda q}{2}-3pq-3\lambda pq+3p^2q-q^3,\\
&\varepsilon_{4,\lambda}^{(S)}(p,q)=q-4\lambda^2q+12\lambda pq+8\lambda^2pq-6p^2q-12\lambda p^2q+4p^3q+2q^3-4pq^3.
\end{aligned}$$

Now, we observe that

$$\begin{aligned}
\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt)&=\left(\sum_{l=0}^{\infty}\varepsilon_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}C_{m,\lambda}(p,q)\frac{t^m}{m!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\varepsilon_{l,\lambda}C_{n-l,\lambda}(p,q)\right)\frac{t^n}{n!}.
\end{aligned} \tag{14}$$

Therefore, we obtain the following theorem:

Theorem 2.1. *For $n\geq 0$, we have*

$$\varepsilon_{n,\lambda}^{(C)}(p,q)=\sum_{l=0}^{n}\binom{n}{l}\varepsilon_{l,\lambda}C_{n-l,\lambda}(p,q)$$

and

$$\varepsilon_{n,\lambda}^{(S)}(p,q)=\sum_{l=0}^{n}\binom{n}{l}\varepsilon_{l,\lambda}S_{n-l,\lambda}(p,q).$$

From (12), we have

$$\begin{aligned}2(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt) &= \left(\sum_{l=0}^{\infty}\varepsilon_{l,\lambda}^{(C)}(p,q)\frac{t^l}{l!}\right)\left((1+\lambda t)^{\frac{1}{\lambda}}+1\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\varepsilon_{l,\lambda}^{(C)}(p,q)(1)_{n-l,\lambda}+\varepsilon_{n,\lambda}^{(C)}(p,q)\right)\frac{t^n}{n!}.\end{aligned} \tag{15}$$

By (6) and (15), we get

$$C_{n,\lambda}(p,q) = \frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}\varepsilon_{l,\lambda}^{(C)}(p,q)(1)_{n-l,\lambda}+\varepsilon_{n,\lambda}^{(C)}(p,q)\right). \tag{16}$$

Therefore, we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$C_{n,\lambda}(p,q) = \frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}\varepsilon_{l,\lambda}^{(C)}(p,q)(1)_{n-l,\lambda}+\varepsilon_{n,\lambda}^{(C)}(p,q)\right),$$

and

$$S_{n,\lambda}(p,q) = \frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}\varepsilon_{l,\lambda}^{(S)}(p,q)(1)_{n-l,\lambda}+\varepsilon_{n,\lambda}^{(S)}(p,q)\right).$$

From (6), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\varepsilon_{n,\lambda}^{(C)}(1-p,q)\frac{t^n}{n!} &= \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{1-p}{\lambda}}\cos(qt)\\ &= \frac{2}{(1+\lambda t)^{-\frac{1}{\lambda}}+1}(1+\lambda t)^{-\frac{p}{\lambda}}\cos(-qt)\\ &= \frac{2}{(1+(-\lambda)(-t))^{-\frac{1}{\lambda}}+1}(1+(-\lambda)(-t))^{-\frac{p}{\lambda}}\cos(-qt)\\ &= \left(\sum_{l=0}^{\infty}\varepsilon_{l,-\lambda}(-1)^l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}C_{m,-\lambda}(p,q)(-1)^m\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\varepsilon_{l,-\lambda}C_{n-l,-\lambda}(p,q)\right)\frac{(-1)^n}{n!}t^n.\end{aligned} \tag{17}$$

Therefore, we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\begin{aligned}\varepsilon_{n,\lambda}^{(C)}(1-p,q) &= (-1)^n\sum_{l=0}^{n}\binom{n}{l}\varepsilon_{l,-\lambda}C_{n-l,-\lambda}(p,q)\\ &= (-1)^n\varepsilon_{n,-\lambda}^{(C)}(p,q),\end{aligned}$$

and

$$\begin{aligned}\varepsilon_{n,\lambda}^{(S)}(1-p,q) &= (-1)^{n+1}\varepsilon_{n,-\lambda}^{(S)}(p,q)\\ &= (-1)^{n+1}\sum_{l=0}^{n}\binom{n}{l}\varepsilon_{l,-\lambda}S_{n-l,-\lambda}(p,q).\end{aligned}$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\varepsilon_{n,\lambda}^{(C)}(p+1,q)\frac{t^n}{n!}&=\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{p+1}{\lambda}}\cos(qt)\\&=\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{p}{\lambda}}((1+\lambda)^{\frac{1}{\lambda}}-1+1)\cos(qt)\\&=2(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt)-\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt)\\&=\sum_{n=0}^{\infty}\left(2C_{n,\lambda}(p,q)-\varepsilon_{n,\lambda}^{(C)}(p,q)\right)\frac{t^n}{n!}.\end{aligned}\tag{18}$$

By comparing the coefficients on the both sides, we get

$$\varepsilon_{n,\lambda}^{(C)}(p+1,q)+\varepsilon_{n,\lambda}^{(C)}(p,q)=2C_{n,\lambda}(p,q),\ (n\geq 0).\tag{19}$$

Therefore, we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\varepsilon_{n,\lambda}^{(C)}(p+1,q)+\varepsilon_{n,\lambda}^{(C)}(p,q)=2C_{n,\lambda}(p,q),$$

and

$$\varepsilon_{n,\lambda}^{(S)}(p+1,q)+\varepsilon_{n,\lambda}^{(S)}(p,q)=2S_{n,\lambda}(p,q).$$

From (6) and (9), we have

$$\sum_{k=0}^{\infty}C_{k,\lambda}(0,q)\frac{t^k}{k!}=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}q^{2m}t^{2m}.\tag{20}$$

Therefore, by Theorem 2.4 and (20), we obtain the following corollary.

Corollary 2.5. *For $n \geq 0$, we have*

$$\varepsilon_{2n,\lambda}^{(C)}(1,q)+\varepsilon_{2n,\lambda}^{(C)}(0,q)=2(-1)^n q^{2n},$$

and

$$\varepsilon_{2n+1,\lambda}^{(S)}(1,q)+\varepsilon_{2n+1,\lambda}^{(S)}(0,q)=2(-1)^n q^{2n+1}.$$

By (6) and (7), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\varepsilon_{n,\lambda}^{(C)}(p+r,q)\frac{t^n}{n!}&=\left(\frac{2(1+\lambda t)^{\frac{p}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\cos(qt)\right)(1+\lambda t)^{\frac{r}{\lambda}}\\&=\left(\sum_{l=0}^{\infty}\varepsilon_{l,\lambda}^{(C)}(p,q)\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}(r)_{k,\lambda}\frac{t^k}{k!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\varepsilon_{k,\lambda}^{(C)}(p,q)(r)_{n-k,\lambda}\right)\frac{t^n}{n!}.\end{aligned}\tag{21}$$

Therefore, by comparing the coefficients on the both sides, we obtain the following theorem.

Theorem 2.6. *For $n \geq 0, r \in \mathbb{N}$, we have*

$$\varepsilon_{n,\lambda}^{(C)}(p+r,q) = \sum_{k=0}^{n} \binom{n}{k} \varepsilon_{k,\lambda}^{(C)}(p,q)(r)_{n-k,\lambda},$$

and

$$\varepsilon_{n,\lambda}^{(S)}(p+r,q) = \sum_{k=0}^{n} \binom{n}{k} \varepsilon_{k,\lambda}^{(S)}(p,q)(r)_{n-k,\lambda}.$$

Taking $r = 1$ in Theorem 2.6, we obtain the following corollary.

Corollary 2.7. *For $n \geq 0$, we have*

$$2C_{n,\lambda}(p,q) = \varepsilon_{n,\lambda}^{(C)}(p,q) + \sum_{k=0}^{n} \binom{n}{k} \varepsilon_{k,\lambda}^{(C)}(p,q)(1)_{n-k,\lambda},$$

and

$$2S_{n,\lambda}(p,q) = \varepsilon_{n,\lambda}^{(S)}(p,q) + \sum_{k=0}^{n} \binom{n}{k} \varepsilon_{k,\lambda}^{(S)}(p,q)(1)_{n-k,\lambda}.$$

From Corollary 2.7, we note that

$$\varepsilon_{n,\lambda}^{(C)}(0,q) + \sum_{k=0}^{n} \binom{n}{k} \varepsilon_{k,\lambda}^{(C)}(0,q)(1)_{n-k,\lambda} = \begin{cases} 0 & \text{if } n = 2m+1, \\ 2(-1)^m q^{2m} & \text{if } n = 2m, \end{cases} \tag{22}$$

and

$$\varepsilon_{n,\lambda}^{(S)}(0,q) + \sum_{k=0}^{n} \binom{n}{k} \varepsilon_{k,\lambda}^{(S)}(0,q)(1)_{n-k,\lambda} = \begin{cases} 2(-1)^m q^{2m+1} & \text{if } n = 2m+1, \\ 0 & \text{if } n = 2m. \end{cases} \tag{23}$$

By (12), we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\partial}{\partial p} \varepsilon_{n,\lambda}^{(C)}(p,q) \frac{t^n}{n!} &= \frac{\partial}{\partial p} \left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} (1+\lambda t)^{\frac{p}{\lambda}} \cos(qt) \right) \\
&= \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} \frac{1}{\lambda} \log(1+\lambda t)(1+\lambda t)^{\frac{p}{\lambda}} \cos(qt) \\
&= \left(\sum_{l=0}^{\infty} \varepsilon_{l,\lambda}^{(C)}(p,q) \frac{t^l}{l!} \right) \left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \lambda^{m-1} t^m \right) \\
&= \sum_{n=1}^{\infty} \left(\sum_{l=0}^{n-1} \frac{\varepsilon_{l,\lambda}^{(C)}(p,q)}{l!} \frac{n!}{n-l} (-1)^{n-l-1} \lambda^{n-l-1} \right) \frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty} \left(n \sum_{l=0}^{n-1} \binom{n-1}{l} \varepsilon_{l,\lambda}^{(C)}(p,q)(-1)^{n-l-1} \lambda^{n-l-1} \frac{(n-l-1)!}{n-l} \right) \frac{t^n}{n!}.
\end{aligned} \tag{24}$$

Comparing the coefficients on the both sides of (24), we have

$$\frac{\partial}{\partial p} \varepsilon_{n,\lambda}^{(C)}(p,q) = n \sum_{l=0}^{n-1} \binom{n-1}{l} \varepsilon_{l,\lambda}^{(C)}(p,q)(-1)^{n-l-1} \lambda^{n-l-1} \frac{(n-l-1)!}{n-l}. \tag{25}$$

By the same method, we easily get

$$\left(\frac{\partial}{\partial p}\right)^{2k} \varepsilon_{n,\lambda}^{(C)}(p,q) = \begin{cases} (-1)^k \varepsilon_{n-2k,\lambda}^{(C)}(p,q)\frac{n!}{(n-2k)!} & \text{if } n \geq 2k, \\ 0 & \text{if } n < 2k. \end{cases} \tag{26}$$

Similarly we have

$$\frac{\partial}{\partial q}\varepsilon_{n,\lambda}^{(S)}(p,q) = n\varepsilon_{n-1,\lambda}^{(C)}(p,q), (n \geq 1).$$

Now, we consider the Sheffer type degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt) = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(C)}(p,q)\frac{t^n}{n!}, \tag{27}$$

and

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{p}{\lambda}}\sin(qt) = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(S)}(p,q)\frac{t^n}{n!}. \tag{28}$$

Note that $\beta_{n,\lambda}^{(C)}(p,0) = \beta_{n,\lambda}(p)$ are the Carlitz's degenerate Bernoulli polynomials. The Sheffer type degenerate Bernoulli polynomials can be determined explicitly. A few of them are

$$\begin{aligned}
&\beta_{0,\lambda}^{(C)}(p,q) = 1, \quad \beta_{1,\lambda}^{(C)}(p,q) = -\frac{1}{2}+\frac{\lambda}{2}+p, \\
&\beta_{2,\lambda}^{(C)}(p,q) = \frac{1}{6}-\frac{\lambda^2}{6}-p+p^2-q^2, \\
&\beta_{3,\lambda}^{(C)}(p,q) = -\frac{\lambda}{4}+\frac{\lambda^3}{4}+\frac{p}{2}+\frac{3\lambda p}{2}-\frac{3p^2}{2}-\frac{3\lambda p^2}{2}+p^3+\frac{3q^2}{2}-\frac{3\lambda q^2}{2}-3pq^2, \\
&\beta_{4,\lambda}^{(C)}(p,q) = -\frac{1}{30}+\frac{2\lambda^2}{3}-\frac{19\lambda^4}{30}-2\lambda p-4\lambda^2 p+p^2+6\lambda p^2+4\lambda^2p^2-2p^3-4\lambda p^3 \\
&\qquad +p^4-q^2+\lambda^2q^2+6pq^2-6p^2q^2+q^4,
\end{aligned}$$

and

$$\begin{aligned}
&\beta_{0,\lambda}^{(S)}(p,q) = 0, \quad \beta_{1,\lambda}^{(S)}(p,q) = q, \\
&\beta_{2,\lambda}^{(S)}(p,q) = -q+\lambda q+2pq, \\
&\beta_{3,\lambda}^{(S)}(p,q) = \frac{q}{2}-\frac{\lambda^2 q}{2}-3pq+3p^2q-q^3, \\
&\beta_{4,\lambda}^{(S)}(p,q) = -\lambda q+\lambda^3q+2pq+6\lambda pq-6p^2q-6\lambda p^2q+4p^3q+2q^3-2\lambda q^3-4pq^3.
\end{aligned}$$

From (27), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(C)}(p,q)\frac{t^n}{n!} &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt) \\
&= \left(\sum_{l=0}^{\infty}\beta_{n,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}C_{m,\lambda}(p,q)\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda}C_{n-l,\lambda}(p,q)\right)\frac{t^n}{n!}.
\end{aligned} \tag{29}$$

Comparing the coefficients on the both sides of (29), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$\beta_{n,\lambda}^{(C)}(p,q) = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,\lambda} C_{n-l,\lambda}(p,q),$$

and

$$\beta_{n,\lambda}^{(S)}(p,q) = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,\lambda} S_{n-l,\lambda}(p,q).$$

By replacing p by $1-p$ in (27), we get

$$\begin{aligned}\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(C)}(1-p,q)\frac{t^n}{n!} &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{1-p}{\lambda}}\cos(qt) \\ &= \frac{t}{1-(1+\lambda t)^{-\frac{1}{\lambda}}}(1+\lambda t)^{-\frac{p}{\lambda}}\cos(qt) \\ &= \sum_{n=0}^{\infty} \beta_{n,-\lambda}(p,q)(-1)^n \frac{t^n}{n!}.\end{aligned} \tag{30}$$

Therefore, we obtain the following theorem.

Theorem 2.9. *For $n \geq 0$, we have*

$$\beta_{n,\lambda}^{(C)}(1-p,q) = (-1)^n \beta_{n,-\lambda}^{(C)}(p,q),$$

and

$$\beta_{n,\lambda}^{(S)}(1-p,q) = (-1)^{n+1} \beta_{n,-\lambda}^{(S)}(p,q).$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(C)}(p+1,q)\frac{t^n}{n!} &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{p+1}{\lambda}}\cos(qt) \\ &= t(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt) + \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt) \\ &= \sum_{n=1}^{\infty} nC_{n-1,\lambda}(p,q)\frac{t^n}{n!} + \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(C)}(p,q)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(nC_{n-1,\lambda}(p,q) + \beta_{n,\lambda}^{(C)}(p,q)\right)\frac{t^n}{n!}.\end{aligned} \tag{31}$$

Thus, by (31), we get

$$\beta_{n,\lambda}^{(C)}(p+1,q) = nC_{n-1,\lambda}(p,q) + \beta_{n,\lambda}^{(C)}(p,q), (n \geq 1). \tag{32}$$

Therefore, by (32), we obtain the following theorem.

Theorem 2.10. *For $n \geq 1$, we have*

$$\beta_{n,\lambda}^{(C)}(p+1,q) - \beta_{n,\lambda}^{(C)}(p,q) = nC_{n-1,\lambda}(p,q),$$

and

$$\beta_{n,\lambda}^{(S)}(p+1,q) - \beta_{n,\lambda}^{(S)}(p,q) = nS_{n-1,\lambda}(p,q).$$

By (27), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\partial}{\partial p}\beta_{n,\lambda}^{(C)}(p,q)\frac{t^n}{n!} &= \frac{\partial}{\partial p}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt)\right)\\ &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\frac{1}{\lambda}\log(1+\lambda t)(1+\lambda t)^{\frac{p}{\lambda}}\cos(qt)\\ &= \left(\sum_{l=0}^{\infty}\beta_{l,\lambda}^{(C)}(p,q)\frac{t^l}{l!}\right)\left(\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\lambda^{m-1}t^m\right)\\ &= \sum_{n=1}^{\infty}\left(n\sum_{l=0}^{n-1}\binom{n-1}{l}\beta_{l,\lambda}^{(C)}(p,q)(-1)^{n-l-1}\lambda^{n-l-1}\frac{(n-l-1)!}{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{33}$$

Comparing the coefficients on the both sides of (33), we obtain

$$\frac{\partial}{\partial p}\beta_{n,\lambda}^{(C)}(p,q) = n\sum_{l=0}^{n-1}\binom{n-1}{l}\beta_{l,\lambda}^{(C)}(p,q)(-1)^{n-l-1}\lambda^{n-l-1}\frac{(n-l-1)!}{n-l}.$$

3. Distribution of zeros of the Sheffer type degenerate Bernoulli and Euler polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the Sheffer type degenerate Bernoulli and Euler polynomials. By using the computer, the Sheffer type degenerate Bernoulli and Euler polynomials can be determined explicitly. We display the shapes of the Sheffer type degenerate Bernoulli and Euler polynomials and investigate the zeros of the Sheffer type degenerate Bernoulli and Euler polynomials. We investigate the beautiful zeros of the Sheffer type degenerate Bernoulli and Euler polynomials by using a computer. We plot the zeros of the Sheffer type degenerate Euler polynomials $\varepsilon_{n,\lambda}^{(C)}(p,q)$(Figure 1).

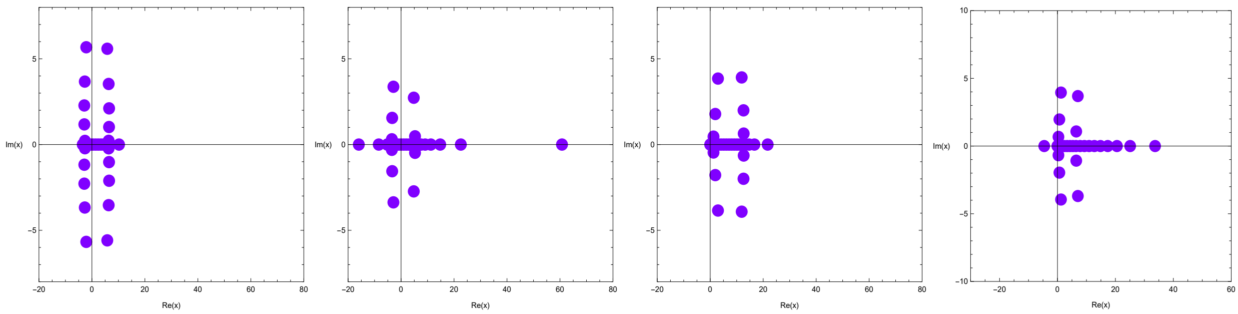

Figure 1: Zeros of $\varepsilon_{n,\lambda}^{(C)}(p,q)$

In Figure 1 (first from left), we choose $n = 30$, $\lambda = \frac{1}{10}$, and $q = \frac{1}{3}$.
In Figure 1 (second from left), we choose $n = 30$, $\lambda = \frac{1}{10}$, and $q = 3$.
In Figure 1 (second from right), we choose $n = 30$, $\lambda = \frac{5}{10}$, and $q = \frac{1}{3}$.
In Figure 1 (first from right), we choose $n = 30$, $\lambda = \frac{5}{10}$, and $q = 3$.
We plot the zeros of the Sheffer type degenerate Euler polynomials $\varepsilon_{n,\lambda}^{(C)}(p,q)$(Figure 2).

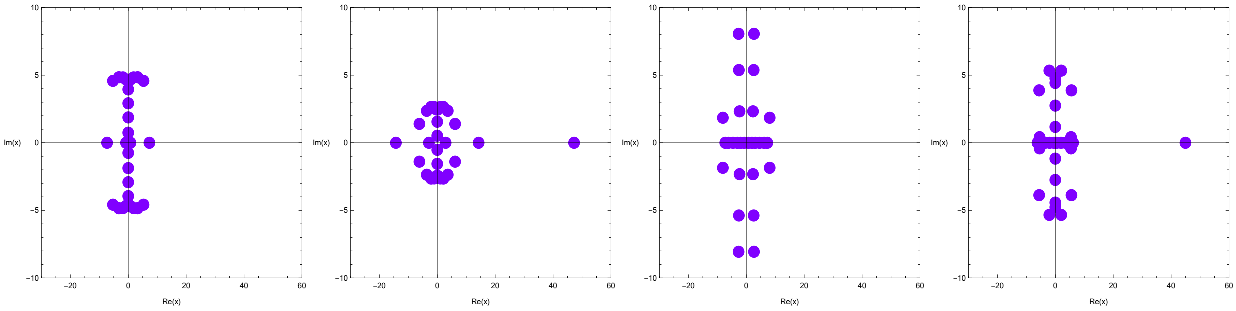

Figure 2: Zeros of $\varepsilon_{n,\lambda}^{(C)}(p,q)$

In Figure 2 (first form left), we choose $n = 30$, $\lambda = \frac{1}{10}$, and $p = \frac{1}{3}$.

In Figure 2 (second from left), we choose $n = 30$, $\lambda = \frac{1}{10}$, and $p = 3$.

In Figure 2 (second from right), we choose $n = 30$, $\lambda = \frac{5}{10}$, and $p = \frac{1}{3}$.

In Figure 2 (first from right), we choose $n = 30$, $\lambda = \frac{5}{10}$, and $p = 3$.

Our numerical results for approximate solutions of real zeros of the Sheffer type degenerate Euler polynomials $\varepsilon_{n,\lambda}^{(C)}(p,q)$ are displayed(Table 1).

Table 1. Numbers of real and complex zeros of $\varepsilon_{n,\frac{1}{10}}^{(C)}(p,\frac{1}{3})$

degree n	real zeros	complex zeros
1	1	0
2	2	0
3	3	0
4	4	0
5	5	0
6	4	2
7	5	2
8	4	4
9	5	4
10	6	4
11	7	4
12	6	6
13	7	6
14	8	6

Our numerical results for approximate solutions of real zeros of the Sheffer type degenerate Bernoulli polynomials $\beta_{n,\lambda}^{(S)}(p,q)$ are displayed(Table 2).

Table 2. Numbers of real and complex zeros of $\beta^{(S)}_{n,\frac{1}{10}}(\frac{1}{3}, q)$

degree n	real zeros	complex zeros
1	1	0
2	1	0
3	1	2
4	1	2
5	3	2
6	1	4
7	3	4
8	1	6
9	3	6
10	3	6
11	3	8
12	3	8
13	3	10
14	3	10

Stacks of zeros of the Sheffer type degenerate Euler polynomials $\varepsilon^{(C)}_{n,\lambda}(p, q)$ for $1 \leq n \leq 30$ from a 3-D structure are presented(Figure 3).

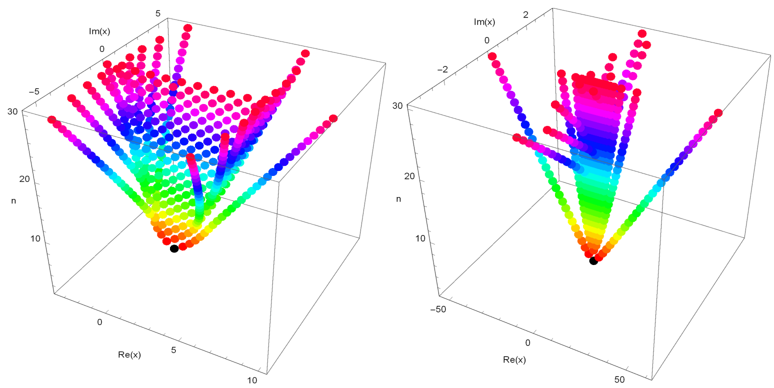

Figure 3: Stacks of zeros of $\varepsilon^{(C)}_{n,\lambda}(p, q), 1 \leq n \leq 30$

In Figure 3 (left), we choose $n = 30$, $\lambda = \frac{1}{10}$, and $p = \frac{1}{3}$.
In Figure 3 (right), we choose $n = 30$, $\lambda = \frac{1}{10}$, and $p = 3$.
Figure 4 presents Real zeros of the Sheffer type degenerate Bernoulli polynomials $\beta^{(S)}_{n,\lambda}(p, q)$ for $1 \leq n \leq 30$.

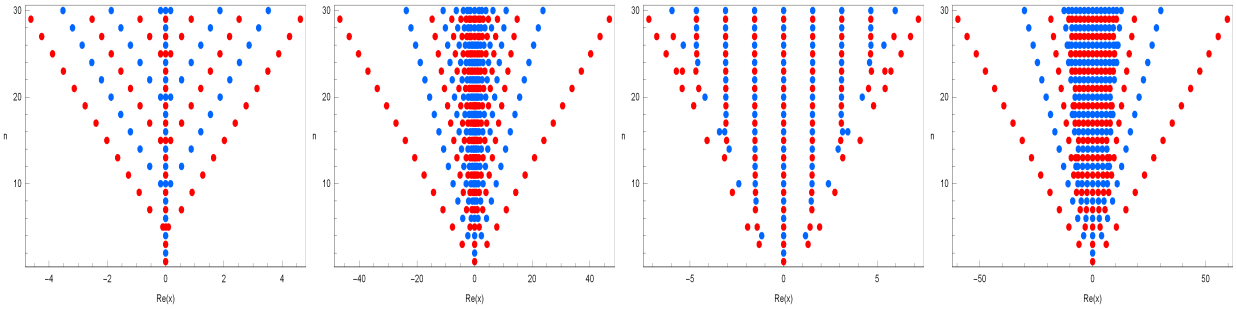

Figure 4: Real zeros of $\beta^{(S)}_{n,\lambda}(p,q)$

In Figure 4 (first form left), we choose $\lambda = \frac{1}{10}$ and $p = \frac{1}{3}$.

In Figure 4 (second from right), we choose $\lambda = \frac{1}{10}$ and $p = 3$.

In Figure 4 (second from right), we choose $\lambda = \frac{5}{10}$ and $p = -\frac{1}{3}$.

In Figure 4 (first from right), we choose $\lambda = \frac{5}{10}$ and $p = -3$.

We observe a remarkable regular structure of the complex roots of the Sheffer type degenerate Euler polynomials $\varepsilon^{(C)}_{n,\lambda}(p,q)$. We also hope to verify a remarkable regular structure of the complex roots of the Sheffer type degenerate Euler polynomials $\varepsilon^{(C)}_{n,\lambda}(p,q)$. Next, we calculated an approximate solution satisfying $\varepsilon^{(C)}_{n,\lambda}(p,q) = 0, p \in \mathbb{R}$. The results are given in Table 3.

Table 3. Approximate solutions of $\varepsilon^{(C)}_{n,\frac{1}{10}}(p,\frac{1}{3})$

degree n	x
1	0.50000
2	−0.053002, 1.1530
3	−0.43003, 0.56949, 1.6605
4	−0.70000, 0.10330, 1.1171, 2.0796
5	−0.87294, −0.31745, 0.66143, 1.5928, 2.4362
6	0.20892, 1.2078, 1.9886, 2.7497
7	−0.24161, 0.75991, 1.7584, 2.2652, 3.0423

Next, we calculated an approximate solution satisfying $\beta^{(S)}_{n,\lambda}(p,q) = 0, q \in \mathbb{R}$. The results are given in

Table 4.

Table 4. Approximate solutions of $\beta^{(S)}_{n,\frac{1}{10}}(\frac{1}{3},q)$

degree n	x
1	0.00000
2	0.00000
3	0.00000
4	0.00000
5	−0.095082, 0.00000, 0.095082
6	0.00000
7	−0.54377, 0.00000, 0.54377

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