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Dual Mixed Complex Brightness Integrals

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Abstract. In this paper, we define the dual mixed complex brightness integrals and establish related Brunn-Minkowski type inequality, Aleksandrov-Fenchel inequality, cyclic inequality and monotonicity inequality, respectively. As applications, we give the analogous version of the differences inequalities for the dual mixed complex brightness integrals.

1. Introduction and main results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of star bodies (about the origin) in $\mathbb{R}^n$, we write $\mathcal{S}_o^n$. Let S^{n-1} denote the unit sphere and $V(K)$ denote the n-dimensional volume of the body K. For the standard unit ball U in $\mathbb{R}^n$, its volume $V(U) = \omega_n$.

The projection bodies were introduced by Minkowski at the turn of the previous century. For each $K \in \mathcal{K}^n$, the projection body, ΠK, of K is an origin-symmetric convex body whose support function is defined by (see [7])

$$h(\Pi K, u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| dS(K, v),$$

for all $u \in S^{n-1}$. Here $S(K, \cdot)$ denotes the surface area measure of K.

The mixed brightness of convex bodies first were given by Lutwak [23]. After, associated with the notion of the projection bodies and the mixed brightness, Li and Zhu [21] introduced the mixed-brightness integrals as follows: For $K_1, \dots, K_n \in \mathcal{K}^n$, the mixed-brightness integrals, $D(K_1, \dots, K_n)$, is defined by

$$D(K_1, \dots, K_n) = \frac{1}{n}\int_{S^{n-1}} \delta(K_1, u) \cdots \delta(K_n, u) dS(u),$$

where $\delta(K, u) = \frac{1}{2}h(\Pi K, u)$ is the half brightness of $K \in \mathcal{K}^n$ in the direction u and for all $u \in S^{n-1}$. Convex bodies $K_1, \dots, K_n$ are said to have similar brightness if there exist constants $\lambda_1, \dots, \lambda_n > 0$ such that $\lambda_1\delta(K_1, u) = \cdots = \lambda_n\delta(K_n, u)$ for all $u \in S^{n-1}$.

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For the mixed brightness integrals, Zhao [38] established the greatest upper bound for the product of the mixed brightness-integrals of a convex body and its polar dual. Whereafter, Zhou, Wang and Feng [41] established some Brunn-Minkowski type inequalities for the mixed brightness integrals. Recently, using the general L_p-projection bodies (see [9, 22, 31, 32, 35]), Yan and Wang [37] defined the general L_p-mixed brightness integrals and established some inequalities of general L_p-mixed brightness integrals.

The theory of real convex bodies goes back to ancient times and continues to be a very active field now. Until recently, the situation with complex convex bodies began to attract attention (see [2, 4, 12–15, 17, 26, 42, 43]). Some classical concepts of convex geometry in real vector space were extended to complex cases, such as complex projection bodies (see [3, 20, 29, 39]), complex difference bodies (see [1]), complex intersection bodies (see [16, 30, 36, 40]), complex centroid bodies (see [10, 19]) and mixed complex brightness integrals (see [18]).

The notion of intersection body was introduced by Lutwak [24]: For $K \in \mathcal{S}_o^n$, the intersection body, IK, of K is a star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by $u^\perp$, the hyperplane orthogonal to u, i.e., for all $u \in S^{n-1}$,

$$\rho(IK, u) = V_{n-1}(K \cap u^\perp),$$

where V_{n-1} denotes $(n-1)$-dimensional volume.

The intersection bodies belong to dual Brunn-Minkowski theory. In 2006, Haberl and Ludwig [8] introduced the L_p-intersection bodies. Further, they [8] defined the asymmetric L_p-intersection bodies. Based on this notion, Wang and Li [33, 34] defined the general L_p-intersection body, (also see [25, 28]). The family of intersection bodies and mixed intersection bodies are valuable in geometry analysis, many important results were obtained (see [7, 27]).

Let $\mathcal{S}(\mathbb{C}^n)$ and $\mathcal{S}_o(\mathbb{C}^n)$ respectively denote the set of star bodies (with respect to the origin) and the set of origin-symmetric star bodies in $\mathbb{C}^n$. The real vector space $\mathbb{R}^n$ of real dimension n is replaced by a complex vector space $\mathbb{C}^n$ of dimension n. Koldobsky, Paouris and Zymonopoulou [16] identified $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ used the standard mapping

$$\xi = (\xi_1, \ldots, \xi_n) = (\xi_{11} + i\xi_{12}, \ldots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}).$$

The unit ball B in $\mathbb{C}^n$ is given by

$$B = \{\xi \in \mathbb{C}^n : \sum_{i=1}^{n} (\xi_{i1}^2 + \xi_{i2}^2) \le 1\}.$$

For the unit sphere in $\mathbb{R}^{2n}$, we write S^{2n-1}. The volume of the unit ball B in $\mathbb{C}^n$ is denoted by ω_{2n}.

For $\xi \in S^{2n-1}$, the complex hyperplane H_ξ is denoted by

$$H_\xi = \{x \in \mathbb{C}^n : (z, \xi) = \sum_{k=1}^{n} z_k \overline{\xi}_k = 0\}.$$

Here $\overline{\xi}$ is the complex conjugate of ξ, the complex hyperplane through the origin, perpendicular to ξ: Under the standard mapping from $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ the hyperplane H_ξ turns into a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \ \ and \ \ \xi^\perp = (-\xi_{12}, \xi_{11}, \ldots, -\xi_{n2}, \xi_{n1}).$$

The orthogonal two-dimensional subspace $H_\xi^\perp$ has orthonormal basis $\{\xi, \xi^\perp\}$. A star body K in $\mathbb{R}^{2n}$ is a complex star body if and only if for every $\xi \in S^{2n-1}$, the section $K \cap H_\xi^\perp$ is a two-dimensional Euclidean circle with radius function $\rho(K, \xi)$.

Recently, Koldobsky, Paouris and Zymonopoulou [16] firstly introduced the notion of mixed complex intersection bodies as follows: For $K, L \in \mathcal{S}_o(\mathbb{C}^n)$, K is the complex intersection body of L and write $K = I^{\mathbb{C}}L$ if for every $\xi \in S^{2n-1}$,

$$V_2(K \cap H_\xi^\perp) = V_{2n-2}(L \cap H_\xi). \tag{1}$$

Here $V_2(K\cap H_\xi^\perp)$ denote the 2-dimensional volume of $K\cap H_\xi^\perp$ and $V_{2n-2}(L\cap H_\xi)$ denote the $(2n-2)$-dimensional volume of $L\cap H_\xi$.

Since $K\cap H_\xi^\perp$ is the 2-dimensional Euclidean circle with radius $\rho(K,\xi)$ for each $\xi\in S^{2n-1}$ (see [16]), using the polar coordinates transform and (1), Koldobsky et al. also gave the radial function of complex intersection body $I^C L$, by

$$\rho(I^C L,\xi)^2=\frac{1}{\pi}V_{2n-2}(L\cap H_\xi)=\frac{1}{2\pi(n-1)}\int_{S^{2n-1}\bigcap H_\xi}\rho_L(u)^{2n-2}du, \tag{2}$$

where du is the standard spherical Lebesgue measure on $S^{2n-1}\bigcap H_\xi$.

In particular, by $I^C B=\sqrt{\frac{\omega_{2n-2}}{\pi}}B$ (see page 1642 of [36]), since for every $\xi\in S^{2n-1}$, by (2), Wang et.al. (see page 422 of [30]) obtained

$$\rho(I^C B,\xi)^2=\frac{1}{2\pi(n-1)}\int_{S^{2n-1}\bigcap H_\xi}du=\frac{\omega_{2n-2}}{\pi}. \tag{3}$$

In this article, similar to the definition of half brightness, based on the notion of complex intersection bodies, we define the dual half complex brightness as follows:

Definition 1.1. *For $K\in\mathcal{S}_o(\mathbb{C}^n)$ and $\xi\in\mathcal{S}^{2n-1}$, the dual half complex brightness, $\widetilde{\delta}^C(K,\xi)$, of K is defined by*

$$\widetilde{\delta}^C(K,\xi)=\frac{1}{2}\rho(I^C K,\xi)^2. \tag{4}$$

According to dual half complex brightness, we give the dual mixed complex brightness integrals of complex star bodies as follows:

Definition 1.2. *For $K_1,\dots,K_{2n}\in\mathcal{S}_o(\mathbb{C}^n)$, the dual mixed complex brightness integral, $\widetilde{D}^C(K_1,\dots,K_{2n})$, of $K_1,\dots,K_{2n}$ is defined by*

$$\widetilde{D}^C(K_1,\dots,K_{2n})=\frac{1}{2n}\int_{S^{2n-1}}\widetilde{\delta}^C(K_1,\xi)\cdots\widetilde{\delta}^C(K_{2n},\xi)dS(\xi). \tag{5}$$

Complex star bodies $K_1,\dots,K_{2n}$ are said to have similar dual complex brightness if there exist constants $\lambda_1,\dots,\lambda_{2n}>0$, such that $\lambda_1\widetilde{\delta}^C(K_1,\xi)=\cdots=\lambda_{2n}\widetilde{\delta}^C(K_{2n},\xi)$ for all $\xi\in\mathcal{S}^{2n-1}$.

Let $\underbrace{K_1,\dots,K_{2n-i}}_{2n-i}=K$ and $\underbrace{K_{2n-i+1},\dots,K_{2n}}_{i}=L$ $(i=0,\dots,2n)$ in (5), we write $\widetilde{D}^C_i(K,L)=\widetilde{D}^C(\underbrace{K,\dots,K}_{2n-i},\underbrace{L,\dots,L}_{i})$. More general, for any real i, if $K,L\in\mathcal{S}_o(\mathbb{C}^n)$, then the ith dual mixed complex brightness integrals, $\widetilde{D}^C_i(K,L)$, of K and L is given by

$$\widetilde{D}^C_i(K,L)=\frac{1}{2n}\int_{S^{2n-1}}\widetilde{\delta}^C(K,\xi)^{2n-i}\widetilde{\delta}^C(L,\xi)^i dS(\xi). \tag{6}$$

For $L=B$ in (6), by (3) and (4), notice that $\widetilde{\delta}^C(B,\xi)=\frac{1}{2}\rho(I^C B,\xi)^2=\frac{\omega_{2n-2}}{2\pi}$, we remark $\widetilde{D}^C_i(K,B)=\widetilde{D}^C_i(K)$, then (6) yields

$$\widetilde{D}^C_i(K)=\frac{\omega^i_{2n-2}}{2^{i+1}\pi^i n}\int_{S^{2n-1}}\widetilde{\delta}^C(K,\xi)^{2n-i}dS(\xi), \tag{7}$$

where $\widetilde{D}^C_i(K)$ is called the ith dual mixed complex brightness integrals of K.

For $L=K$ in (6), write $\widetilde{D}^C_i(K,L)=\widetilde{D}^C(K)$, which is called the dual complex brightness integral of K. Clearly,

$$\widetilde{D}^C(K)=\frac{1}{2n}\int_{S^{2n-1}}\widetilde{\delta}^C(K,\xi)^{2n}dS(\xi). \tag{8}$$

Obviously, by (5), (6), (7) and (8), we have

$$\widetilde{D}^C(K,\dots,K)=\widetilde{D}^C(K), \qquad \widetilde{D}_0^C(K)=\widetilde{D}^C(K); \tag{9}$$

$$\widetilde{D}_0^C(K,L)=\widetilde{D}^C(K), \qquad \widetilde{D}_{2n}^C(K,L)=\widetilde{D}^C(L). \tag{10}$$

In this paper, we establish several inequalities for dual mixed complex brightness integrals. First, we obtain the following Brunn-Minkowski type inequality.

Theorem 1.1. *If $K, L \in \mathcal{S}_o(\mathbb{C}^n)$ and $i \in \mathbb{R}$, then for $i < 2n-1$,*

$$\widetilde{D}_i^C(K\widehat{+}_C L)^{\frac{1}{2n-i}} \le \widetilde{D}_i^C(K)^{\frac{1}{2n-i}} + \widetilde{D}_i^C(L)^{\frac{1}{2n-i}}; \tag{11}$$

for $i > 2n-1$ and $i \neq 2n$,

$$\widetilde{D}_i^C(K\widehat{+}_C L)^{\frac{1}{2n-i}} \ge \widetilde{D}_i^C(K)^{\frac{1}{2n-i}} + \widetilde{D}_i^C(L)^{\frac{1}{2n-i}}. \tag{12}$$

In each case, equality holds if and only if K and L have similar dual complex brightness.

Here, $K\widehat{+}_C L$ is the complex radial Blaschke linear combination of $K, L \in \mathcal{S}(\mathbb{C}^n)$.

Next, we obtain the following Aleksandrov-Fenchel type inequality.

Theorem 1.2. *If $K_1,\dots,K_{2n} \in \mathcal{S}_o(\mathbb{C}^n)$, $1 < m \le 2n$, then*

$$\widetilde{D}^C(K_1,\dots,K_{2n})^m \le \prod_{i=1}^{m} \widetilde{D}^C(K_1,\dots,K_{2n-m},\underbrace{K_{2n-i+1},\dots,K_{2n-i+1}}_{m}), \tag{13}$$

equality holds if and only if $K_{2n-m+1}, K_{2n-m+2},\dots,K_{2n}$ are all of similar dual complex brightness.

Specially, taking $m = 2n$ in Theorem 1.2 and using (9), we obtain the following result.

Corollary 1.1. *If $K_1,\dots,K_{2n} \in \mathcal{S}_o(\mathbb{C}^n)$, then*

$$\widetilde{D}^C(K_1,\dots,K_{2n})^{2n} \le \widetilde{D}^C(K_1)\cdots\widetilde{D}^C(K_{2n}),$$

equality holds if and only if $K_1, K_2,\dots,K_{2n}$ are all of similar dual complex brightness.

Further, we establish the following cyclic inequality for the ith dual mixed complex brightness integrals.

Theorem 1.3. *If $K, L \in \mathcal{S}_o(\mathbb{C}^n)$, $i, j, k \in \mathbb{R}$ and $i < j < k$, then*

$$\widetilde{D}_j^C(K,L)^{k-i} \le \widetilde{D}_i^C(K,L)^{k-j}\widetilde{D}_k^C(K,L)^{j-i}, \tag{14}$$

equality holds if and only if K and L have similar dual complex brightness.

In particular, if $i = 0$ and $k = 2n$ in Theorem 1.3, then by (10), we have the following Minkowski type inequality for the ith dual mixed complex brightness integrals.

Corollary 1.2. *If $K, L \in \mathcal{S}_o(\mathbb{C}^n)$, $j \in \mathbb{R}$ and $0 < j < 2n$, then*

$$\widetilde{D}_j^C(K,L)^{2n} \le \widetilde{D}^C(K)^{2n-j}\widetilde{D}^C(L)^j, \tag{15}$$

equality holds if and only if K and L have similar dual complex brightness. For $j = 0$ or $j = 2n$, equality always holds in (15).

In addition, let $L = B$ in Theorem 1.3, we may obtain the following result.

Corollary 1.3. *If $K, L \in \mathcal{S}_o(\mathbb{C}^n)$, $i, j, k \in \mathbb{R}$ and $i < j < k$, then*

$$\widetilde{D}_j^C(K)^{k-i} \le \widetilde{D}_i^C(K)^{k-j}\widetilde{D}_k^C(K)^{j-i},$$

equality holds if and only if K have similar dual complex brightness, i.e., K has constant dual complex brightness.

Finally, we establish monotonicity inequalities for the ith dual mixed complex brightness integrals as follows:

Theorem 1.4. *If $K, L \in \mathcal{S}_o(\mathbb{C}^n)$, real number $i, j \neq 0$ and $i < j$, then*

$$\left(\frac{\widetilde{D}_i^C(K,L)}{\widetilde{D}^C(K)}\right)^{\frac{1}{i}} \leq \left(\frac{\widetilde{D}_j^C(K,L)}{\widetilde{D}^C(K)}\right)^{\frac{1}{j}}, \tag{16}$$

equality holds if and only if K and L have similar dual complex brightness.

Theorem 1.5. *If $K, L \in \mathcal{S}_o(\mathbb{C}^n)$, real number $i, j \neq 0$ and $i < j$, then*

$$\left(\frac{\widetilde{D}_{2n-i}^C(K,L)}{\widetilde{D}^C(L)}\right)^{\frac{1}{i}} \leq \left(\frac{\widetilde{D}_{2n-j}^C(K,L)}{\widetilde{D}^C(L)}\right)^{\frac{1}{j}}, \tag{17}$$

equality holds if and only if K and L have similar dual complex brightness.

This paper is organized as follows. In Section 2, we collect some basic concepts that will be used in the proofs of our results. In Section 3, we complete the proofs of Theorems 1.1–1.5. Finally, in Section 4, according to Theorem 1.1, Theorem 1.2 and Theorem 1.3, we establish the Brunn-Minkowski, Aleksandrov-Fenchel and cycle type inequalities of differences, which are related to the dual mixed complex brightness integrals, respectively.

2. Notations and Background Materials

For $K \in \mathcal{K}^n$, its support function, $h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$, is defined by (see [7, 27])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

If K be a compact star-shaped set (about the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, is defined by (see [7, 27])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \ \ x \in \mathbb{R}^n \setminus \{0\}. \tag{18}$$

If ρ_K is positive and continuous, K will be called a star body (respect to the origin).

By (18), we easily see that if $K \subseteq L$, then

$$\rho(K, \cdot) \leq \rho(L, \cdot). \tag{19}$$

For star bodies K, L and $\lambda, \mu \geq 0$ (not both zero), the radial Blaschke linear combination, $\lambda \cdot K \widehat{+} \mu \cdot L$, of K and L is defined by (see [7, 27])

$$\rho(\lambda \cdot K \widehat{+} \mu \cdot L, \cdot)^{n-1} = \lambda \rho(K, \cdot)^{n-1} + \mu \rho(L, \cdot)^{n-1}. \tag{20}$$

Above definition (20) was extended to the complex case by Wu et al. (see[36]). For $K, L \in \mathcal{S}(\mathbb{C}^n)$ and $\lambda, \mu \geq 0$ (not both zero), the complex radial Blaschke linear combination, $\lambda \cdot K \widehat{+}_C \mu \cdot L$, of K and L as the star body whose radial function is given by:

$$\rho(\lambda \cdot K \widehat{+}_C \mu \cdot L, \cdot)^{2n-2} = \lambda \rho(K, \cdot)^{2n-2} + \mu \rho(L, \cdot)^{2n-2}. \tag{21}$$

3. Proofs of the Theorems

Proof of Theorem 1.1. For $K, L \in \mathcal{S}_o(\mathbb{C}^n)$ and $i \in \mathbb{R}$, let $p = 2n - i$, since $p > 1$, thus we have that $i \le 2n - 1$. Hence, by (7), (4), (2), (21) and Minkowski integral inequality [11], we obtain

$$
\begin{aligned}
&\widetilde{D}^C_{2n-p}(K\widehat{+}_C L)^{\frac{1}{p}}\\
&= \Big[\frac{\omega_{2n-2}^{2n-p}}{2^{2n-p+1}\pi^{2n-p}n}\int_{S^{2n-1}}\widetilde{\delta}^C(K\widehat{+}_C L,\xi)^p dS(\xi)\Big]^{\frac{1}{p}}\\
&= \Big[\frac{\omega_{2n-2}^{2n-p}}{2^{2n-p+1}\pi^{2n-p}n}\int_{S^{2n-1}}(\frac{1}{2}\rho(I^C(K\widehat{+}_C L),\xi))^{2p} dS(\xi)\Big]^{\frac{1}{p}}\\
&= \Big[\frac{\omega_{2n-2}^{2n-p}}{2^{2n-p+1}\pi^{2n-p}n}\int_{S^{2n-1}}\Big(\frac{1}{2^2}\cdot\frac{1}{2\pi(n-1)}\int_{S^{2n-1}\bigcap H_\xi}\rho(K\widehat{+}_C L,u)^{2n-2}du\Big)^p dS(\xi)\Big]^{\frac{1}{p}}\\
&= \Big[\frac{\omega_{2n-2}^{2n-p}}{2^{2n-p+1}\pi^{2n-p}n}\int_{S^{2n-1}}\Big(\frac{1}{2^2}\cdot\frac{1}{2\pi(n-1)}\int_{S^{2n-1}\bigcap H_\xi}(\rho(K,u)^{2n-2}+\rho(L,u)^{2n-2})du\Big)^p dS(\xi)\Big]^{\frac{1}{p}}\\
&= \Big[\frac{\omega_{2n-2}^{2n-p}}{2^{2n-p+1}\pi^{2n-p}n}\int_{S^{2n-1}}[\widetilde{\delta}^C(K,\xi)+\widetilde{\delta}^C(L,\xi)]^p dS(\xi)\Big]^{\frac{1}{p}}\\
&\le \Big[\frac{\omega_{2n-2}^{2n-p}}{2^{2n-p+1}\pi^{2n-p}n}\int_{S^{2n-1}}[\widetilde{\delta}^C(K,\xi)]^p dS(\xi)\Big]^{\frac{1}{p}} + \Big[\frac{\omega_{2n-2}^{2n-p}}{2^{2n-p+1}\pi^{2n-p}n}\int_{S^{2n-1}}[\widetilde{\delta}^C(L,\xi)]^p dS(\xi)\Big]^{\frac{1}{p}}\\
&= \widetilde{D}^C_{2n-p}(K)^{\frac{1}{p}} + \widetilde{D}^C_{2n-p}(L)^{\frac{1}{p}}.
\end{aligned}
\tag{22}
$$

Let $2n - p = i$ in (22), then inequality (11) is given.

The equality condition of Minkowski integral inequality imply that equality holds in inequality (22) if and only if $\widetilde{\delta}^C(K,\xi) = \lambda\widetilde{\delta}^C(L,\xi)$, where λ is a constant, i.e., K and L have similar dual complex brightness.

Similar to the above method, for $i > 2n - 1$ and $i \neq 2n$, inequality (12) can be obtained from (7), (4), (2), (21) and inverse Minkoeski integral inequality [11]. □

Lemma 3.3 ([6, 11]). *If $f_0, f_1, \dots, f_m$ are (strictly) positive continuous functions defined on S^{n-1} and $\lambda_1, \dots, \lambda_m$ are positive constants the sum of whose reciprocals is unity, then*

$$
\int_{S^{n-1}} f_0(u)\cdots f_m(u)dS(u) \le \prod_{i=1}^m\Big(\int_{S^{n-1}} f_0(u)f_i^{\lambda_i}(u)dS(u)\Big)^{\frac{1}{\lambda_i}}, \tag{23}
$$

with equality if and only if there exist positive constants $\alpha_1, \alpha_2, \dots, \alpha_m$ such that $\alpha_1 f_1^{\lambda_1}(u) = \cdots = \alpha_m f_m^{\lambda_m}(u)$ for all $u \in S^{n-1}$.

Proof of Theorem 1.2. For $K_1, \dots, K_{2n} \in \mathcal{S}_o(\mathbb{C}^n)$, taking

$$
\begin{aligned}
&\lambda_i = m \quad (1 \le i \le m),\\
&f_0(\xi) = \widetilde{\delta}^C(K_1,\xi)\cdots\widetilde{\delta}^C(K_{2n-m},\xi) \quad (if\ m = 2n,\ then\ f_0 = 1),\\
&f_i(\xi) = \widetilde{\delta}^C(K_{2n-i+1},\xi) \quad (1 \le i \le m),
\end{aligned}
$$

then by (23) we obtain

$$
\begin{aligned}
&\int_{S^{2n-1}}\widetilde{\delta}^C(K_1,\xi)\cdots\widetilde{\delta}^C(K_{2n},\xi)dS(\xi)\\
&\le \prod_{i=1}^m\Big(\int_{S^{2n-1}}\widetilde{\delta}^C(K_1,\xi)\cdots\widetilde{\delta}^C(K_{2n-m},\xi)\underbrace{\widetilde{\delta}^C(K_{2n-i+1},\xi)\cdots\widetilde{\delta}^C(K_{2n-i+1},\xi)}_{m}dS(\xi)\Big)^{\frac{1}{m}},
\end{aligned}
\tag{24}
$$

from (5) and (24), we deduce

$$\widetilde{D}^C(K_1,\ldots,K_{2n})^m \le \prod_{i=1}^{m} \widetilde{D}^C(K_1,\ldots,K_{2n-m},\underbrace{K_{2n-i+1},\ldots,K_{2n-i+1}}_{m}).$$

This is just inequality (13).

According to the equality condition of (23), we see that equality holds in (24) if and only if there exist positive constants $\lambda_1,\lambda_2,\ldots,\lambda_m$ such that

$$\lambda_1\widetilde{\delta}^C(K_{2n-m+1},\xi)^m = \lambda_2\widetilde{\delta}^C(K_{2n-m+2},\xi)^m = \cdots = \lambda_m\widetilde{\delta}^C(K_{2n},\xi)^m,$$

for all $\xi \in S^{2n-1}$. Thus equality holds in (13) if and only if $K_{2n-m+1},K_{2n-m+2},\ldots,K_{2n}$ are all of similar dual complex brightness. □

Proof of Theorem 1.3. For $K,L \in \mathcal{S}_o(\mathbb{C}^n)$, if $i<j<k$, then $\frac{k-j}{k-i}+\frac{j-i}{k-i}=1$ and $\frac{k-i}{k-j}>1$, thus from (6) and Hölder's integral inequality [11], we obtain

$$\begin{aligned}
&\widetilde{D}_i^C(K,L)^{\frac{k-j}{k-i}}\widetilde{D}_k^C(K,L)^{\frac{j-i}{k-i}}\\
&=\left(\frac{1}{2n}\int_{S^{2n-1}}\widetilde{\delta}^C(K,\xi)^{2n-i}\widetilde{\delta}^C(L,\xi)^i dS(\xi)\right)^{\frac{k-j}{k-i}}\times\left(\frac{1}{2n}\int_{S^{2n-1}}\widetilde{\delta}^C(K,\xi)^{2n-k}\widetilde{\delta}^C(L,\xi)^k dS(\xi)\right)^{\frac{j-i}{k-i}}\\
&=\left(\frac{1}{2n}\int_{S^{2n-1}}(\widetilde{\delta}^C(K,\xi)^{\frac{(2n-i)(k-j)}{k-i}}\widetilde{\delta}^C(L,\xi)^{\frac{i(k-j)}{k-i}})^{\frac{k-i}{k-j}}dS(\xi)\right)^{\frac{k-j}{k-i}}\times\left(\frac{1}{2n}\int_{S^{2n-1}}(\widetilde{\delta}^C(K,\xi)^{\frac{(2n-k)(j-i)}{k-i}}\widetilde{\delta}^C(L,\xi)^{\frac{k(j-i)}{k-i}})^{\frac{k-i}{j-i}}dS(\xi)\right)^{\frac{j-i}{k-i}}\\
&\ge\frac{1}{2n}\int_{S^{2n-1}}\widetilde{\delta}^C(K,\xi)^{2n-j}\widetilde{\delta}^C(L,\xi)^j dS(\xi)\\
&=\widetilde{D}_j^C(K,L).
\end{aligned}$$

This yields inequality (14).

According to the equality condition of Hölder's integral inequality, we know that equality holds in (14) if and only if there exists a constant $\lambda>0$ such that

$$(\widetilde{\delta}^C(K,\xi)^{\frac{(2n-i)(k-j)}{k-i}}\widetilde{\delta}^C(L,\xi)^{\frac{i(k-j)}{k-i}})^{\frac{k-i}{k-j}}=\lambda(\widetilde{\delta}^C(K,\xi)^{\frac{(2n-k)(j-i)}{k-i}}\widetilde{\delta}^C(L,\xi)^{\frac{k(j-i)}{k-i}})^{\frac{k-i}{j-i}},$$

i.e., $\widetilde{\delta}^C(K,\xi)=\lambda\widetilde{\delta}^C(L,\xi)$ for all $\xi\in S^{2n-1}$. Thus, equality holds in (14) if and only if K and L have similar dual complex brightness. □

Proof of Theorem 1.4. For $K,L\in\mathcal{S}_o(\mathbb{C}^n)$, by (6), we obtain

$$\begin{aligned}
\widetilde{D}_i^C(K,L)&=\frac{1}{2n}\int_{S^{2n-1}}\widetilde{\delta}^C(K,\xi)^{2n-i}\widetilde{\delta}^C(L,\xi)^i dS(\xi)\\
&=\frac{1}{2n}\int_{S^{2n-1}}(\widetilde{\delta}^C(K,\xi)^{2n-j}\widetilde{\delta}^C(L,\xi)^j)^{\frac{i}{j}}(\widetilde{\delta}^C(K,\xi)^{2n})^{\frac{j-i}{j}}dS(\xi).
\end{aligned}\tag{25}$$

If $0<i<j$, then $\frac{i}{j}+\frac{j-i}{j}=1$ and $\frac{j}{i}>1$, thus according to Hölder's integral inequality [11], (25), (6) and (8), we obtain that

$$\begin{aligned}
\widetilde{D}_i^C(K,L)&\le\left(\frac{1}{2n}\int_{S^{2n-1}}\widetilde{\delta}^C(K,\xi)^{2n-j}\widetilde{\delta}^C(L,\xi)^j dS(\xi)\right)^{\frac{i}{j}}\times\left(\frac{1}{2n}\int_{S^{2n-1}}\widetilde{\delta}^C(K,\xi)^{2n}dS(\xi)\right)^{\frac{j-i}{j}}\\
&=\widetilde{D}_j^C(K,L)^{\frac{i}{j}}\widetilde{D}^C(K)^{\frac{j-i}{j}}.
\end{aligned}\tag{26}$$

Since $i > 0$, thus by (26), we get

$$\Big(\frac{\widetilde{D}_i^C(K,L)}{\widetilde{D}^C(K)}\Big)^{\frac{1}{i}} \leq \Big(\frac{\widetilde{D}_j^C(K,L)}{\widetilde{D}^C(K)}\Big)^{\frac{1}{j}}.$$

This yields the desired inequality (16).

According to the equality condition of Hölder's integral inequality, we know that equality holds in (16) if and only if K and L have similar dual complex brightness.

Similar to the above method, if $i < 0 < j$ or $i < j < 0$, then $\frac{j}{i} < 0$ or $0 < \frac{j}{i} < 1$. Thus, by (6), (8) and inverse Hölder's integral inequality [11], we know that inequality (26) is reversed. From this, notice that $i < 0$, we can prove the inequality (16) is true. □

Proof of Theorem 1.5. For $K, L \in \mathcal{S}_o(\mathbb{C}^n)$, by (6), we have

$$\begin{aligned}\widetilde{D}_{2n-i}^C(K,L) &= \frac{1}{2n}\int_{S^{2n-1}} \widetilde{\delta}^C(K,\xi)^i \widetilde{\delta}^C(L,\xi)^{2n-i} dS(\xi)\\ &= \frac{1}{2n}\int_{S^{2n-1}} (\widetilde{\delta}^C(K,\xi)^j \widetilde{\delta}^C(L,\xi)^{2n-j})^{\frac{i}{j}} (\widetilde{\delta}^C(L,\xi)^{2n})^{\frac{j-i}{j}} dS(\xi). \end{aligned} \tag{27}$$

If $0 < i < j$, then $\frac{i}{j} + \frac{j-i}{j} = 1$ and $\frac{j}{i} > 1$, thus by Hölder's integral inequality [11], (27), (6) and (8), we get that

$$\begin{aligned}\widetilde{D}_{2n-i}^C(K,L) &\leq \Big(\frac{1}{2n}\int_{S^{2n-1}} \widetilde{\delta}^C(K,\xi)^j \widetilde{\delta}^C(L,\xi)^{2n-j} dS(\xi)\Big)^{\frac{i}{j}} \times \Big(\frac{1}{2n}\int_{S^{2n-1}} \widetilde{\delta}^C(L,\xi)^{2n} dS(\xi)\Big)^{\frac{j-i}{j}}\\ &= \widetilde{D}_{2n-j}^C(K,L)^{\frac{i}{j}} \widetilde{D}^C(L)^{\frac{j-i}{j}}. \end{aligned} \tag{28}$$

Since $i > 0$, thus by (28), we get

$$\Big(\frac{\widetilde{D}_{2n-i}^C(K,L)}{\widetilde{D}^C(L)}\Big)^{\frac{1}{i}} \leq \Big(\frac{\widetilde{D}_{2n-j}^C(K,L)}{\widetilde{D}^C(L)}\Big)^{\frac{1}{j}}.$$

This deduces the desired inequality (17).

According to the equality condition of Hölder's integral inequality, we know that equality holds in (17) if and only if K and L have similar dual complex brightness.

Similar to the above method, if $i < 0 < j$ or $i < j < 0$, then $\frac{j}{i} < 0$ or $0 < \frac{j}{i} < 1$. Thus, by (6), (8) and inverse Hölder's integral inequality [11], we know that inequality (28) is reversed. From this, notice that $i < 0$, we can prove the inequality (17) is true. □

4. Differences type inequalities

In this part, as the applications of Theorem 1.1-1.3 and their equality conditions, we give the analogous version of the differences inequalities for the dual mixed complex brightness integrals. Firstly, we establish Brunn-Minkowski type inequality of differences inequality which is related dual mixed complex brightness integrals as follows:

Theorem 4.1. *If $K, L, M, N \in \mathcal{S}_o(\mathbb{C}^n)$ and $i \in \mathbb{R}$, and $M \subseteq K$, $N \subseteq L$, K and L have similar dual complex brightness. For $i < 2n-1$, then*

$$[\widetilde{D}_i^C(K\widetilde{+}_C L) - \widetilde{D}_i^C(M\widetilde{+}_C N)]^{\frac{1}{2n-i}} \geq [\widetilde{D}_i^C(K) - \widetilde{D}_i^C(M)]^{\frac{1}{2n-i}} + [\widetilde{D}_i^C(L) - \widetilde{D}_i^C(N)]^{\frac{1}{2n-i}}, \tag{29}$$

for $i > 2n-1$ and $i \neq 2n$, inequality (29) is reversed. Equality holds in (29) if and only if M and N have similar dual complex brightness and $(\widetilde{D}_i^C(K), \widetilde{D}_i^C(M)) = \lambda(\widetilde{D}_i^C(L), \widetilde{D}_i^C(N))$, where λ is a constant. For $i = 2n-1$, (29) is identical.

Lemma 4.1 (Bellman's inequality [5]). *If* $\mathbf{a} = \{a_1, \dots, a_n\}$ *and* $\mathbf{b} = \{b_1, \dots, b_n\}$ *be two series of positive real numbers. If* $a_1^p - \sum_{i=2}^n a_i^p > 0$, $b_1^p - \sum_{i=2}^n b_i^p > 0$, *then for* $p > 1$,

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{\frac{1}{p}} \le \left((a_1 + b_1)^p - \sum_{i=2}^n (a_i + b_i)^p\right)^{\frac{1}{p}}, \tag{30}$$

for $p < 0$ *or* $0 < p < 1$, *inequality (30) is reversed. Equality holds in (30) if and only if* $\mathbf{a} = c\mathbf{b}$, *where c is a constant.*

Proof of Theorem 4.1. For $K, L, M, N \in \mathcal{S}_o(\mathbb{C}^n)$, if $i < 2n-1$, by (11), then

$$\widetilde{D}_i^C(M \widehat{+}_C N)^{\frac{1}{2n-i}} \le \widetilde{D}_i^C(N)^{\frac{1}{2n-i}} + \widetilde{D}_i^C(M)^{\frac{1}{2n-i}}, \tag{31}$$

with equality if and only if M and N have similar dual complex brightness. Since K and L have similar dual complex brightness, thus according to the equality condition of inequality (11), we have

$$\widetilde{D}_i^C(K \widehat{+}_C L)^{\frac{1}{2n-i}} = \widetilde{D}_i^C(K)^{\frac{1}{2n-i}} + \widetilde{D}_i^C(L)^{\frac{1}{2n-i}}. \tag{32}$$

Due to $M \subseteq K$, $N \subseteq L$, by (7), (4), (2) and (19), we deduce

$$\widetilde{D}_i^C(K) \ge \widetilde{D}_i^C(M), \qquad \widetilde{D}_i^C(L) \ge \widetilde{D}_i^C(N), \qquad \widetilde{D}_i^C(K \widehat{+}_C L) \ge \widetilde{D}_i^C(M \widehat{+}_C N),$$

from these, notice that $2n - i > 1$ ($i < 2n-1$), and according to (31), (32) and (30), we obtain

$$\begin{aligned}
&(\widetilde{D}_i^C(K \widehat{+}_C L) - \widetilde{D}_i^C(M \widehat{+}_C N))^{\frac{1}{2n-i}} \\
\ge& \left[\left(\widetilde{D}_i^C(K)^{\frac{1}{2n-i}} + \widetilde{D}_i^C(L)^{\frac{1}{2n-i}}\right)^{2n-i} - \left(\widetilde{D}_i^C(M)^{\frac{1}{2n-i}} + \widetilde{D}_i^C(N)^{\frac{1}{2n-i}}\right)^{2n-i}\right]^{\frac{1}{2n-i}} \\
\ge& \left(\widetilde{D}_i^C(K) - \widetilde{D}_i^C(M)\right)^{\frac{1}{2n-i}} + \left(\widetilde{D}_i^C(L) - \widetilde{D}_i^C(N)\right)^{\frac{1}{2n-i}}.
\end{aligned}$$

This yields inequality (29).

Along the same line, for $i > 2n-1$ and $i \ne 2n$, the reversed inequality of (29) can be deduced directly via of follows from (12) and the reversed case of (30).

By the equality conditions of inequalities (11) and (30), we see that equality holds in (29) if and only if M and N have similar dual complex brightness and there exists constant λ such that $(\widetilde{D}_i^C(K), \widetilde{D}_i^C(M)) = \lambda(\widetilde{D}_i^C(L), \widetilde{D}_i^C(N))$. For $i = 2n-1$, (29) is identical. □

Next, we establish Aleksandrov-Fenchel inequality of differences forms as follows.

Theorem 4.2. *If* $K_1, \dots, K_{2n} \in \mathcal{S}_o(\mathbb{C}^n)$, $L_1, \dots, L_{2n} \in \mathcal{S}_o(\mathbb{C}^n)$ *and* $L_i \subseteq K_i$ $(1 \le i \le 2n)$, $K_{2n-m+1}, K_{2n-m+2}, \dots, K_{2n}$ *are all of similar dual complex brightness and* $1 < m \le 2n$, *then*

$$\begin{aligned}
&\left(\widetilde{D}^C(K_1, \dots, K_{2n}) - \widetilde{D}^C(L_1, \dots, L_{2n})\right)^m \\
\ge& \prod_{i=1}^m \left(\widetilde{D}^C(K_1, \dots, K_{2n-m}, K_{2n-i+1}, \dots, K_{2n-i+1}) - \widetilde{D}^C(L_1, \dots, L_{2n-m}, L_{2n-i+1}, \dots, L_{2n-i+1})\right),
\end{aligned} \tag{33}$$

with equality if and only if $L_{2n-m+1}, L_{2n-m+2}, \dots, L_{2n}$ *are all of similar dual complex brightness and*

$$\begin{aligned}
&[\widetilde{D}^C(K_1, \dots, K_{2n-m}, K_{2n}, \dots, K_{2n}), \dots, \widetilde{D}^C(K_1, \dots, K_{2n-m}, K_{2n-m+1}, \dots, K_{2n-m+1})] \\
=& \lambda[\widetilde{D}^C(L_1, \dots, L_{2n-m}, L_{2n}, \dots, L_{2n}), \dots, \widetilde{D}^C(L_1, \dots, L_{2n-m}, L_{2n-m+1}, \dots, L_{2n-m+1})],
\end{aligned}$$

where λ *is a constant.*

Lemma 4.2 ([11]). *If $c_i > 0,\ b_i > 0,\ c_i > b_i,\ i = 1, \cdots, n$, then*

$$\Big(\prod_{i=1}^{n}(c_i - b_i)\Big)^{\frac{1}{n}} \le \Big(\prod_{i=1}^{n} c_i\Big)^{\frac{1}{n}} - \Big(\prod_{i=1}^{n} b_i\Big)^{\frac{1}{n}}, \tag{34}$$

with equality if and only if $\frac{c_1}{b_1} = \frac{c_2}{b_2} = \cdots = \frac{c_n}{b_n}$.

Proof of Theorem 4.2. For $L_1, \ldots, L_{2n} \in \mathcal{S}_o(\mathbb{C}^n)$, from (13), we know

$$\widetilde{D}^C(L_1, \ldots, L_{2n})^m \le \prod_{i=1}^{m} \widetilde{D}^C(L_1, \ldots, L_{2n-m}, L_{2n-i+1}, \ldots, L_{2n-i+1}), \tag{35}$$

with equality if and only if $L_{2n-m+1}, L_{2n-m+2}, \ldots, L_{2n}$ are all of similar dual complex brightness. Since the bodies $K_{2n-m+1}, K_{2n-m+2}, \ldots, K_{2n}$ are all of similar dual complex brightness, thus by (13), we obtain

$$\widetilde{D}^C(K_1, \ldots, K_{2n})^m = \prod_{i=1}^{m} \widetilde{D}^C(K_1, \ldots, K_{2n-m}, K_{2n-i+1}, \ldots, K_{2n-i+1}). \tag{36}$$

Notice that $L_i \subseteq K_i$ $(1 \le i \le 2n)$ and associated with (5), (4), (2) and (19), we have

$$\widetilde{D}^C(K_1, \ldots, K_{2n}) \ge \widetilde{D}^C(L_1, \ldots, L_{2n}), \tag{37}$$

taking $K_{2n-m+1} = \cdots = K_{2n} = K_{2n-i+1}$, $L_{2n-m+1} = \cdots = L_{2n} = L_{2n-i+1}$ in (37), we obtain

$$\widetilde{D}^C(K_1, \ldots, K_{2n-m}, K_{2n-i+1}, \ldots, K_{2n-i+1}) \ge \widetilde{D}^C(L_1, \ldots, L_{2n-m}, L_{2n-i+1}, \ldots, L_{2n-i+1}).$$

From these, and according to (35), (36) and (34), we deduce

$$\begin{aligned}
&\Big(\widetilde{D}^C(K_1, \ldots, K_{2n}) - \widetilde{D}^C(L_1, \ldots, L_{2n})\Big)^m \\
\ge &\Big[\Big(\prod_{i=1}^{m} \widetilde{D}^C(K_1, \ldots, K_{2n-m}, K_{2n-i+1}, \ldots, K_{2n-i+1})\Big)^{\frac{1}{m}} - \Big(\prod_{i=1}^{m} \widetilde{D}^C(L_1, \ldots, L_{2n-m}, L_{2n-i+1}, \ldots, L_{2n-i+1})\Big)^{\frac{1}{m}}\Big]^m \\
\ge &\prod_{i=1}^{m} \Big(\widetilde{D}^C(K_1, \ldots, K_{2n-m}, K_{2n-i+1}, \ldots, K_{2n-i+1}) - \widetilde{D}^C(L_1, \ldots, L_{2n-m}, L_{2n-i+1}, \ldots, L_{2n-i+1})\Big).
\end{aligned}$$

By the equality conditions of inequalities (35) and (34), we see that equality holds in (33) if and only if $L_{2n-m+1}, L_{2n-m+2}, \ldots, L_{2n}$ are all of similar dual complex brightness and

$$\begin{aligned}
&[\widetilde{D}^C(K_1, \ldots, K_{2n-m}, K_{2n}, \ldots, K_{2n}), \ldots, \widetilde{D}^C(K_1, \ldots, K_{2n-m}, K_{2n-m+1}, \ldots, K_{2n-m+1})] \\
= &\lambda[\widetilde{D}^C(L_1, \ldots, L_{2n-m}, L_{2n}, \ldots, L_{2n}), \ldots, \widetilde{D}^C(L_1, \ldots, L_{2n-m}, L_{2n-m+1}, \ldots, L_{2n-m+1})],
\end{aligned}$$

where λ is a constant. □

Finally, we also obtain a new cycle type inequality for the differences of dual mixed complex brightness integral.

Theorem 4.3. *If $K, L, M, N \in \mathcal{S}_o(\mathbb{C}^n)$, $i, j, k \in \mathbb{R}$ and $M \subseteq K$, $N \subseteq L$, K and L have similar dual complex brightness and $0 \le i < j < k$, then*

$$\Big(\widetilde{D}^C_j(K, L) - \widetilde{D}^C_j(M, N)\Big)^{k-i} \ge \Big(\widetilde{D}^C_k(K, L) - \widetilde{D}^C_k(M, N)\Big)^{j-i}\Big(\widetilde{D}^C_i(K, L) - \widetilde{D}^C_i(M, N)\Big)^{k-j}, \tag{38}$$

with equality if and only if M and N have similar dual complex brightness and there exists a constant λ such that $(\widetilde{D}^C_i(K, L), \widetilde{D}^C_i(M, N)) = \lambda(\widetilde{D}^C_k(K, L), \widetilde{D}^C_k(M, N))$.

Lemma 4.3 (Popviciu's inequality [11]). *If $p > 0$, $q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\mathbf{a} = \{a_1, \ldots, a_n\}$ and $\mathbf{b} = \{b_1, \ldots, b_n\}$ be two series of positive real numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$, $b_1^q - \sum_{i=2}^n b_i^q > 0$, then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \leq a_1 b_1 - \sum_{i=2}^n a_i b_i, \tag{39}$$

with equality if and only if $\mathbf{a} = c\mathbf{b}$, where c is a constant.

Proof of Theorem 4.3. For $K, L, M, N \in \mathcal{S}_o(\mathbb{C}^n)$, $i, j, k \in \mathbb{R}$ and $0 \leq i < j < k$, by (14), then

$$\widetilde{D}_j^C(M,N)^{k-i} \leq \widetilde{D}_i^C(M,N)^{k-j} \widetilde{D}_k^C(M,N)^{j-i}, \tag{40}$$

with equality if and only if M and N have similar dual complex brightness. Since K and L have similar dual complex brightness, thus according to the equality condition of inequality (14), we have

$$\widetilde{D}_j^C(K,L)^{k-i} = \widetilde{D}_i^C(K,L)^{k-j} \widetilde{D}_k^C(K,L)^{j-i}. \tag{41}$$

Hence, by (40) and (41), we get

$$\widetilde{D}_j^C(K,L) - \widetilde{D}_j^C(M,N) \geq \widetilde{D}_i^C(K,L)^{\frac{k-j}{k-i}} \widetilde{D}_k^C(K,L)^{\frac{j-i}{k-i}} - \widetilde{D}_i^C(M,N)^{\frac{k-j}{k-i}} \widetilde{D}_k^C(M,N)^{\frac{j-i}{k-i}}, \tag{42}$$

with equality if and only if M and N have similar dual complex brightness. Notice that $M \subseteq K$, $N \subseteq L$, by (6), (4), (2) and (19), we obtain

$$\widetilde{D}_i^C(K,L) \geq \widetilde{D}_i^C(M,N), \quad \widetilde{D}_k^C(K,L) \geq \widetilde{D}_k^C(M,N).$$

From these, and notice that $\frac{k-i}{k-j} > 0$, $\frac{k-i}{j-i} > 0$ and $\frac{k-j}{k-i} + \frac{j-i}{k-i} = 1$, thus according to (39) we have

$$\widetilde{D}_i^C(K,L)^{\frac{k-j}{k-i}} \widetilde{D}_k^C(K,L)^{\frac{j-i}{k-i}} - \widetilde{D}_i^C(M,N)^{\frac{k-j}{k-i}} \widetilde{D}_k^C(M,N)^{\frac{j-i}{k-i}} \geq \left(\widetilde{D}_i^C(K,L) - \widetilde{D}_i^C(M,N)\right)^{\frac{k-j}{k-i}} \left(\widetilde{D}_k^C(K,L) - \widetilde{D}_k^C(M,N)\right)^{\frac{j-i}{k-i}}.$$

This and (42) give inequality (38).

By the equality conditions of inequalities (42) and (39), we see that equality holds in (38) if and only if M and N have similar dual complex brightness and there exists a constant λ such that $(\widetilde{D}_i^C(K,L), D_i^C(M,N)) = \lambda(\widetilde{D}_k^C(K,L), \widetilde{D}_k^C(M,N))$. □

In particular, if $L = N = B$ in Theorem 4.3, by (7) the following result is obvious.

Corollary 4.1. *If $K, M \in \mathcal{S}_o(\mathbb{C}^n)$, $i, j, k \in \mathbb{R}$ and $M \subseteq K$, K have constant dual complex brightness and $i < j < k$, then*

$$\left(\widetilde{D}_j^C(K) - \widetilde{D}_j^C(M)\right)^{k-i} \geq \left(\widetilde{D}_i^C(K) - \widetilde{D}_i^C(M)\right)^{k-j} \left(\widetilde{D}_k^C(K) - \widetilde{D}_k^C(M)\right)^{j-i},$$

with equality if and only if M have constant dual complex brightness.

If $i = 0$ and $k = 2n$ in (38), then by (10), we have the following cycle Minkowski type inequality for the differences of dual mixed complex brightness integrals.

Corollary 4.2. *If $K, L, M, N \in \mathcal{S}_o(\mathbb{C}^n)$, $M \subseteq K$, $N \subseteq L$, K and L have similar dual complex brightness, $j \in \mathbb{R}$ and $0 < j < 2n$, then*

$$\left(\widetilde{D}_j^C(K,L) - \widetilde{D}_j^C(M,N)\right)^{2n} \geq \left(\widetilde{D}^C(K) - \widetilde{D}^C(M)\right)^{2n-j} \left(\widetilde{D}^C(L) - \widetilde{D}^C(N)\right)^{j},$$

with equality if and only if M and N have similar dual complex brightness and there exists a constant λ such that $(\widetilde{D}^C(K), \widetilde{D}^C(M)) = \lambda(\widetilde{D}^C(L), \widetilde{D}^C(N))$.

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