Filomat 33:19 (2019), 6141–6150
https://doi.org/10.2298/FIL1919141M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Group Invertibility in Rings

Nadica Mihajlović[a], Dragan S. Djordjević[b]

[a]*University of Priština temporarilly settled in Kosovska Mitrovica, Faculty of Technical Sciences, Kneza Miloša 7, 38220 Kosovska Mitrovica, Serbia*
[b]*University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia*

Abstract. We prove some results for the group inverse of elements in a unital ring. Thus, some results from (C. Deng, Electronic J. Linear Algebra 31 (2016)) are extended to more general settings.

1. Introduction

Let R be a ring with the unit 1. We use R^{-1} and $R^{\bullet}$, respectively, to denote the set of all idempotents of R.

We use the following convention on 2×2 matrices induced by projections in rings. Let $x \in R$ and $p, q \in R^{\bullet}$. Then

$$x = pxq + px(1-q) + (1-p)xq + (1-p)x(1-q) \equiv \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}_{p,q},$$

with

$$x_{11} = pxq, \; x_{12} = px(1-q), \; x_{21} = (1-p)xq, \; x_{22} = (1-p)x(1-q).$$

We use $R^{\#}$ and R^D, R^d, respectively, to denote the set of all group invertible and Drazin invertible elements in R (see for example [2]). If $a \in R^D$, then a^D is the Drazin invrse of a. If $\operatorname{ind}(a) \leq 1$, then $a^D = a^{\#}$ reduces to the group inverse of a. It is well-known that $\operatorname{ind}(a) = 0$ if and only if $a \in R^{-1}$ and in this case $a^D = a^{-1}$.

In this paper we extend some operator results from [1] to elements of an arbitrary ring with unit.

If $M \subset R$, then

$$M^{\circ} = \{x \in R : Mx = \{0\}\} \quad \text{and} \quad {}^{\circ}M = \{x \in R : xM = \{0\}\}.$$

We prove the following auxilliary results.

Lemma 1.1. *Let R be a ring with identity, $t \in R$ and $p \in R^{\bullet}$. Then the following hold:*
(1) *$pt = t$ if and only if $tR \subset pR$;*
(2) *$tp = t$ if and only if $t^0 \supset p^0$.*

2010 *Mathematics Subject Classification*. Primary 16B99; Secondary 47A05, 15A09.
Keywords. Group inverse; matrices over a ring.
Received: 06 November 2018; Accepted: 07 December 2018
Communicated by Dijana Mosić
The second autor is supported by the Ministry of Education, Science and Technological developmet of Republic of Serbia, grant no. 174007.
Email addresses: nadica.milenkovic@pr.ac.rs (Nadica Mihajlović), dragandjordjevic70@gmail.com (Dragan S. Djordjević)

Proof. (1) Let $pt = t$, and $tr \in tR$ for some $r \in R$. Then $tr = ptr \in pR$, so $tR \subset pR$

On the other hand, let $tR \subset pR$. Since $t \in tR$, we have $t \in pR$, so $t = pr$ for some $r \in R$. Then $pt = ppr = pr = t$.

(2) Let $tp = t$ and $x \in p^0$. Then $px = 0$, $tpx = 0$, $tx = 0$ and $x \in t^0$. Hence, $t^0 \supset p^0$.

On the other hand, let $t^0 \supset p^0$. Since $1 \in R$, we get $1 - p \in p^0$ and $1 - p \in t^0$. Now, $t(1 - p) = 0$ implies $t = tp$. □

If $t \in R^d$, then $t^\pi = 1 - tt^d$ is the spectral idempotent of t. If R is a Banach algebra, then p can be obtained by the functional calculus.

Similarity in rings is defined in a standard way. Two elements $t, b \in R$ are similar, in the notation $t \sim b$, if there exists some invertible $s \in R$ such that $t = s^{-1}bs$.

Lemma 1.2. *Let $a, b \in R$.*

If ba is group invertible, then ab is Drazin invertible with $\mathrm{ind}(ab) \leq 2$ *and* $(ab)^D = a[(ba)^\sharp]^2 b$.

If both ab and ba are group invertible then $(ab)^\sharp = a[(ba)^\sharp]^2 b$, $(ab)^\sharp a = a(ba)^\sharp$ *and* $b(ab)^\sharp = (ba)^\sharp b$.

Proof. Let $x = a[(ba)^\sharp]^2 b$. Clearly,

$$xabx = a[(ba)^\sharp]^2 baba[(ba)^\sharp]^2 b = a(ba)^\sharp (ba)^\sharp b = a[(ba)^\sharp]^2 b = x,$$

$$abx = aba[(ba)^\sharp]^2 b = a(ba)^\sharp b,$$

$$xab = a[(ba)^\sharp]^2 bab = a(ba)^\sharp b,$$

$$(ab)^3 x = (ab)^3 a[(ba)^\sharp]^2 b = (ab)^2 a(ba)^\sharp b = abab = (ab)^2.$$

Hence, $x = (ab)^D$ and $\mathrm{ind}(ab) \leq 2$.

Moreover, if ab and ba are group invertible, then

$$(ab)^\sharp = (ab)^D = a[(ba)^\sharp]^2 b,$$

$$(ab^\sharp a) = a[(ba)^\sharp]^2 ba = a(ba^\sharp),$$

$$b(ab)^\sharp = ba[(ba)^\sharp]^2 b = (ba)^\sharp b.$$

□

2. Main results

In this section we prove main results of this paper.

Theorem 2.1. *Let R be a ring, $x \in R$, $p \in R^\bullet$, and*

$$x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}_{p,p}.$$

The following assertions hold:

(i) Assume that $d^\sharp$ exists (resp., $a^\sharp$ exists). Then $x^\sharp$ exists if and only if $a^\sharp$ exists (resp., $d^\sharp$ exists) and $a^\pi b d^\pi = 0$.

(ii) Assume $a^\sharp$ and $d^\sharp$ exists. Then $x^\sharp$ exists if and only if $a^\pi b d^\pi = 0$. *In this case,*

$$x^\sharp = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}_{p,p}^\sharp = \begin{pmatrix} a^\sharp & y \\ 0 & d^\sharp \end{pmatrix}_{p,p},$$

where

$$y = (a^\sharp)^2 b d^\pi + a^\pi b (d^\sharp)^2 - a^\sharp b d^\sharp.$$

Proof. Part (1)

$\Longrightarrow$: Assume that $x^\sharp$ and $d^\sharp$ exist. For

$$x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}_{p,p}$$

take

$$x_1 = \begin{pmatrix} y & z \\ 0 & d^\sharp \end{pmatrix}_{p,p}$$

Hence,

$$xx_1x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} y & z \\ 0 & d^\sharp \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} ay & az + bd^\sharp \\ 0 & dd^\sharp \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$
$$= \begin{pmatrix} aya & ayb + azd + bd^\sharp d \\ 0 & dd^\sharp d \end{pmatrix}.$$

We have $xx_1x = x$ if and only if

$$\begin{pmatrix} aya & ayb + azd + bd^\sharp d \\ 0 & dd^\sharp d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

So, $aya = a$. Moreovever,

$$x_1xx_1 = \begin{pmatrix} y & z \\ 0 & d^\sharp \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} y & z \\ 0 & d^\sharp \end{pmatrix} = \begin{pmatrix} ya & yb + zd \\ 0 & d^\sharp d \end{pmatrix}\begin{pmatrix} y & z \\ 0 & d^\sharp \end{pmatrix}$$
$$= \begin{pmatrix} yay & yaz + ybd^\sharp + zdd^\sharp \\ 0 & d^\sharp dd^\sharp \end{pmatrix}$$

We have $x_1xx_1 = x_1$ if and only if

$$\begin{pmatrix} yay & yaz + ybd^\sharp + zdd^\sharp \\ 0 & d^\sharp dd^\sharp \end{pmatrix} = \begin{pmatrix} y & z \\ 0 & d^\sharp \end{pmatrix}.$$

Hence, $yay = y$. We also calculate

$$xx_1 = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} y & z \\ 0 & d^\sharp \end{pmatrix} = \begin{pmatrix} ay & az + bd^\sharp \\ 0 & dd^\sharp \end{pmatrix},$$

and

$$x_1x = \begin{pmatrix} y & z \\ 0 & d^\sharp \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} ya & yb + zd \\ 0 & d^\sharp d \end{pmatrix}.$$

We have $xx_1 = x_1x$ if and only if

$$\begin{pmatrix} ay & az + bd^\sharp \\ 0 & dd^\sharp \end{pmatrix} = \begin{pmatrix} ya & yb + zd \\ 0 & d^\sharp d \end{pmatrix}.$$

Hence, $ay = ya$. Since $aya = a$, $yay = y$ and $ay = ya$, we obtain $y = a^\sharp$.

Notice that by now we have:

$$ayb + azd + bd^\sharp d = b, \quad yaz + ybd^\sharp + zdd^\sharp = z, \quad az + bd^\sharp = yb + zd.$$

We get

$$a(yb+zd)=b-bd^\sharp d,\quad a(az+bd^\sharp)=b-bd^\sharp d,$$

$$a^\sharp aaz+a^\sharp abd^\sharp=a^\sharp b-a^\sharp bd^\sharp d,\quad az+a^\sharp abd^\sharp=a^\sharp b-a^\sharp bd^\sharp d,$$

$$a(az+bd^\sharp)=aa^\sharp b-aa^\sharp bd^\sharp d,\quad b-bd^\sharp d=aa^\sharp b-aa^\sharp bd^\sharp d.$$

The last equality is equivalent to $a^\pi bd^\pi=0$.

$\Longleftarrow$: Assume that both $a^\sharp$ and $d^\sharp$ exists and $a^\pi bd^\pi=0$. Let

$$x=\begin{pmatrix} a & b\\ 0 & d\end{pmatrix},\quad z=\begin{pmatrix} a^\sharp & y\\ 0 & d^\sharp\end{pmatrix}.$$

Then

$$xzx=\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\begin{pmatrix} a^\sharp & y\\ 0 & d^\sharp\end{pmatrix}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}=\begin{pmatrix} aa^\sharp & ay+bd^\sharp\\ 0 & dd^\sharp\end{pmatrix}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}$$
$$=\begin{pmatrix} aa^\sharp a & aa^\sharp b+ayd+bd^\sharp d\\ 0 & dd^\sharp d\end{pmatrix}=\begin{pmatrix} a & aa^\sharp b+ayd+bd^\sharp d\\ 0 & d\end{pmatrix}.$$

We have $xzx=x$ if and only if

$$\begin{pmatrix} a & aa^\sharp b+ayd+bd^\sharp d\\ 0 & d\end{pmatrix}=\begin{pmatrix} a & b\\ 0 & d\end{pmatrix},$$

i.e.

$$aa^\sharp b+ayd+bd^\sharp d=b. \tag{1}$$

We also have

$$zxz=\begin{pmatrix} a^\sharp & y\\ 0 & d^\sharp\end{pmatrix}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\begin{pmatrix} a^\sharp & y\\ 0 & d^\sharp\end{pmatrix}=\begin{pmatrix} a^\sharp a & a^\sharp b+yd\\ 0 & d^\sharp d\end{pmatrix}\begin{pmatrix} a^\sharp & y\\ 0 & d^\sharp\end{pmatrix}$$
$$=\begin{pmatrix} a^\sharp aa^\sharp & a^\sharp ay+a^\sharp bd^\sharp+ydd^\sharp\\ 0 & d^\sharp dd^\sharp\end{pmatrix}=\begin{pmatrix} a^\sharp & a^\sharp ay+a^\sharp bd^\sharp+ydd^\sharp\\ 0 & d^\sharp\end{pmatrix},$$

We conclude $zxz=z$ if and only if

$$\begin{pmatrix} a^\sharp & a^\sharp ay+a^\sharp bd^\sharp+ydd^\sharp\\ 0 & d^\sharp\end{pmatrix}=\begin{pmatrix} a^\sharp & y\\ 0 & d^\sharp\end{pmatrix},$$

i.e.

$$a^\sharp ay+a^\sharp bd^\sharp+ydd^\sharp=y. \tag{2}$$

Notice that

$$xz=\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\begin{pmatrix} a^\sharp & y\\ 0 & d^\sharp\end{pmatrix}=\begin{pmatrix} aa^\sharp & ay+bd^\sharp\\ 0 & dd^\sharp\end{pmatrix},$$

and

$$zx=\begin{pmatrix} a^\sharp & y\\ 0 & d^\sharp\end{pmatrix}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}=\begin{pmatrix} a^\sharp a & a^\sharp b+yd\\ 0 & d^\sharp d\end{pmatrix}.$$

We have $xz=zx$ if and only if

$$\begin{pmatrix} aa^\sharp & ay+bd^\sharp\\ 0 & dd^\sharp\end{pmatrix}=\begin{pmatrix} a^\sharp a & a^\sharp b+yd\\ 0 & d^\sharp d\end{pmatrix},$$

i.e.

$$ay + bd^\sharp = a^\sharp b + yd. \tag{3}$$

Since $a^\pi b d^\pi = 0$, we obtain

$$\begin{aligned}&(1 - aa^\sharp)b(1 - dd^\sharp) = 0,\\ &(b - aa^\sharp b)(1 - dd^\sharp) = 0,\\ &b - bdd^\sharp - aa^\sharp b + aa^\sharp bdd^\sharp = 0,\\ &b = aa^\sharp b + bdd^\sharp - aa^\sharp bdd^\sharp\end{aligned} \tag{4}$$

Multiplying the equality (2) by a from the left side and by d from the right side, we get

$$\begin{aligned}&aa^\sharp ayd + aa^\sharp bd^\sharp d + aydd^\sharp d = ayd, \quad ayd + aa^\sharp bd^\sharp d + ayd = ayd,\\ &ayd = -aa^\sharp bd^\sharp d.\end{aligned}$$

Now, equality (1) becomes

$$aa^\sharp b - aa^\sharp bd^\sharp d + bd^\sharp d = b.$$

In the same way, multiplying equality (1) by $a^\sharp$ from the left side and by $d^\sharp$ from the right side, we get

$$\begin{aligned}&a^\sharp aa^\sharp bd^\sharp + a^\sharp aydd^\sharp + a^\sharp bd^\sharp dd^\sharp = a^\sharp bd^\sharp,\\ &a^\sharp bd^\sharp + a^\sharp aydd^\sharp + a^\sharp bd^\sharp = a^\sharp bd^\sharp, \quad a^\sharp bd^\sharp = -a^\sharp aydd^\sharp.\end{aligned}$$

Now, equality (2) becomes

$$a^\sharp ay - a^\sharp aydd^\sharp + ydd^\sharp = y.$$

Similarly, multiplying equality (3) by $a^\sharp$ from the left side, we get

$$a^\sharp ay + a^\sharp bd^\sharp = (a^\sharp)^2 b + a^\sharp yd.$$

The last equality and equality (2) give

$$(a^\sharp)^2 b + a^\sharp yd + ydd^\sharp = y. \tag{5}$$

Now, we have $ay + bd^\sharp = a^\sharp b + yd$ (which is (3)), so we get

$$\begin{aligned}&a \cdot (2) + (1) \cdot d^\sharp = ay + bd^\sharp = a^\sharp b + yd = a^\sharp \cdot (1) + (2) \cdot d\\ &= a(a^\sharp ay + a^\sharp bd^\sharp + ydd^\sharp) + (aa^\sharp b + ayd + bd^\sharp d)d^\sharp\\ &= a^\sharp(aa^\sharp + ayd + bd^\sharp d) + (a^\sharp ay + a^\sharp bd^\sharp + ydd^\sharp)d,\\ &aa^\sharp ay + aa^\sharp bd^\sharp + aydd^\sharp + aa^\sharp bd^\sharp + aydd^\sharp + bd^\sharp dd^\sharp\\ &= a^\sharp aa^\sharp b + a^\sharp ayd + a^\sharp bd^\sharp d + a^\sharp ayd + a^\sharp bd^\sharp d + ydd^\sharp d,\end{aligned}$$

and

$$ay + 2aydd^\sharp + 2aa^\sharp bd^\sharp + bd^\sharp = a^\sharp b + 2a^\sharp ayd + 2a^\sharp bd^\sharp d + yd.$$

From equality (3) we get

$$2aa^\sharp bd^\sharp + 2aydd^\sharp = 2a^\sharp ayd + 2a^\sharp bd^\sharp d, \quad 2aa^\sharp(bd^\sharp - yd) = 2(a^\sharp b - ay)dd^\sharp,$$

$$2aa^\sharp(a^\sharp b - ay) = 2(a^\sharp b - ay)dd^\sharp, \quad 2a^\sharp b - 2ay = 2(bd^\sharp - yd)dd^\sharp,$$

$$2a^\sharp b - 2ay = 2bd^\sharp - 2yd, \quad a^\sharp b + yd = bd^\sharp + ay.$$

Multiplying equality (3) by $a^\sharp$ from the left side, we get

$$a^\sharp ay + a^\sharp bd^\sharp = (a^\sharp)^2 b + a^\sharp yd,$$

and from (2) we get

$$y - ydd^\sharp = a^\sharp ay + a^\sharp bd^\sharp, \quad y - ydd^\sharp = (a^\sharp)^2 b + a^\sharp yd, \quad y = (a^\sharp)^2 b + a^\sharp yd + ydd^\sharp.$$

Multiplying the last equality by $(1 - dd^\sharp)$ from the right side, we get

$$y(1 - dd^\sharp) = (a^\sharp)^2 b(1 - dd^\sharp) + a^\sharp yd(1 - dd^\sharp) + ydd^\sharp(1 - dd^\sharp),$$

$$y - ydd^\sharp = (a^\sharp)^2 bd^\pi + a^\sharp y(d - ddd^\sharp) + y(dd^\sharp - dd^\sharp dd^\sharp),$$

$$y - ydd^\sharp = (a^\sharp)^2 bd^\pi, \quad y = (a^\sharp)^2 bd^\pi + ydd^\sharp.$$

Now, multiplying equality (3) by $d^\sharp$ from the right side we obtain

$$ayd^\sharp + b(d^\sharp)^2 = a^\sharp bd^\sharp + ydd^\sharp,$$

From equality (2) we get

$$a^\sharp bd^\sharp + ydd^\sharp = y - a^\sharp ay, \quad ayd^\sharp + b(d^\sharp)^2 = y - a^\sharp ay,$$

$$y = ayd^\sharp + b(d^\sharp)^2 + a^\sharp ay.$$

Multiplying the last equality by $(1 - aa^\sharp)$ from the left side, we get

$$(1 - aa^\sharp)y = (1 - aa^\sharp)ayd^\sharp + (1 - aa^\sharp)b(d^\sharp)^2 + (1 - aa^\sharp)a^\sharp ay,$$

$$a^\pi y = (a - aa^\sharp a)yd^\sharp + a^\pi b(d^\sharp)^2 + (a^\sharp a - aa^\sharp a^\sharp a)y,$$

$$a^\pi y = a^\pi b(d^\sharp)^2, \quad (1 - aa^\sharp)y = a^\pi b(d^\sharp)^2, \quad y - aa^\sharp y = a^\pi b(d^\sharp)^2,$$

$$y = a^\pi b(d^\sharp)^2 + aa^\sharp y. \tag{6}$$

Since $(a^\sharp)^2 b + a^\sharp yd + ydd^\sharp = y$, we obtain

$$(a^\sharp)^2 b + a^\sharp yd = y(1 - dd^\sharp),$$
$$(a^\sharp)^2 b(1 - dd^\sharp) + a^\sharp yd(1 - dd^\sharp) = y(1 - dd^\sharp)(1 - dd^\sharp),$$
$$(a^\sharp)^2 bd^\pi = y(1 - dd^\sharp),$$

$$y = (a^\sharp)^2 bd^\pi + ydd^\sharp. \tag{7}$$

From (6) and (7) we get

$$y = a^\pi b(d^\sharp)^2 + aa^\sharp[(a^\sharp)^2 bd^\pi + ydd^\sharp], \quad y = a^\pi b(d^\sharp)^2 + (a^\sharp)^2 bd^\pi + aa^\sharp ydd^\sharp,$$
$$y = a^\pi b(d^\sharp)^2 + (a^\sharp)^2 bd^\pi - a^\sharp bd^\sharp.$$

Part (2)

$\Longleftarrow$: Assume that both $a^\sharp$ and $d^\sharp$ exist and $a^\pi bd^\pi = 0$. Thus $x^\sharp$ exists. Let

$$z = \begin{pmatrix} a^\sharp & y \\ 0 & d^\sharp \end{pmatrix},$$

where $y = (a^\sharp)^2 bd^\pi + a^\pi b(d^\sharp)^2 - a^\sharp bd^\sharp$. We have

$$xzx = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} a^\sharp & y \\ 0 & d^\sharp \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} aa^\sharp & ay + bd^\sharp \\ 0 & dd^\sharp \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$
$$= \begin{pmatrix} aa^\sharp a & aa^\sharp b + ayd + bd^\sharp d \\ 0 & dd^\sharp d \end{pmatrix} = \begin{pmatrix} a & aa^\sharp b + ayd + bd^\sharp d \\ 0 & d \end{pmatrix}.$$

We have $xzx = x$ if and only if

$$\begin{pmatrix} a & aa^\sharp b + ayd + bd^\sharp d \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

i.e. $aa^\sharp b + ayd + bd^\sharp d = b$. We compute as follows

$$\begin{aligned} &aa^\sharp b + ayd + bd^\sharp d = aa^\sharp b + a[(a^\sharp)^2 bd^\pi + a^\pi b(d^\sharp)^2 - a^\sharp bd^\sharp]d + bdd^\sharp \\ &= aa^\sharp b + a^\sharp b(1 - dd^\sharp)d + a(1 - aa^\sharp)b(d^\sharp)^2 d - aa^\sharp bd^\sharp d + bd^\sharp d \\ &= aa^\sharp b - aa^\sharp bd^\sharp d + bd^\sharp d. \end{aligned}$$

Now, from $(1 - aa^\sharp)b(1 - dd^\sharp) = 0$ we get

$$b - bdd^\sharp - aa^\sharp b + aa^\sharp bdd^\sharp = 0,$$

i.e.

$$aa^\sharp b + bdd^\sharp - aa^\sharp bdd^\sharp = b.$$

Therefore, $xzx = x$.

We have

$$zxz = \begin{pmatrix} a^\sharp & y \\ 0 & d^\sharp \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} a^\sharp & y \\ 0 & d^\sharp \end{pmatrix} = \begin{pmatrix} a^\sharp a & a^\sharp b + yd \\ 0 & d^\sharp d) \end{pmatrix}\begin{pmatrix} a^\sharp & y \\ 0 & d^\sharp \end{pmatrix}$$
$$= \begin{pmatrix} a^\sharp aa^\sharp & a^\sharp ay\sharp + a^\sharp bd^\sharp + ydd^\sharp \\ 0 & d^\sharp dd^\sharp \end{pmatrix} = \begin{pmatrix} a^\sharp & a^\sharp ay + a^\sharp bd^\sharp + ydd^\sharp \\ 0 & d^\sharp \end{pmatrix}.$$

Hnce, $zxz = z$ if and only if

$$\begin{pmatrix} a^\sharp & a^\sharp ay + a^\sharp bd^\sharp + ydd^\sharp \\ 0 & d^\sharp \end{pmatrix} = \begin{pmatrix} a^\sharp & y \\ 0 & d^\sharp \end{pmatrix},$$

i.e. $a^\sharp ay + a^\sharp bd^\sharp + ydd^\sharp = y$. We compute as follows:

$$\begin{aligned} &a^\sharp a[(a^\sharp)^2 bd^\pi + a^\pi b(d^\sharp)^2 - a^\sharp bd^\sharp] + a^\sharp bd^\sharp + [(a^\sharp)^2 bd^\pi + a^\pi b(d^\sharp)^2 - a^\sharp bd^\sharp]dd^\sharp \\ &= (a^\sharp)^2 bd^\pi + a^\pi b(d^\sharp)^2 - a^\sharp bd^\sharp, (a^\sharp)^2 bd^\pi + a^\sharp a(1 - aa^\sharp)b(d^\sharp)^2 - a^\sharp bd^\sharp + a^\sharp bd^\sharp \\ &+ (a^\sharp)^2 b(1 - dd^\sharp)dd^\sharp + a^\pi b(d^\sharp)^2 - a^\sharp bd^\sharp = y, \end{aligned}$$

and $(a^\sharp)^2 bd^\pi + a^\pi b(d^\sharp)^2 - a^\sharp bd^\sharp = y$. Therefore, $zxz = z$.

We have

$$xz = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} a^\sharp & y \\ 0 & d^\sharp \end{pmatrix} = \begin{pmatrix} aa^\sharp & ay + bd^\sharp \\ 0 & dd^\sharp \end{pmatrix},$$

$$zx = \begin{pmatrix} a^\sharp & y \\ 0 & d^\sharp \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a^\sharp a & a^\sharp b + yd \\ 0 & d^\sharp d \end{pmatrix}.$$

Now, $xz = zx$ if and only if

$$\begin{pmatrix} aa^\sharp & ay + bd^\sharp \\ 0 & dd^\sharp \end{pmatrix} = \begin{pmatrix} a^\sharp a & a^\sharp b + yd \\ 0 & d^\sharp d \end{pmatrix},$$

i.e. $ay + bd^\sharp = a^\sharp b + yd$. We compute as follows:

$$\begin{aligned} ay + bd^\sharp &= a[(a^\sharp)^2 bd^\pi + a^\pi b(d^\sharp)^2 - a^\sharp bd^\sharp] + bd^\sharp \\ &= a^\sharp bd^\pi + a(1 - aa^\sharp)b(d^\sharp)^2 - aa^\sharp bd^\sharp + bd^\sharp \\ &= a^\sharp bd^\pi - aa^\sharp bd^\sharp + bd^\sharp = a^\sharp b(1 - dd^\sharp) - aa^\sharp bd^\sharp + bd^\sharp \\ &= a^\sharp b - a^\sharp bdd^\sharp - aa^\sharp bd^\sharp + bd^\sharp \\ &= a^\sharp b(1 - dd^\sharp) + bd^\sharp(1 - aa^\sharp) = a^\sharp bd^\pi + bd^\sharp a^\pi, \end{aligned}$$

and

$$\begin{aligned} a^\sharp b + yd &= a^\sharp b + [(a^\sharp)^2 bd^\pi + a^\pi b(d^\sharp)^2 - a^\sharp bd^\sharp]d \\ &= a^\sharp b + (a^\sharp)^2 b(1 - dd^\sharp)d + a^\pi bd^\sharp - a^\sharp bd^\sharp d \\ &= a^\sharp b + (1 - aa^\sharp)bd^\sharp - a^\sharp bd^\sharp d \\ &= a^\sharp b + bd^\sharp - aa^\sharp bd^\sharp - a^\sharp bd^\sharp d \\ &= a^\sharp b(1 - d^\sharp d) + (1 - aa^\sharp)bd^\sharp = a^\sharp bd^\pi + a^\pi bd^\sharp. \end{aligned}$$

Therefore, $xz = zx$ and

$$x^\sharp = z = \begin{pmatrix} a^\sharp & y \\ 0 & d^\sharp \end{pmatrix},$$

where $y = (a^\sharp)^2 bd^\pi + a^\pi b(d^\sharp)^2 - a^\sharp bd^\sharp$.

$\Longrightarrow$: Assume that $a^\sharp, d^\sharp, x^\sharp$ exists. Then the result follows from the part (1). □

Theorem 2.2. *Let $a, b \in R$. If any two of the following hold, then the remaining one also holds:*
(1) *$(ab)^\sharp$ exists;*
(2) *$(ba)^\sharp$ exists;*
(3) *$ab \sim ba$.*

Proof. (1), (1) ⇒ (3): Let ab and ba be group invertible, $p = (ab)^\pi = 1 - ab(ab)^\sharp$ and $q = (ba)^\pi = (1 - ba(ba)^\sharp)$. Then ab, ba, a and b have matrix forms

$$ab = \begin{pmatrix} x_{11} & 0 \\ 0 & 0 \end{pmatrix}_{1-p,1-p}, \quad ba = \begin{pmatrix} y_{11} & 0 \\ 0 & 0 \end{pmatrix}_{1-q,1-q},$$
$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{1-p,1-q}, \quad b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}_{1-q,1-p}.$$

Since $q = 1 - ba(ba)^\sharp = 1 - b(ab)^\sharp a$ (by Lemma 2.3), $aq = a - ab(ab)^\sharp a = pa$, i.e.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{1-p,1-q} \begin{pmatrix} 0 & 0 \\ 0 & 1-q \end{pmatrix}_{1-q,1-q} = \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_{1-p,1-p} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{1-p,1-q}$$

we get

$$\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix}$$

Hence, $a_{12} = 0, a_{21} = 0$ and

$$a = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}.$$

Similarly, $qb = bp$, which implies that $b_{12} = 0, b_{21} = 0$ and

$$b = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}.$$

Now,

$$ab = \begin{pmatrix} a_{11}b_{11} & 0 \\ 0 & a_{22}b_{22} \end{pmatrix}$$

and

$$ba = \begin{pmatrix} b_{11}a_{11} & 0 \\ 0 & b_{22}a_{22} \end{pmatrix}.$$

Thus, $x_{11} = a_{11}b_{11}$ and $y_{11} = b_{11}a_{11}$ are invertible, $a_{22}b_{22} = 0$ and $b_{22}a_{22} = 0$, i.e.

$$(ab)^{\#} = \begin{pmatrix} (a_{11}b_{11})^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad (ba)^{\#} = \begin{pmatrix} (b_{11}a_{11})^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $a_{11}b_{11}$ and $b_{11}a_{11}$ are invertible, we see that b_{11} is invertible.

Let

$$s = \begin{pmatrix} b_{11} & 0 \\ 0 & 1-p \end{pmatrix}_{1-q,1-p}.$$

Then $sab = bas$, i.e. $ab \sim ba$.

The implications (1),(3) $\Rightarrow$ (2) and (2),(3) $\Rightarrow$ (1) are obvious. □

Theorem 2.3. *Let $a, b, ab \in R$ be group invertible. Then $(ab)^{\#} = b^{\#}a^{\#}$ if and only if $(1-a^{\pi})ba^{\pi} = 0, b^{\#}(1-a^{\pi}) = (ab)^{\#}a$. In addition, if a, b, ba^{π} are group invertible, then the following are equivalent:*

(1) $(ab)^{\#} = b^{\#}a^{\#}$;

(2) $(ba)^{\#} = a^{\#}b^{\#}$;

(3) $a = \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}_{1-p,1-p}$, $b = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}_{1-p,1-p}$ and $b_{11}^{\#} = (a_{11}b_{11})^{\#}a_{11}$, with respect to the decomposition $1 = p + (1-p)$, where $p = 1 - aa^{\#}$ and a_{11} is invertible;

(4) $a = \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}_{1-p,1-p}$, $\begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}_{1-p,1-p}$ and $b_{11}^{\#} = a_{11}(b_{11}a_{11})^{\#}$, with respect to the decomposition $1 = p + (1-p)$, where $p = 1 - aa^{\#}$ and a_{11} is invertible.

Proof. Part one.

$\Rightarrow$ Since a and b are group invertible, $a, a^{\#}, b$ and $b^{\#}$ have the forms:

$$a = \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}_{1-p,1-p}, a^{\#} = c, \quad b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}_{1-p,1-p}, \quad b^{\#} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}_{1-p,1-p},$$

respectively. Since

$$ab = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} \\ 0 & 0 \end{pmatrix}_{1-p,1-p}$$

is group invertible, we get

$$(1 - a_{11}b_{11}(a_{11}b_{11})^{\#})a_{11}b_{12} = 0$$

and

$$(ab)^{\#} = \begin{pmatrix} (a_{11}b_{11})^{\#} & [(a_{11}b_{11})^{\#}]^2 a_{11}b_{12} \\ 0 & 0 \end{pmatrix}$$

From $ab^{\#} = b^{\#}a^{\#}$ we get

$$\begin{pmatrix} (a_{11}b_{11})^{\#} & [(a_{11}b_{11})^{\#}]^2 a_{11}b_{12} \\ 0 & 0 \end{pmatrix} = c.$$

It follows that $c_{21} = 0$, $c_{11}a_{11}^{-1} = (a_{11}b_{11})^{\#}$, so $c_{11} = (a_{11}b_{11})^{\#}a_{11}$, and $[(a_{11}b_{11})^{\#}]^2 a_{11}b_{12} = 0$.

So, $b_{12} = a_{11}^{-1}a_{11}b_{12} = a_{11}^{-1}[a_{11}b_{11}(a_{11}b_{11})^{\sharp}a_{11}b_{12}] = a_{11}^{-1}(a_{11}b_{11})^2[(a_{11}b_{11})^{\sharp}]^2a_{11}b_{12} = 0$. Note that $a^{\pi} = 1 - aa^{\sharp} = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}_{1-p,1-p}$. We get $(1-a^{\pi})ba^{\pi} = \begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix}_{1-p,1-p} = 0$, $b^{\sharp}(1-a^{\pi}) = \begin{pmatrix} (a_{11}b_{11})^{\sharp}a_{11} & 0 \\ 0 & 0 \end{pmatrix}_{1-p,1-p} = (ab)^{\sharp}a$.

$\Leftarrow$ On the other hand, if $(1-a^{\pi})ba^{\pi} = 0$, then $b_{12} = 0$ and $(ab)^{\sharp} = \begin{pmatrix} (a_{11}b_{11})^{\sharp} & 0 \\ 0 & 0 \end{pmatrix}$. If $b^{\sharp}(1-a^{\pi}) = (ab)^{\sharp}a$, then $c_{11} = (a_{11}b_{11})^{\sharp}a_{11}$ i $c_{21} = 0$. Hence, $(ab)^{\sharp} = b^{\sharp}a^{\sharp}$.

Part two.

Now, assume that a, b, ab, ba^{π} are group invertible.

(1) $\Rightarrow$ (3): Note that $(ab)^{\sharp} = b^{\sharp}a^{\sharp}$ if and only if $a, a^{\sharp}, b, b^{\sharp}$ have the forms:

$a = \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$, $a^{\sharp} = \begin{pmatrix} a_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}$, $b^{\sharp} = \begin{pmatrix} (a_{11}b_{11})^{\sharp}a_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix}$, respectively. Since ba^{π} is group invertible, b_{22} is group invertible, and hence b_{11} is group invertible, $b_{22}^{\pi}b_{21}b_{11}^{\pi} = 0$ and

$b^{\sharp} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}^{\sharp} = \begin{pmatrix} b_{11}^{\sharp} & 0 \\ y & b_{22}^{\sharp} \end{pmatrix} = \begin{pmatrix} (a_{11}b_{11})^{\sharp}a_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix}$ where $y = b_{22}^{\pi}b_{21}(b_{11}^{\sharp})^2 + (b_{22}^{\sharp})^2b_{21}b_{11}^{\pi} - b_{22}^{\sharp}b_{21}b_{11}^{\sharp}$. It follows that $b_{11}^{\sharp} = (a_{11}b_{11})^{\sharp}a_{11}$ and $y = 0$.

Now, we have $b_{22}yb_{11}^{\pi} = 0$, $b_{22}b_{22}^{\sharp}b_{21}b_{11}^{\pi}0$, $b_{21}b_{11}^{\pi} = 0$, $b_{22}^{\pi}yb_{11}^2 = 0$. Hence, $b_{22}^{\pi}b_{21}b_{11}^{\sharp}b_{11} = 0$, so $b_{22}^{\pi}b_{21} = 0$, $b_{22}yb_{11} = 0$, $b_{22}b_{22}^{\sharp}b_{21}b_{11}^{\sharp}b_{11} = 0$, $b_{22}b_{22}^{\sharp}b_{21}b_{11}^{\sharp}b_{11} = 0$. Hence, $b_{21} = 0$ and $b = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}$.

(3) $\Rightarrow$ (1): It is clear.

(2) $\Leftrightarrow$ (4): This is similar to the proof (1) $\Leftrightarrow$ (3). □

References

[1] C. Deng, *On the group invertibility of operators*, Electronic J. Linear Algebra 31 (2016), 492-510.

[2] D. S. Djordjević, V. Rakočević, *Lectures on gerenalized inverses*, University of Niš, Faculty of Sciences and Mathematics, 2009.