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Some New Harmonic Mappings Convex in One Direction and their Convolution

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Abstract. In the present article, we construct a new family of locally univalent and sense preserving harmonic mappings by considering a suitable transformation of normalized univalent analytic functions defined in the open unit disc $\mathbb{D}$. We present necessary and sufficient conditions for the functions of this new family to be univalent. Apart from studying properties of this new family, results about the convolutions or Hadamard products of functions from this family with some suitable analytic or harmonic mappings are proved by introducing a new technique which can also be used to simplify the proofs of earlier known results on convolutions of harmonic mappings. The technique presented also enables us to generalize existing such results.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane and S the usual class of univalent analytic functions $f : \mathbb{D} \to \mathbb{C}$, such that $f(0) = 0, f'(0) = 1$. Further, let S_H denote the class of univalent harmonic functions $f : \mathbb{D} \to \mathbb{C}$, such that $f(0) = 0$ and $f_z(0) = 1$. It is known that $f \in S_H$ can be uniquely represented as $f = h + \overline{g}$, where h and g are analytic functions in $\mathbb{D}$. Lewy [10] proved that a harmonic function $f = h+\overline{g}$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if the Jacobian $J_f(z)$ of f, defined by

$$J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2,$$

is positive in $\mathbb{D}$. This is equivalent to the existence of an analytic function $w(z) = \frac{g'(z)}{h'(z)}$, defined on $\mathbb{D}$, such that $|w(z)| < 1$ in $\mathbb{D}$. Here w is referred to as dilatation of f. For more details on planar harmonic mappings see [4].

A domain E in $\mathbb{C}$ is said to be convex in the direction $\psi, 0 \le \psi < \pi$, if every line parallel to the line through 0 and $e^{i\psi}$ has an empty or connected intersection with E. Let $K(\psi)$ and $K_H(\psi)$ be the respective subclasses of S and S_H, whose members map $\mathbb{D}$ onto the domain convex in the direction of $\psi, 0 \le \psi < \pi$. Functions in $K(0)$ or $K_H(0)$ are said to be convex in the direction of the real axis or simply *CHD* functions. Similarly, functions in $K(\pi/2)$ or $K_H(\pi/2)$ are referred to as functions convex in the direction of the imaginary axis. Let

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S^*, K and C be the usual subclasses of S consisting of starlike (with respect to the origin), convex and close to convex functions, respectively. We denote by S_H^*, K_H and C_H, respectively, the corresponding subclasses of S_H. Note that, a domain is convex if and only if it is convex in every direction. Further, it is known that $K(\psi) \subset C$ and $K_H(\psi) \subset C_H$.

The convolution or Hadamard product of two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$, is denoted by $f * F$ and is defined as $(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n$. In the harmonic case, for $f(z) = h + \overline{g} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$ and $F(z) = H + \overline{G} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B}_n \overline{z}^n$, we define their convolution as,

$$(f*F)(z) = (h*H)(z) + \overline{(g*G)(z)} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \overline{z}^n.$$

In 1984, Clunie and Shiel-Small [2] introduced a technique, called *shear construction,* to construct new univalent harmonic functions on the open unit disc $\mathbb{D}$ and provided an interesting example of a univalent harmonic mapping given by

$$L_0(z) = \frac{I(z) + zI'(z)}{2} + \overline{\frac{I(z) - zI'(z)}{2}}$$

where $I(z) = z/(1-z)$. The mapping L_0 is now popularly known as standard right half plane mapping which maps the open unit disc $\mathbb{D}$ onto the region $\{w : Re w > -1/2\}$ in the complex plane.

Stacey Muir [12], in 2012, defined a transformation $T_\alpha[f]$ on an analytic function f satisfying $f(0) = 0$ and $f'(0) = 1$ to generate a new harmonic function given by

$$T_\alpha[f](z) = \frac{f(z) + \alpha z f'(z)}{1+\alpha} + \overline{\frac{f(z) - \alpha z f'(z)}{1+\alpha}}, \tag{1}$$

where $\alpha > 0$ is some real number and proved that
(i) $T_\alpha[f]$ is locally univalent and convex in the direction of the imaginary axis if and only if f is convex.
(ii) $T_\alpha[f] \in K_H$ if and only if $f \in DCP$, where DCP is the class of direction convexity preserving functions. (Note that a function $f \in S$ is direction convexity preserving if it preserves the class $K(\psi), \psi$ fixed, under convolution.)

In the present article, we define a general transformation to generate a new family of locally univalent and sense preserving maps, which contains transformation of Stacey Muir as a particular case.
For $f \in S$, define

$$C_{\alpha,h}[f](z) = \frac{f(z) + \alpha(h*f)(z)}{1+\alpha} + \overline{\frac{f(z) - \alpha(h*f)(z)}{1+\alpha}}, \alpha > 0, z \in \mathbb{D}, \tag{2}$$

where $h : \mathbb{D} \to \mathbb{C}$ is any analytic function with $h(0) = 0$ and $h'(0) = 1$. For example, in $C_{\alpha,h}[f]$ we can take $h = h_i, i = 1, 2, 3, 4$, where h_i are analytic functions given below.

$$h_1(z) = \sum_{n=1}^{\infty} n z^n, \qquad h_2(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n, \qquad h_3(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n, \qquad h_4(z) = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n.$$

Obviously,

$$h_1 * f = zf', \qquad h_2 * f = \frac{1}{2}[f + zf'], \qquad h_3 * f = \int_0^z \frac{f(\xi)}{\xi} d\xi, \qquad h_4 * f = \frac{2}{z} \int_0^z f(\xi) d\xi.$$

Note that, $h_3 * f$ and $h_4 * f$ are famous Alexander and Libera transformations of f, respectively (see [1] and [11]) and $C_{\alpha,h_1}[f] = T_\alpha[f]$.

In Section 3, we establish a necessary and sufficient condition for $C_{\alpha,h}[f]$ to be locally univalent and sense preserving in $\mathbb{D}$. Further we also prove here that if $f \in K$ then for $i = 1, 2, 3, 4$, $C_{\alpha,h_i}[f] \in S_H$ and are convex in the direction of the imaginary axis.

It is known that the class S_H is not closed under convolutions with functions from the class K i.e. for $F = H + \overline{G} \in S_H, \psi \in K$, the convolution $\psi * F = \psi * H + \overline{\psi * G}$ need not belong to S_H (see Example 3.2 [13]). In Theorem 4.1, we show that if $f \in K$, then $\psi * C_{\alpha,h}[f] \in S_H$ for every $\psi \in K$.

It is interesting to explore the properties of convolutions of harmonic functions as they are quite different from those of analytic functions. Many fruitful convolution results have been established in the recent past (see [3, 5, 6, 8, 13, 16–18]). We mention below few of them. Consider the vertical strip mappings $f_\eta = h_\eta + \overline{g_\eta}$ where

$$h_\eta(z) + g_\eta(z) = \frac{1}{2i \sin \eta} \log\left(\frac{1 + ze^{i\eta}}{1 + ze^{-i\eta}}\right), \pi/2 \le \eta < \pi. \tag{3}$$

In 2001, Dorff [5] proved the following result.

Theorem 1.1. *Let $f_1 = h_1 + \overline{g_1} \in K_H$ be any right half plane mapping with $h_1(z) + g_1(z) = z/(1 - z)$ and f_η be as given in (3). Then $f_1 * f_\eta \in S_H$ and is convex in the direction of real axis provided $f_1 * f_\eta$ is locally univalent and sense preserving.*

In 2012, Dorff et al. [6] established the following result.

Theorem 1.2. *Let f_η be as given by (3) with $g'_\eta/h'_\eta = e^{i\theta}z^n (\theta \in \mathbb{R})$. Further let $F_0 = H_0 + \overline{G_0}$, where $H_0(z) + G_0(z) = z/(1 - z)$ and $\frac{G'_0}{H'_0} = -z$, be the standard right half plane mapping. Then for $n = 1, 2, F_0 * f_\eta \in S_H$ and is convex in the direction of the real axis.*

In 2015, Kumar et al. [8] considered more general class of right half plane mappings $F_a = H_a + \overline{G_a}$, given by

$$H_a(z) + G_a(z) = \frac{z}{1 - z}, \qquad \frac{G'_a(z)}{H'_a(z)} = \frac{a - z}{1 - az}, a \in (-1, 1) \tag{4}$$

and generalized Theorem 1.2 as under.

Theorem 1.3. *Let F_a be the harmonic mappings as given in (4)and f_η be the harmonic mappings as given in (3) with $g'_\eta/h'_\eta = e^{i\theta}z^n (n \in \mathbb{N}, \theta \in \mathbb{R})$. Then $F_a * f_\eta \in S_H$ and is convex in the direction of the real axis for all $a \in [\frac{n-2}{n+2}, 1)$.*

In 2017, Liu et al. [17] considered the mappings $T_\alpha[f]$ given by Stacey Muir [12](mentioned above) and presented following results.

Theorem 1.4. *Let $T_\alpha[I]$ and f_η be harmonic mappings as given in (1) and (3), respectively with $g'_\eta/h'_\eta = e^{i\theta}z^n (n \in \mathbb{N}, \theta \in \mathbb{R})$. Then*
(a) $T_\alpha[I] * f_\eta \in S^0_H$ and is convex in the direction of the real axis for $n = 1, \pi/2 \le \eta < \pi$ and $0 < \alpha \le 2$;
(b) $T_\alpha[I] * f_\eta \in S^0_H$ and is convex in the direction of the real axis for $\eta = \pi/2$ and $0 < \alpha \le 2/n$.

In 2018, Liu et al. [18] further generalized the result given in Theorem 1.4(b) by replacing the condition $\eta = \pi/2$ with $\pi/2 \le \eta < \pi$ and proved the following result.

Theorem 1.5. *Let $T_\alpha[I(z)]$ and f_η be harmonic mappings as given in (1) and (3), respectively with $g'_\eta/h'_\eta = e^{i\theta}z^n (n \in \mathbb{N}, \theta \in \mathbb{R})$. Then $T_\alpha[I(z)] * f_\eta \in S^0_H$ and is convex in the direction of the real axis for $0 < \alpha \le 2/n$.*

In the proofs of Theorem 1.3, Theorem 1.4 and Theorem 1.5, authors have used Cohn's rule or Schur- Cohn algorithm and Gauss- Lucas Theorem, which involve, to some extent, lengthy computations. In Section 4 of this article, we present a different and simple technique, which enables us to prove more general results on convolutions of $C_{\alpha,h}[f]$ and f_η. Theorem 1.4 and Theorem 1.5 follow as particular cases of our result.

2. Preliminaries

We shall need the following results to prove our main theorems in subsequent sections.

Lemma 2.1. *Let ψ and G be analytic in $\mathbb{D}$ with $\psi(0) = G(0) = 0$. If ψ is convex and G is starlike, then for each analytic function F satisfying $ReF(z) > 0$ in $\mathbb{D}$, we have*

$$Re\frac{(\psi * FG)(z)}{(\psi * G)(z)} > 0, z \in \mathbb{D}.$$

Lemma 2.2. *Let a harmonic mapping $f = h + \overline{g}$ be locally univalent in $\mathbb{D}$. Then f is univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of α if and only if $h - e^{2i\alpha}g$ is a univalent analytic mapping of $\mathbb{D}$ onto a domain convex in the direction of α.*

Lemma 2.3. *Let f be analytic function in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) \neq 0$ and let*

$$\phi(z) = \frac{z}{(1 + ze^{i\theta})(1 + ze^{-i\theta})},$$

where $\theta \in \mathbb{R}$. If

$$Re\left[\frac{zf'(z)}{\phi(z)}\right] > 0, \quad \text{for all } z \in \mathbb{D},$$

then f is convex in the direction of the real axis.

Lemma 2.1. is due to Ruscheweyh and Sheil Small [15], Lemma 2.2. is due to Clunie and Shiel-Small [2] and Lemma 2.3 is due to Pommerenke [14].

3. Local Univalence and Univalence of $C_{\alpha,h}[f]$

We start this section by proving a necessary and sufficient condition for $C_{\alpha,h}[f]$ to be locally univalent and sense preserving in the open unit disc $\mathbb{D}$.

Theorem 3.1. *Let $f \in S$ and h any analytic functions. Then the function $C_{\alpha,h}[f]$ defined in (2) is locally univalent and sense preserving in $\mathbb{D}$ if and only if*

$$Re\left(\frac{(h * zf')(z)}{zf'(z)}\right) > 0, z \in \mathbb{D}.$$

Proof. Let w denote the dilatation of $C_{\alpha,h}[f]$. Thus $C_{\alpha,h}[f]$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if

$$|w(z)| = \left|\frac{f'(z) - \alpha(h * f)'(z)}{f'(z) + \alpha(h * f)'(z)}\right| < 1, z \in \mathbb{D},$$

which is equivalent (because $\alpha > 0$) to

$$Re\left(\frac{z(h * f)'(z)}{zf'(z)}\right) = Re\left(\frac{(h * zf')(z)}{zf'(z)}\right) > 0, z \in \mathbb{D}. \quad \square$$

Theorem 3.1 and Lemma 2.2 immediately lead to the following result.

Theorem 3.2. *Let f and h be analytic functions such that $Re(h * zf')(z)/zf'(z)) > 0$ in $\mathbb{D}$. Then,$C_{\alpha,h}[f] \in S_H$ and is convex in the direction of the imaginary axis if and only if f is convex in the direction of the imaginary axis.*

Theorem 3.3. *If $f \in K$, then for $i = 1, 2, 3, 4$, $C_{\alpha,h_i}[f] \in S_H$ and is convex in the direction of the imaginary axis.*

Proof. In view of Theorem 3.1 and Theorem 3.2, to prove our result, it is enough to prove that for $i = 1, 2, 3, 4$,

$$Re\left\{\frac{(h_i * f)'(z)}{f'(z)}\right\} > 0, z \in \mathbb{D}.$$

Now, $i = 1$ is the case already proved by Stacey Muir [12].
We have $h_2 * f = \frac{1}{2}[f + zf']$. So,

$$Re\left\{\frac{(h_2 * f)'(z)}{f'(z)}\right\} = \frac{1}{2} + \frac{1}{2}Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \tag{5}$$

as $f \in K$.
We have $(h_3 * f)'(z) = f(z)/z$. Hence,

$$Re\left\{\frac{(h_3 * f)'(z)}{f'(z)}\right\} = Re\left\{\frac{f(z)}{zf'(z)}\right\} > 0,$$

as $f \in K \subset S^*$.
Note that $h_4 * h_2 = z/(1 - z)$. So,

$$Re\left\{\frac{(h_4 * f)'}{f'}\right\} = Re\left\{\frac{(h_4 * f)'}{(h_4 * h_2 * f)'}\right\} = Re\left\{\frac{h_4 * zf'}{h_4 * z(h_2 * f)'}\right\} = Re\left\{\frac{h_4 * zf'}{h_4 * \frac{(h_2 * f)'}{f'}zf'}\right\}$$

Now, $f \in K$ implies $zf' \in S^*$ and $Re(h_2 * f)'/f' > 0$, by (5). Also, $h_4 \in K$ (see [9]). Hence, in view of Lemma 2.1,

$$Re\left\{\frac{h_4 * zf'}{h_4 * \frac{(h_2 * f)'}{f'}zf'}\right\} > 0, z \in \mathbb{D}. \quad \square$$

4. Convolution Properties of the Family $C_{\alpha,h}[f]$

In this section, we have investigated convolution properties of $C_{\alpha,h}[f]$ with some other functions - both analytic and harmonic. The Example 3.2 presented by Nagpal and Ravichandran [13] clearly shows that for $F \in S_H$, $\psi * F$ need not be in S_H for each $\psi \in K$. In the following theorem, we prove that, for each $\psi \in K$, $\psi * C_{\alpha,h}[f] \in S_H$, provided $f \in K$.

Theorem 4.1. *Let $\psi \in K$ and $C_{\alpha,h}[f]$ as given by (2) be locally univalent and sense preserving in $\mathbb{D}$. Then $\psi * C_{\alpha,h}[f] \in S_H$ and is convex in the direction of the imaginary axis provided $f \in K$.*

Proof. We note that,

$$\psi * C_{\alpha,h}[f] = \frac{\psi * f + \alpha(\psi * h * f)}{1 + \alpha} + \overline{\frac{\psi * f - \alpha(\psi * h * f)}{1 + \alpha}}.$$

Now, $C_{\alpha,h}[f]$ is locally univalent and sense preserving in $\mathbb{D}$ implies $Re[(h * zf')/zf'] > 0$ in $\mathbb{D}$, by Theorem 3.1. Therefore, in view of Lemma 2.1,

$$Re\left(\frac{z(\psi * h * f)'}{z(\psi * f)'}\right) = Re\left(\frac{\psi * h * zf'}{\psi * zf'}\right) = Re\left(\frac{\psi * (\frac{h * zf'}{zf'})zf'}{\psi * zf'}\right) > 0, z \in \mathbb{D}, \tag{6}$$

because $\psi \in K$ and $zf' \in S^*$(as $f \in K$). In view of Theorem 3.1 and Theorem 3.2, proof shall follow provided $\psi * f$ is convex in the direction of the imaginary axis, which is true as $\psi * f \in K$. $\square$

In the next part of this section, a different and simple technique is introduced to prove the following result which generalizes the results given in Theorem 1.4 and Theorem 1.5. For this, consider a harmonic mapping $f_\eta = h_\eta + \overline{g_\eta} \in S_H^0$ where

$$h_\eta(z) + g_\eta(z) = k(z), \frac{g'_\eta}{h'_\eta} = e^{i\theta} z^n (\theta \in \mathbb{R}, n \in \mathbb{N}). \tag{7}$$

Here $k \in K$ is such that

$$zk'(z) = \frac{z}{(1 + ze^{i\eta})(1 + ze^{-i\eta})}, \eta \in \mathbb{R}. \tag{8}$$

Theorem 4.2. *Let $C_{\alpha,h_i}[I], i = 1,2$ be as defined in (2), where $h_1(z) = \sum_{n=1}^{\infty} nz^n$ and $h_2(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n$. If $f_\eta = h_\eta + \overline{g_\eta} \in S_H^0$ is given by (7), then*
(a) $C_{\alpha,h_1}[I] * f_\eta \in S_H^0$ and is convex in the direction of the real axis for $0 < \alpha \le 2/n$;
(b) $C_{\alpha,h_2}[I] * f_\eta \in S_H^0$ and is convex in the direction of the real axis for $0 < \alpha \le 4/n$.

To prove Theorem 4.2, we need following two lemmas which we prove here first.

Lemma 4.3. *Let $F_1 = H_1 + \overline{G_1}$ and $F_2 = H_2 + \overline{G_2}$ be two functions in S_H, with $H_1(z) + G_1(z) = \beta z/(1 - z), H_2(z) + G_2(z) = k(z)$, where $\beta > 0$ and*

$$zk'(z) = \frac{z}{(1 + ze^{i\eta})(1 + ze^{-i\eta})}$$

for some $\eta \in \mathbb{R}$. Then $F_1 * F_2 \in S_H$ and is convex in the direction of the real axis provided $F_1 * F_2$ is locally univalent and sense preserving in $\mathbb{D}$.

Proof. Let

$$P_i = \frac{1 - G'_i/H'_i}{1 + G'_i/H'_i}, i = 1, 2.$$

Then $F_i = H_i + \overline{G_i} \in S_H$, implies that, $|G'_i/H'_i| < 1$ in $\mathbb{D}$ and consequently

$$Re P_i(z) > 0, z \in \mathbb{D}, \tag{9}$$

for $i = 1, 2$. Further, let

$$L_1 = (H_1 + G_1) * (H_2 - G_2) = H_1 * H_2 - H_1 * G_2 + G_1 * H_2 - G_1 * G_2$$

$$L_2 = (H_1 - G_1) * (H_2 + G_2) = H_1 * H_2 + H_1 * G_2 - G_1 * H_2 - G_1 * G_2.$$

Then

$$\frac{1}{2}[L_1 + L_2] = H_1 * H_2 - G_1 * G_2.$$

In view of Lemma 2.2, it is enough to show that $L_1 + L_2$ is convex in the direction of the real axis. Now, we have

$$zL'_1 = z[(H_1 + G_1) * (H_2 - G_2)]' = z(H'_2 + G'_2)\left(\frac{1 - G'_2/H'_2}{1 + G'_2/H'_2}\right) = \beta zk' P_2(z).$$

Hence, in view of (9) and the fact that $\beta > 0$,

$$Re\left(\frac{zL'_1(z)}{zk'(z)}\right) > 0, z \in \mathbb{D}. \tag{10}$$

Next, consider

$$zL'_2 = z[(H_1 - G_1) * (H_2 + G_2)]' = z\left(\frac{1 - G'_1/H'_1}{1 + G'_1/H'_1}\right)(H'_1 + G'_1) * k = \beta[\frac{z}{(1 - z)^2}]P_1(z) * k.$$

As $k \in K$ (because $zk' \in S^*$ ([14], Theorem 4)), therefore, by Lemma 2.1, we get

$$Re\left(\frac{zL_2'(z)}{zk'(z)}\right) = \beta Re\left(\frac{k(z) * P_1(z)\frac{z}{(1-z)^2}}{k(z) * \frac{z}{(1-z)^2}}\right) > 0, z \in \mathbb{D}. \tag{11}$$

So, from (10) and (11), we have

$$Re\left(\frac{z[L_1'(z) + L_2'(z)]}{zk'(z)}\right) > 0, z \in \mathbb{D}$$

and therefore, $L_1 + L_2$ is convex in the direction of the real axis by Lemma 2.4. □

Lemma 4.4. *Let s and s' be real numbers with $s' - s > 0$. Then for $w \in \mathbb{C}$,*

$$\left|\frac{s + w}{s' + w}\right| < 1$$

if and only if

$$Re(w) > -\left(\frac{s + s'}{2}\right).$$

Proof. We can easily see that

$$\left|\frac{s + w}{s' + w}\right| < 1$$

if and only if

$$s^2 + |w|^2 + 2sRe(w) < s'^2 + |w|^2 + 2s'Re(w).$$

This is equivalent to

$$Re(w) > -\left(\frac{s + s'}{2}\right)$$

as $s' - s > 0$. □

Proof of Theorem 4.2. (a) From (2), we have

$$C_{\alpha,h_1}[I](z) = \frac{I(z) + \alpha z I'(z)}{1 + \alpha} + \overline{\frac{I(z) - \alpha z I'(z)}{1 + \alpha}}$$

and

$$(C_{\alpha,h_1}[I] * f_\eta)(z) = \frac{1}{1 + \alpha}\left[h_\eta(z) + \alpha z h_\eta'(z)\right] + \overline{\frac{1}{1 + \alpha}\left[g_\eta(z) - \alpha z g_\eta'(z)\right]}.$$

In view of Lemma 4.3, it is enough to prove that $C_{\alpha,h_1}[I] * f_\eta$ is locally univalent and sense preserving in $\mathbb{D}$ and we know that $C_{\alpha,h_1}[I] * f_\eta$ is locally univalent and sense preserving in $\mathbb{D}$ if its dilatation

$$W(z) = \frac{(1 - \alpha)g_\eta'(z) - \alpha z g_\eta''(z)}{(1 + \alpha)h_\eta'(z) + \alpha z h_\eta''(z)} \tag{12}$$

is having modulus value less than 1. From $g_\eta'(z) = e^{i\theta} z^n h_\eta'(z)$, we get $g_\eta''(z) = e^{i\theta} z^n h_\eta''(z) + n e^{i\theta} z^{n-1} h_\eta'(z)$. Substituting these values of g_η' and g_η'' in (12), we have

$$W(z) = -e^{i\theta} z^n \left[\frac{((n + 1)\alpha - 1)h_\eta'(z) + \alpha z h_\eta''(z)}{(1 + \alpha)h_\eta'(z) + \alpha z h_\eta''(z)}\right].$$

For $\alpha = 2/n$, we have $W(z) = -e^{i\theta} z^n$ and hence $|W| < 1$ for $\alpha = 2/n$. In view of Lemma 4.4,

$$\left|\frac{((n + 1)\alpha - 1)h_\eta'(z) + \alpha z h_\eta''(z)}{(1 + \alpha)h_\eta'(z) + \alpha z h_\eta''(z)}\right| < 1$$

if and only if $Re\left(\frac{zh''_\eta(z)}{h'_\eta(z)}\right) > -\left(\frac{n+2}{2}\right)$ for $0 < \alpha < 2/n$. So it is enough to prove that $Re\left(\frac{zh''_\eta(z)}{h'_\eta(z)}\right) > -\left(\frac{n+2}{2}\right)$ for $0 < \alpha < 2/n$. As $h_\eta(z) + g_\eta(z) = k(z)$ and $g'_\eta/h'_\eta = e^{i\theta}z^n$, using simple calculations, we have

$$Re\left\{\frac{2zh''_\eta(z)}{h'_\eta(z)} + n + 2\right\} = 2Re\left(1 + \frac{zk''(z)}{k'(z)}\right) + nRe\left(\frac{1 - e^{i\theta}z^n}{1 + e^{i\theta}z^n}\right) > 0,$$

because $k \in K$. The proof is now complete.
(b) Proof of this part is similar to proof of part (a), hence omitted. □

Remark 4.5. 1. *We note that in the proofs of Theorem 4.2(a), we need not to put any restrictions on the values of η except that $\eta \in \mathbb{R}$. Also from equation (8), we immediately get*

$$k(z) = \frac{1}{2i\sin\eta}\log\left(\frac{1 + ze^{i\eta}}{1 + ze^{-i\eta}}\right),$$

for $\eta \neq m\pi, m = 0, \pm1, \pm2, \pm3, ...$ Therefore, by taking values of η in the interval $[\pi/2, \pi)$, we get Theorem 1.5.

2. *By taking $n = 1$ and restricting η in $[\pi/2, \pi)$, we get Theorem 1.4(a). By setting $\eta = \pi/2$ in Theorem 4.2(a), we deduce Theorem 1.4(b).*

For $\eta = m\pi$, from equation (8), we have

$$k(z) = \left\{ \begin{array}{ll} \frac{z}{1+z} & : \text{if m is even} \\ \frac{z}{1-z} & : \text{if m is odd} \end{array} \right\}$$

Therefore in view of Theorem 4.2(a), we get following results.

Theorem 4.6. *Let $C_{\alpha,h_1}[I]$ be as defined in (2), where $h_1(z) = \sum_{n=1}^{\infty} nz^n$.*
(a) If $f_\eta = h_\eta + \overline{g_\eta} \in S_H^0$ with $h_\eta + g_\eta = z/(1-z)$ and $\frac{g'_\eta}{h'_\eta} = e^{i\theta}z^n (\theta \in \mathbb{R}, n \in \mathbb{N})$. Then $C_{\alpha,h_1}[I] * f_\eta \in S_H^0$ and is convex in the direction of the real axis for $0 < \alpha \leq 2/n$;
(b) If $f_\eta = h_\eta + \overline{g_\eta} \in S_H^0$ with $h_\eta + g_\eta = z/(1+z)$ and $\frac{g'_\eta}{h'_\eta} = e^{i\theta}z^n (\theta \in \mathbb{R}, n \in \mathbb{N})$. Then $C_{\alpha,h_1}[I] * f_\eta \in S_H^0$ and is convex in the direction of the real axis for $0 < \alpha \leq 2/n$.

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