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Permanence and Asymptotic Behaviors of Stochastic Competitive Lotka-Volterra System with Markov Switching and Lévy Noise

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Abstract. This paper concerns the dynamics of a stochastic competitive Lotka-Volterra system with Markov switching and Lévy noise. The results show that stochastic permanence and extinction are characterized by two parameters $\mathcal{B}_1$ and $\mathcal{B}_2$: if $\mathcal{B}_1\mathcal{B}_2 \neq 0$, then the system is either stochastically permanent or extinctive. That is, it is extinctive if and only if $\mathcal{B}_1 < 0$ and $\mathcal{B}_2 < 0$; otherwise, it is stochastically permanent. Some existing results are included as special cases.

1. Introduction

Recently, stochastic population systems driven by white noise have been received great attention (see e.g. [1–11]). The stochastic two-species competitive Lotka-Volterra system can be expressed as follows:

$$\begin{cases} \mathrm{d}x(t) = x(t)\left\{\left[r_1 - a_{11}x(t) - a_{12}y(t)\right]\mathrm{d}t + \sigma_1 \mathrm{d}W(t)\right\}, \\ \mathrm{d}y(t) = y(t)\left\{\left[r_2 - a_{21}x(t) - a_{22}y(t)\right]\mathrm{d}t + \sigma_2 \mathrm{d}W(t)\right\}, \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ are the densities of the two species at time t, r_1 and $r_2 > 0$ are the intrinsic growth rates, a_{11} and $a_{22} > 0$ are the intra-specific competition rates, a_{12} and $a_{21} > 0$ are the interspecific competition rates. $W(t)$ is a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Jiang et al. [4] and Li et al. [5] studied the persistence, extinction, global attractivity and stationary distribution of system (1).

However, in the real world population systems often suffer sudden environmental perturbations which cannot be described by white noise, for instance, earthquakes, hurricanes, planting, harvesting, etc (see e.g. [12–18]). Bao et al. (see [17, 18]) pointed out that introducing Lévy jumps into the underlying population system may be a reasonable way to describe these phenomena. Liu et al. [15] investigated the following

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stochastic competitive Lotka-Volterra system with Lévy noise:

$$\begin{cases} \mathrm{d}x(t) = x(t^-)\left\{[r_1 - a_{11}x(t^-) - a_{12}y(t^-)]\,\mathrm{d}t + \sigma_1 \mathrm{d}W(t) + \int_{\mathbb{Z}} \gamma_1(\mu)\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \\ \mathrm{d}y(t) = y(t^-)\left\{[r_2 - a_{21}x(t^-) - a_{22}y(t^-)]\,\mathrm{d}t + \sigma_2 \mathrm{d}W(t) + \int_{\mathbb{Z}} \gamma_2(\mu)\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \end{cases} \tag{2}$$

where $x(t^-)$ and $y(t^-)$ stand for, respectively, the left limits of $x(t)$ and $y(t)$. N is a Poisson counting measure with characteristic measure λ on a measurable subset $\mathbb{Z}$ of $[0, +\infty)$ with $\lambda(\mathbb{Z}) < +\infty$ and $\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu) = N(\mathrm{d}t, \mathrm{d}\mu) - \lambda(\mathrm{d}\mu)\mathrm{d}t$. $\gamma_i(\mu) > -1$ ($\mu \in \mathbb{Z}$) are bounded functions ($i = 1, 2$). Liu et al. [15] established sufficient and necessary conditions of persistence in mean and extinction for each population. However, they did not investigate stochastic permanence of system (2).

On the other hand, parameters in some population systems may suffer abrupt changes, for example, some authors (see e.g. [8, 10]) pointed out that the growth rates of some species in summer will be much different from those in winter, and one can use a continuous-time Markov chain with a finite state space to describe these abrupt changes (see e.g. [12, 15, 19]). Especially, Takeuchi et al. [20] investigated a predator-prey Lotka-Volterra system with regime switching and revealed the significant effect of environmental noise on the population system: both its subsystems develop periodically but switching between them makes them become neither permanent nor dissipative (see e.g. [20–22]).

Motivated by above discussions, in this paper we study stochastic permanence and asymptotic behaviors of the following stochastic competitive Lotka-Volterra system with Markov switching and Lévy noise:

$$\begin{cases} \mathrm{d}x(t) = x(t^-)\left\{[r_1(\rho(t)) - a_{11}(\rho(t))x(t^-) - a_{12}(\rho(t))y(t^-)]\,\mathrm{d}t + \sigma_1(\rho(t))\mathrm{d}W(t) + \int_{\mathbb{Z}} \gamma_1(\mu, \rho(t))\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \\ \mathrm{d}y(t) = y(t^-)\left\{[r_2(\rho(t)) - a_{21}(\rho(t))x(t^-) - a_{22}(\rho(t))y(t^-)]\,\mathrm{d}t + \sigma_2(\rho(t))\mathrm{d}W(t) + \int_{\mathbb{Z}} \gamma_2(\mu, \rho(t))\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \end{cases} \tag{3}$$

where $\rho(t)$ is a right-continuous Markov chain on $(\Omega, \mathcal{F}, P)$, taking values in a finite state space $\mathbb{S} = \{1, 2, ..., S\}$. System (3) is operated as follows: If $\rho(0) = i_0$, then system (3) obeys

$$\begin{cases} \mathrm{d}x(t) = x(t^-)\left\{[r_1(i) - a_{11}(i)x(t^-) - a_{12}(i)y(t^-)]\,\mathrm{d}t + \sigma_1(i)\mathrm{d}W(t) + \int_{\mathbb{Z}} \gamma_1(\mu, i)\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \\ \mathrm{d}y(t) = y(t^-)\left\{[r_2(i) - a_{21}(i)x(t^-) - a_{22}(i)y(t^-)]\,\mathrm{d}t + \sigma_2(i)\mathrm{d}W(t) + \int_{\mathbb{Z}} \gamma_2(\mu, i)\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \end{cases} \tag{4}$$

with $i = i_0$ until time τ_1 when $\rho(t)$ jumps to i_1 from i_0; system (3) will then obey system (4) with $i = i_1$ from τ_1 until τ_2 when $\rho(t)$ jumps to i_2 from i_1. System (3) will go on switching as long as the Markov chain jumps. That is to say, system (3) can be regarded as system (4) switching from one to another in accordance with the law of the Markov chain. The different systems (4) ($i \in \mathbb{S}$) are therefore referred to as the subsystems of system (3). If the switching between environmental regimes disappears, in other words, $\rho(t)$ has only one state, then system (3) degenerates into system (4).

2. Global positive solutions

Throughout this paper, the generator $\Gamma = (\gamma_{ij})_{S\times S}$ of $\rho(t)$ is given by

$$P\{\rho(t+\varsigma) = j \mid \rho(t) = i\} = \begin{cases} \gamma_{ij}\varsigma + o(\varsigma), & i \neq j, \\ 1 + \gamma_{ij}\varsigma + o(\varsigma), & i = j, \end{cases} \tag{5}$$

where $\varsigma > 0$. Here γ_{ij} represents the transition rate from i to j and $\gamma_{ij} \geq 0$ if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that $\rho(t)$, $W(t)$ and N are mutually independent. As a standing hypothesis we also assume that

$\rho(t)$ is *irreducible*. Under this hypothesis, system (3) can switch from any regime to any other regime and $\rho(t)$ has a unique stationary probability distribution $\pi = (\pi_1, \pi_2, ..., \pi_S) \in \mathbb{R}^{1\times S}$ which can be determined by solving the following linear equation:

$$\pi\Gamma = 0, \tag{6}$$

subject to

$$\sum_{i=1}^{S} \pi_i = 1 \ \textit{and} \ \pi_i > 0, \ \forall i \in \mathbb{S}. \tag{7}$$

In this paper, we impose the following assumptions:

(H$_1$) $r_j(i) > 0, \ a_{jk}(i) > 0$ and there exist $\gamma_j^*(i) \geq \gamma_{j*}(i) > -1$ such that
$\gamma_{j*}(i) \leq \gamma_j(\mu, i) \leq \gamma_j^*(i) \ (\mu \in \mathbb{Z}), \ \forall i \in \mathbb{S}, \ j,k = 1,2.$

(H$_2$) For some $i \in \mathbb{S}, \ r_j(i) > 0, \ a_{jk}(i) > 0$ and there exist $\gamma_j^*(i) \geq \gamma_{j*}(i) > -1$ such that
$\gamma_{j*}(i) \leq \gamma_j(\mu, i) \leq \gamma_j^*(i) \ (\mu \in \mathbb{Z}), \ j,k = 1,2.$

(H$_3$) For some $j \in \mathbb{S}, \ \gamma_{ij} > 0, \ \forall i \neq j.$

(H$_4$) $r_1(i) > 0$ and $a_{11}(i) > 0, \ \forall i \in \mathbb{S}.$

For convenience, define

$$\begin{cases} \mathbb{R}_+^2 = \left\{a = (a_1, a_2) \in \mathbb{R}^2 \mid a_i > 0, \ i = 1,2\right\}, \ X(t) = (x(t), y(t))^{\mathrm{T}}, \ |X(t)| = \sqrt{x^2(t) + y^2(t)}, \\ \langle \nu(t)\rangle = t^{-1}\displaystyle\int_0^t \nu(s)\mathrm{d}s, \ \langle \nu(t)\rangle^* = \limsup_{t\to+\infty}\langle \nu(t)\rangle, \ \langle \nu(t)\rangle_* = \liminf_{t\to+\infty}\langle \nu(t)\rangle, \\ \overline{r_j} = \max\limits_{i\in\mathbb{S}}\left\{r_j(i)\right\}, \ \underline{r_j} = \min\limits_{i\in\mathbb{S}}\left\{r_j(i)\right\}, \ r = \max\limits_{j=1,2}\left\{\overline{r_j}\right\}, \ \overline{a_{jk}} = \max\limits_{i\in\mathbb{S}}\left\{a_{jk}(i)\right\}, \ \underline{a_{jk}} = \min\limits_{i\in\mathbb{S}}\left\{a_{jk}(i)\right\} \ (j,k = 1,2), \\ \sigma = \max\limits_{i\in\mathbb{S}, \, j=1,2}\left\{\left|\sigma_j(i)\right|\right\}, \ \gamma^\star = \max\limits_{i\in\mathbb{S}, \, j=1,2}\left\{|\gamma_{j*}(i)|, |\gamma_j^*(i)|\right\}, \\ B_j(i) = r_j(i) - \frac{\sigma_j^2(i)}{2} - \displaystyle\int_{\mathbb{Z}}\left[\gamma_j(\mu, i) - \ln\left(1 + \gamma_j(\mu, i)\right)\right]\lambda(\mathrm{d}\mu), \ B_j^*(i) = r_j(i) - \frac{\sigma_j^2(i)}{2}, \\ B(i) = \min\limits_{j=1,2}\left\{B_j(i)\right\}, \ \mathcal{B}_j = \displaystyle\sum_{i=1}^{S}\pi_i B_j(i), \ \mathcal{B}_j^* = \sum_{i=1}^{S}\pi_i B_j^*(i), \ \mathcal{B} = \sum_{i=1}^{S}\pi_i B(i). \end{cases} \tag{8}$$

Theorem 2.1. *Under* (**H$_1$**). *For any initial value $X(0) \in \mathbb{R}_+^2$, system (3) has a unique global solution $X(t) \in \mathbb{R}_+^2$ on $t \geq 0$ a.s.*

Proof. Consider the following stochastic differential equation:

$$\begin{cases} \mathrm{d}u(t) = \left[B_1(\rho(t)) - a_{11}(\rho(t))\mathrm{e}^{u(t)} - a_{12}(\rho(t))\mathrm{e}^{v(t)}\right]\mathrm{d}t + \sigma_1(\rho(t))\mathrm{d}W(t) + \displaystyle\int_{\mathbb{Z}}\ln\left[1 + \gamma_1(\mu, \rho(t))\right]\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu), \\ \mathrm{d}v(t) = \left[B_2(\rho(t)) - a_{21}(\rho(t))\mathrm{e}^{u(t)} - a_{22}(\rho(t))\mathrm{e}^{v(t)}\right]\mathrm{d}t + \sigma_2(\rho(t))\mathrm{d}W(t) + \displaystyle\int_{\mathbb{Z}}\ln\left[1 + \gamma_2(\mu, \rho(t))\right]\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu), \\ u(0) = \ln x(0), \ v(0) = \ln y(0). \end{cases} \tag{9}$$

Since the coefficients of system (9) are locally Lipschitz continuous, from [23] and [24] we observe that system (9) admits a unique local solution $(u(t), v(t))^{\mathrm{T}}$ on $t \in [0, \tau_e)$ a.s., where τ_e is the explosion time. By Itô's formula, $X(t) = (\mathrm{e}^{u(t)}, \mathrm{e}^{v(t)})^{\mathrm{T}}$ is the unique local solution to system (3) with initial value $X(0) \in \mathbb{R}_+^2$. The proof of its global solution is almost identical to that for systems with Markov switching driven by white noise (see e.g. [7, 19, 25]), and here is omitted. □

3. Extinction

Theorem 3.1. *Under* $(\mathbf{H_1})$*. Let* $X(t)$ *be the solution to system (3) with initial value* $X(0) \in \mathbb{R}^2_+$*.*
(1) If $\mathcal{B}_1 < 0$*, then* $\lim_{t\to+\infty} x(t) = 0$ *a.s. Moreover, if* $\mathcal{B}_2 > 0$*, then*

$$\frac{\mathcal{B}_2}{\overline{a_{22}}} \leq \langle y(t)\rangle_* \leq \langle y(t)\rangle^* \leq \frac{\mathcal{B}_2}{\underline{a_{22}}} \ a.s. \tag{10}$$

(2) If $\mathcal{B}_2 < 0$*, then* $\lim_{t\to+\infty} y(t) = 0$ *a.s. Moreover, if* $\mathcal{B}_1 > 0$*, then*

$$\frac{\mathcal{B}_1}{\overline{a_{11}}} \leq \langle x(t)\rangle_* \leq \langle x(t)\rangle^* \leq \frac{\mathcal{B}_1}{\underline{a_{11}}} \ a.s. \tag{11}$$

Proof. From system (9), we compute

$$\begin{cases} \ln\left(\frac{x(t)}{x(0)}\right) = \int_0^t B_1(\rho(s))\mathrm{d}s - \int_0^t a_{11}(\rho(s))x(s)\mathrm{d}s - \int_0^t a_{12}(\rho(s))y(s)\mathrm{d}s + \sum_{j=1}^{2} M_{1j}(t), \\ \ln\left(\frac{y(t)}{y(0)}\right) = \int_0^t B_2(\rho(s))\mathrm{d}s - \int_0^t a_{21}(\rho(s))x(s)\mathrm{d}s - \int_0^t a_{22}(\rho(s))y(s)\mathrm{d}s + \sum_{j=1}^{2} M_{2j}(t), \end{cases} \tag{12}$$

where, for $j = 1, 2$,

$$M_{j1}(t) = \int_0^t \sigma_j(\rho(s))\mathrm{d}W(s), \ M_{j2}(t) = \int_0^t \int_{\mathbb{Z}} \ln\left[1 + \gamma_j(\mu, \rho(s))\right] \widetilde{N}(\mathrm{d}s, \mathrm{d}\mu). \tag{13}$$

Based on (13), we deduce

$$\begin{cases} \langle M_{j1}(t)\rangle = \int_0^t \sigma_j^2(\rho(s))\mathrm{d}s \leq \sigma^2 t, \\ \langle M_{j2}(t)\rangle = \int_0^t \int_{\mathbb{Z}} \left\{\ln\left[1 + \gamma_j(\mu, \rho(s))\right]\right\}^2 \lambda(\mathrm{d}\mu)\mathrm{d}s \leq \max_{i\in\mathbb{S},\, j=1,2} \left\{\left[\ln\left(1 + \gamma_{j*}(i)\right)\right]^2, \left[\ln\left(1 + \gamma_j^*(i)\right)\right]^2\right\} \lambda(\mathbb{Z})t. \end{cases} \tag{14}$$

By Lemma 3.1 in [17], we obtain

$$\lim_{t\to+\infty} t^{-1} M_{ij}(t) = 0 \ a.s. \ (i, j = 1, 2). \tag{15}$$

According to system (12), we have

$$\begin{cases} t^{-1} \ln\left(\frac{x(t)}{x(0)}\right) = \langle B_1(\rho(t))\rangle - \langle a_{11}(\rho(t))x(t)\rangle - \langle a_{12}(\rho(t))y(t)\rangle + t^{-1} \sum_{j=1}^{2} M_{1j}(t), \\ t^{-1} \ln\left(\frac{y(t)}{y(0)}\right) = \langle B_2(\rho(t))\rangle - \langle a_{21}(\rho(t))x(t)\rangle - \langle a_{22}(\rho(t))y(t)\rangle + t^{-1} \sum_{j=1}^{2} M_{2j}(t). \end{cases} \tag{16}$$

Combining (15) with (16) yields

$$\limsup_{t\to+\infty} t^{-1} \ln x(t) \leq \langle B_1(\rho(t))\rangle^* = \mathcal{B}_1, \quad \limsup_{t\to+\infty} t^{-1} \ln y(t) \leq \langle B_2(\rho(t))\rangle^* = \mathcal{B}_2. \tag{17}$$

Now, we only prove (1), the proof of (2) is analogous. In view of (17), if $\mathcal{B}_1 < 0$, then $\lim_{t\to+\infty} x(t) = 0$ a.s. Hence, for $\forall \varepsilon \in (0, 1)$, there exist $T > 0$ and $\Omega_\varepsilon \in \mathcal{F}$ such that for any $t > T$ and $\omega \in \Omega_\varepsilon$, $P(\Omega_\varepsilon) \geq 1 - \varepsilon$ and $x(t, \omega) \leq \varepsilon$. On the one hand, from the second equation of system (12), we have

$$\ln\left(\frac{y(t)}{y(0)}\right) \leq \int_0^t B_2(\rho(s))\mathrm{d}s - \underline{a_{22}} \int_0^t y(s)\mathrm{d}s + \sum_{j=1}^{2} M_{2j}(t). \tag{18}$$

According to Lemma 2 in [15], we deduce $\langle y(t)\rangle^* \leq \frac{\mathcal{B}_2}{\underline{a_{22}}}$ a.s. On the other hand, for any $t > T$, we deduce

$$\ln\left(\frac{y(t)}{y(0)}\right) \geq \int_0^t B_2(\rho(s))\mathrm{d}s - \overline{a_{21}}\int_0^T x(s)\mathrm{d}s - \overline{a_{21}}\varepsilon(t-T) - \overline{a_{22}}\int_0^t y(s)\mathrm{d}s + \sum_{j=1}^{2} M_{2j}(t). \tag{19}$$

By Lemma 2 in [15] and the arbitrariness of ε, we obtain $\langle y(t)\rangle_* \geq \frac{\mathcal{B}_2}{\overline{a_{22}}}$ a.s. The proof is complete. □

Corollary 3.2. *Under* $(\mathbf{H_2})$. *The solutions of system (4) have the following properties:*
(1) If $B_1(i) < 0$, $B_2(i) < 0$, *then* $\lim_{t\to+\infty} x(t) = 0$, $\lim_{t\to+\infty} y(t) = 0$ *a.s.;*
(2) If $B_1(i) < 0$, $B_2(i) > 0$, *then* $\lim_{t\to+\infty} x(t) = 0$, $\lim_{t\to+\infty}\langle y(t)\rangle = \frac{B_2(i)}{a_{22}(i)}$ *a.s.;*
(3) If $B_1(i) > 0$, $B_2(i) < 0$, *then* $\lim_{t\to+\infty}\langle x(t)\rangle = \frac{B_1(i)}{a_{11}(i)}$, $\lim_{t\to+\infty} y(t) = 0$ *a.s.*

4. Stochastic permanence

Definition 4.1. *System (3) is said to be stochastically permanent, if, for* $\forall\epsilon \in (0, 1)$, *there exist* $\delta_* = \delta_*(\epsilon) > 0$ *and* $\delta^* = \delta^*(\epsilon) > 0$ *such that*

$$\liminf_{t\to+\infty} P\{|X(t)| \geq \delta_*\} \geq 1-\epsilon,\ \liminf_{t\to+\infty} P\{|X(t)| \leq \delta^*\} \geq 1-\epsilon. \tag{20}$$

Remark 4.2. *The above stochastic permanence definition of multi-population system was first proposed by Li et al. [5] and has been intensively applied (see e.g. [6, 17, 22, 26, 27]).*

Lemma 4.3. *Under* $(\mathbf{H_1})$. *Let* $X(t)$ *be the solution to system (3) with initial value* $X(0) \in \mathbb{R}^2_+$, *then for any* $\theta_1 > 0$ *and* $\theta_2 > 0$, *there exists a constant* $K(\theta_1, \theta_2) > 0$ *such that*

$$\limsup_{t\to+\infty} \mathbb{E}\left[x^{\theta_1}(t) + y^{\theta_2}(t)\right] \leq K(\theta_1, \theta_2). \tag{21}$$

Proof. Define $W(x, y) = x^{\theta_1} + y^{\theta_2}$. By Itô's formula, we compute

$$\begin{aligned}
\mathcal{L}[W(x,y)] =& \theta_1 x^{\theta_1}\left[r_1(\rho(t)) - a_{11}(\rho(t))x - a_{12}(\rho(t))y\right] + \tfrac{\theta_1(\theta_1-1)\sigma_1^2(\rho(t))}{2}x^{\theta_1}\\
&+ x^{\theta_1}\int_{\mathbb{Z}}\left\{[1+\gamma_1(\mu,\rho(t))]^{\theta_1} - 1 - \theta_1\gamma_1(\mu,\rho(t))\right\}\lambda(\mathrm{d}\mu)\\
&+ \theta_2 y^{\theta_2}\left[r_2(\rho(t)) - a_{21}(\rho(t))x - a_{22}(\rho(t))y\right] + \tfrac{\theta_2(\theta_2-1)\sigma_2^2(\rho(t))}{2}y^{\theta_2}\\
&+ y^{\theta_2}\int_{\mathbb{Z}}\left\{[1+\gamma_2(\mu,\rho(t))]^{\theta_2} - 1 - \theta_2\gamma_2(\mu,\rho(t))\right\}\lambda(\mathrm{d}\mu)\\
\leq& \theta_1 x^{\theta_1}\left[\overline{r_1} - \underline{a_{11}}x\right] + \theta_2 y^{\theta_2}\left[\overline{r_2} - \underline{a_{22}}y\right] + \tfrac{\sigma^2}{2}\left(\theta_1^2 x^{\theta_1} + \theta_2^2 y^{\theta_2}\right)\\
&+ x^{\theta_1}\int_{\mathbb{Z}}\max_{i\in\mathbb{S}}\left\{\left[1+\gamma_1^*(i)\right]^{\theta_1} - 1 - \theta_1\gamma_{1*}(i)\right\}\lambda(\mathrm{d}\mu)\\
&+ y^{\theta_2}\int_{\mathbb{Z}}\max_{i\in\mathbb{S}}\left\{\left[1+\gamma_2^*(i)\right]^{\theta_2} - 1 - \theta_2\gamma_{2*}(i)\right\}\lambda(\mathrm{d}\mu).
\end{aligned} \tag{22}$$

From (22) we observe that there exists a constant $K(\theta_1, \theta_2) > 0$ such that

$$\mathcal{L}[W(x,y)] + W(x,y) \leq K(\theta_1, \theta_2). \tag{23}$$

Based on Itô's formula and (23), we obtain $\mathcal{L}\left[\mathrm{e}^t W(x,y)\right] \leq \mathrm{e}^t K(\theta_1, \theta_2)$. Hence, integrating $\mathrm{d}[\mathrm{e}^t W(x(t), y(t))]$ from 0 to t and then taking expectations lead to

$$\mathrm{e}^t\mathbb{E}\left[x^{\theta_1}(t) + y^{\theta_2}(t)\right] \leq [x(0)]^{\theta_1} + [y(0)]^{\theta_2} + K(\theta_1,\theta_2)\left(\mathrm{e}^t - 1\right), \tag{24}$$

which implies the required assertion (21). □

Now, we are in the position to prove stochastic permanence of system (3). For convenience, let C be a vector or matrix. Denote by $C \gg 0$ all elements of C are positive. Let

$$Y^{S\times S} = \left\{C = (c_{ij})_{S\times S} : c_{ij} \le 0,\, i \ne j\right\}. \tag{25}$$

We shall also need the following two classical results.

Lemma 4.4. *(see Lemma 5.3 in [23]) If $C = (c_{ij})_{S\times S} \in Y^{S\times S}$ has all of its row sums positive, that is, for $\forall i \in \mathbb{S}$, $\sum_{j=1}^{S} c_{ij} > 0$, then $det(C) > 0$.*

Lemma 4.5. *(See Theorem 2.10 in [23]) If $C = (c_{ij})_{S\times S} \in Y^{S\times S}$, then the following statements are equivalent:*
(1) C is a nonsingular M-matrix;
(2) All of the principal minors of C are positive; that is,

$$\begin{vmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & \vdots & & \vdots \\ c_{k1} & c_{k2} & \dots & c_{kk} \end{vmatrix} > 0 \ for\ \forall k \in \mathbb{S}. \tag{26}$$

(3) C is semi-positive; that is, there exists $x \gg 0$ in $\mathbb{R}^{S\times 1}$ such that $Cx \gg 0$.

Lemma 4.6. *Under* $(\mathbf{H_3})$. *If $\mathcal{B} > 0$, then there exists a constant $\theta_0 > 0$ such that for any $0 < \theta < \theta_0$, $G(\theta) = diag\,(\nu_1(\theta), ..., \nu_S(\theta)) - \Gamma$ is a nonsingular M-matrix, where*

$$\begin{cases} \nu_i(\theta) = B(i)\theta - \frac{\sigma^2}{2}\theta^2 - \int_{\mathbb{Z}} \left[\Gamma^{\theta}(i) - 1 - \theta \ln \Gamma(i)\right] \lambda(d\mu), \\ \ln \Gamma(i) = \max\left\{\left|\ln\left(1 + \min_{j=1,2}\left\{\gamma_{j*}(i)\right\}\right)\right|, \left|\ln\left(1 + \max_{j=1,2}\left\{\gamma_j^*(i)\right\}\right)\right|\right\}. \end{cases} \tag{27}$$

Proof. Without loss of generality, let $j = S$, that is, for all $1 \le i \le S-1$, $\gamma_{iS} > 0$. By Appendix A in [28], under $(\mathbf{H_3})$, $\mathcal{B} > 0$ is equivalent to

$$\begin{vmatrix} B(1) & -\gamma_{12} & \dots & -\gamma_{1S} \\ B(2) & -\gamma_{22} & \dots & -\gamma_{2S} \\ \vdots & \vdots & & \vdots \\ B(S) & -\gamma_{S2} & \dots & -\gamma_{SS} \end{vmatrix} > 0. \tag{28}$$

Compute

$$\det G(\theta) = \begin{vmatrix} \nu_1(\theta) & -\gamma_{12} & \dots & -\gamma_{1S} \\ \nu_2(\theta) & \nu_2(\theta) - \gamma_{22} & \dots & -\gamma_{2S} \\ \vdots & \vdots & & \vdots \\ \nu_S(\theta) & -\gamma_{S2} & \dots & \nu_S(\theta) - \gamma_{SS} \end{vmatrix} = \sum_{i=1}^{S} \nu_i(\theta) M_i(\theta), \tag{29}$$

where $M_i(\theta)$ represents the corresponding minor of $\nu_i(\theta)$ in the first column. According to (27), we deduce that $\nu_i(0) = 0$ and $\frac{d\nu_i(\theta)}{d\theta}|_{\theta=0} = B(i)$. Hence,

$$\frac{d}{d\theta}\left[\det G(\theta)\right]|_{\theta=0} = \sum_{i=1}^{S} B(i) M_i(0) = \begin{vmatrix} B(1) & -\gamma_{12} & \dots & -\gamma_{1S} \\ B(2) & -\gamma_{22} & \dots & -\gamma_{2S} \\ \vdots & \vdots & & \vdots \\ B(S) & -\gamma_{S2} & \dots & -\gamma_{SS} \end{vmatrix}. \tag{30}$$

Combining (28) with (30) yields that $\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\det G(\theta)\right]|_{\theta=0} > 0$. Clearly, $\det G(0) = 0$. Hence, there exists a constant $0 < \theta_0 < 1$ such that for any $0 < \theta < \theta_0$, $\det G(\theta) > 0$ and

$$\nu_i(\theta) > -\gamma_{iS},\ 1 \le i \le S-1. \tag{31}$$

For each $k = 1, 2, ..., S-1$, consider the leading principal sub-matrix

$$G_k(\theta) := \begin{pmatrix} \nu_1(\theta) - \gamma_{11} & -\gamma_{12} & \dots & -\gamma_{1k} \\ -\gamma_{21} & \nu_2(\theta) - \gamma_{22} & \dots & -\gamma_{2k} \\ \vdots & \vdots & & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \dots & \nu_k(\theta) - \gamma_{kk} \end{pmatrix} \tag{32}$$

of $G(\theta)$. Obviously, $G_k(\theta) \in Y^{k\times k}$. Moreover, in the light of (31), we compute

$$\nu_i(\theta) - \sum_{j=1}^{k} \gamma_{ij} = \nu_i(\theta) + \sum_{j=k+1}^{S} \gamma_{ij} \ge \nu_i(\theta) + \gamma_{iS} > 0,\ i = 1, 2, ..., k. \tag{33}$$

By Lemma 4.4, $\det G_k(\theta) > 0$. That is to say, all the leading principal minors of $G(\theta)$ are positive. Hence, the required assertion follows from Lemma 4.5. □

Similarly, we obtain

Lemma 4.7. *Under* (**H$_3$**). *If* $\mathcal{B}_1 > 0$, *then there exists a constant* $\overline{\theta_0} > 0$ *such that for any* $0 < \theta < \overline{\theta_0}$, $\overline{G(\theta)} = diag\left(\overline{\nu_1(\theta)}, ..., \overline{\nu_S(\theta)}\right) - \Gamma$ *is a nonsingular M-matrix; If* $\mathcal{B}_2 > 0$, *then there exists a constant* $\widehat{\theta_0} > 0$ *such that for any* $0 < \theta < \widehat{\theta_0}$, $\widehat{G(\theta)} = diag\left(\widehat{\nu_1(\theta)}, ..., \widehat{\nu_S(\theta)}\right) - \Gamma$ *is a nonsingular M-matrix, where*

$$\begin{cases} \overline{\nu_i(\theta)} = B_1(i)\theta - \frac{\sigma_1^2(i)}{2}\theta^2 - \displaystyle\int_{\mathbb{Z}} \left\{[1+\gamma_1(\mu,i)]^{-\theta} - 1 + \theta\ln[1+\gamma_1(\mu,i)]\right\}\lambda(\mathrm{d}\mu), \\ \widehat{\nu_i(\theta)} = B_2(i)\theta - \frac{\sigma_2^2(i)}{2}\theta^2 - \displaystyle\int_{\mathbb{Z}} \left\{[1+\gamma_2(\mu,i)]^{-\theta} - 1 + \theta\ln[1+\gamma_2(\mu,i)]\right\}\lambda(\mathrm{d}\mu). \end{cases}$$

Lemma 4.8. *Let $X(t)$ be the solution to system (3) with initial value $X(0) \in \mathbb{R}_+^2$. If there exists a constant $\theta > 0$ such that $G(\theta)$ is a nonsingular M-matrix, then there exists $H(\theta) > 0$ such that*

$$\limsup_{t\to+\infty} \mathbb{E}\left[(x(t)+y(t))^{-\theta}\right] \le H(\theta). \tag{34}$$

Proof. According to Lemma 4.5 (3), there exists $\mathbf{p} = (p_1, p_2, ..., p_S)^{\mathrm{T}} \gg 0$ such that $G(\theta)\mathbf{p} \gg 0$, namely, for $\forall i \in \mathbb{S}$, $\nu_i(\theta)p_i - \sum_{j=1}^{S}\gamma_{ij}p_j > 0$. Hence, there exists a constant $\kappa > 0$ such that

$$\nu_i(\theta)p_i - \sum_{j=1}^{S}\gamma_{ij}p_j - \kappa p_i > 0,\ \forall i \in \mathbb{S}. \tag{35}$$

Define $V = x + y$, $U = \frac{1}{V}$, $\widetilde{U} = p_i(1+U)^{\theta}$. By Itô's formula, we compute

$$\mathcal{L}\left[\mathrm{e}^{\kappa t}\widetilde{U}\right] = \kappa\mathrm{e}^{\kappa t}\widetilde{U} + \mathrm{e}^{\kappa t}\mathcal{L}\left[\widetilde{U}\right] = \mathrm{e}^{\kappa t}\left\{\kappa\widetilde{U} + \mathcal{L}\left[\widetilde{U}\right]\right\}, \tag{36}$$

where

$$\begin{aligned}\kappa\widetilde{U}+\mathcal{L}\left[\widetilde{U}\right]=&\kappa p_i\,(1+U)^{\theta}+\sum_{j=1}^{S}\gamma_{ij}p_j\,(1+U)^{\theta}-p_i\theta\,(1+U)^{\theta-1}\,U\tfrac{r_1(\rho(t))x+r_2(\rho(t))y}{V}\\&+p_i\theta\,(1+U)^{\theta-1}\,\tfrac{a_{11}(\rho(t))x^2+(a_{12}(\rho(t))+a_{21}(\rho(t)))xy+a_{22}(\rho(t))y^2}{V^2}\\&+p_i\theta\,(1+U)^{\theta-1}\,U\left(\tfrac{\sigma_1(\rho(t))x+\sigma_2(\rho(t))y}{V}\right)^2\\&+p_i\theta\,(1+U)^{\theta-1}\,U\int_{\mathbb{Z}}\tfrac{\gamma_1(\mu,\rho(t))x+\gamma_2(\mu,\rho(t))y}{V}\lambda(\mathrm{d}\mu)\\&+p_i\tfrac{\theta(\theta-1)}{2}\,(1+U)^{\theta-2}\,U^2\left(\tfrac{\sigma_1(\rho(t))x+\sigma_2(\rho(t))y}{V}\right)^2\\&+p_i\int_{\mathbb{Z}}\left[\left(1+\tfrac{1}{V+\gamma_1(\mu,\rho(t))x+\gamma_2(\mu,\rho(t))y}\right)^{\theta}-(1+U)^{\theta}\right]\lambda(\mathrm{d}\mu)\\=:&\mathcal{O}\left[U^{\theta}\right]U^{\theta}+F(U),\end{aligned}\tag{37}$$

where $\lim_{U\to+\infty}\frac{F(U)}{U^{\theta}}=0$ and

$$\begin{aligned}\mathcal{O}\left[U^{\theta}\right]=&\kappa p_i+\sum_{j=1}^{S}\gamma_{ij}p_j-p_i\theta\tfrac{r_1(\rho(t))x+r_2(\rho(t))y}{V}+p_i\theta\left(\tfrac{\sigma_1(\rho(t))x+\sigma_2(\rho(t))y}{V}\right)^2\\&+p_i\theta\int_{\mathbb{Z}}\tfrac{\gamma_1(\mu,\rho(t))x+\gamma_2(\mu,\rho(t))y}{V}\lambda(\mathrm{d}\mu)+p_i\tfrac{\theta(\theta-1)}{2}\left(\tfrac{\sigma_1(\rho(t))x+\sigma_2(\rho(t))y}{V}\right)^2\\&+p_i\int_{\mathbb{Z}}\left[\left(\tfrac{V}{V+\gamma_1(\mu,\rho(t))x+\gamma_2(\mu,\rho(t))y}\right)^{\theta}-1\right]\lambda(\mathrm{d}\mu).\end{aligned}\tag{38}$$

Based on Jensen's inequality and $\mathrm{e}^x=\sum_{n=0}^{+\infty}\frac{x^n}{n!}$, we derive

$$\begin{aligned}\mathcal{O}\left[U^{\theta}\right]\leq&\kappa p_i+\sum_{j=1}^{S}\gamma_{ij}p_j-p_i\theta\min_{j=1,2}\left\{B_j(\rho(t))\right\}+p_i\tfrac{\theta^2\sigma^2}{2}\\&+p_i\int_{\mathbb{Z}}\left[\left(1+\tfrac{\gamma_1(\mu,\rho(t))x+\gamma_2(\mu,\rho(t))y}{V}\right)^{-\theta}-1+\theta\ln\left(1+\tfrac{\gamma_1(\mu,\rho(t))x+\gamma_2(\mu,\rho(t))y}{V}\right)\right]\lambda(\mathrm{d}\mu)\\\leq&\kappa p_i+\sum_{j=1}^{S}\gamma_{ij}p_j-p_iB(i)\theta+p_i\tfrac{\theta^2\sigma^2}{2}+p_i\int_{\mathbb{Z}}\left[\Gamma^{\theta}(i)-1-\theta\ln\Gamma(i)\right]\lambda(\mathrm{d}\mu)\\=&\kappa p_i+\sum_{j=1}^{S}\gamma_{ij}p_j-p_i\nu_i(\theta)<0.\end{aligned}\tag{39}$$

In view of (36), (37) and (39), there exists $\mathcal{H}(\theta)>0$ such that

$$\mathcal{L}\left[\mathrm{e}^{\kappa t}\widetilde{U}(t)\right]\leq\mathcal{H}(\theta)\mathrm{e}^{\kappa t}.\tag{40}$$

Integrating $\mathrm{d}\left[\mathrm{e}^{\kappa t}\widetilde{U}(t)\right]$ from 0 to t and then taking expectations yield

$$\mathbb{E}\left[p_i\mathrm{e}^{\kappa t}\left[1+U(t)\right]^{\theta}\right]-p_i\left[1+U(0)\right]^{\theta}\leq\tfrac{\mathcal{H}(\theta)}{\kappa}\left(\mathrm{e}^{\kappa t}-1\right).\tag{41}$$

According to (41), we deduce

$$\mathbb{E}\left[1+U(t)\right]^{\theta}\leq\tfrac{\mathcal{H}(\theta)}{\kappa\min_{i\in\mathbb{S}}\{p_i\}}+\left(1+\tfrac{1}{x(0)+y(0)}\right)^{\theta}\mathrm{e}^{-\kappa t}.\tag{42}$$

Define $H(\theta)=\frac{\mathcal{H}(\theta)}{\kappa\min_{i\in\mathbb{S}}\{p_i\}}$. From (42) we obtain the required assertion. □

Lemma 4.9. *Let $X(t)$ be the solution to system (3) with initial value $X(0) \in \mathbb{R}^2_+$. If $\mathcal{B}_1 < 0$ and there exists a constant $\theta > 0$ such that $\widehat{G(\theta)}$ is a nonsingular M-matrix, then there exists $\widehat{H(\theta)} > 0$ such that*

$$\limsup_{t\to+\infty} \mathbb{E}\left[y^{-\theta}(t)\right] \le \widehat{H(\theta)}. \tag{43}$$

Proof. Define $\widehat{U} = \frac{1}{y}$. By Itô's formula, we obtain

$$\begin{aligned} \mathrm{d}\left[\widehat{U}\right] =& \widehat{U}\left\{-r_2(\rho(t)) + \sigma_2^2(\rho(t)) + \int_{\mathbb{Z}} \tfrac{\gamma_2^2(\mu,\rho(t))}{1+\gamma_2(\mu,\rho(t))}\lambda(\mathrm{d}\mu) + \tfrac{a_{22}(\rho(t))}{\widehat{U}} + a_{21}(\rho(t))x\right\}\mathrm{d}t \\ &- \sigma_2(\rho(t))\widehat{U}\mathrm{d}W(t) - \int_{\mathbb{Z}} \tfrac{\gamma_2(\mu,\rho(t))}{1+\gamma_2(\mu,\rho(t))}\widehat{U}\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu). \end{aligned} \tag{44}$$

Based on Lemma 4.5 (3), there exists $\widehat{\mathbf{p}} = (\widehat{p_1}, \widehat{p_2}, ..., \widehat{p_S})^{\mathrm{T}} \gg 0$ such that $\widehat{G(\theta)\mathbf{p}} \gg 0$, that is to say, for $\forall i \in \mathbb{S}$, $\widehat{\nu_i(\theta)}\widehat{p_i} - \sum_{j=1}^S \gamma_{ij}\widehat{p_j} > 0$. Thus, there exists a constant $\widehat{\kappa} > 0$ such that

$$\widehat{\nu_i(\theta)}\widehat{p_i} - \sum_{j=1}^{S} \gamma_{ij}\widehat{p_j} - \widehat{\kappa p_i} > 0, \ \forall i \in \mathbb{S}. \tag{45}$$

In the light of (45), for any sufficiently small $\varepsilon > 0$,

$$\widehat{\nu_i(\theta)}\widehat{p_i} - \sum_{j=1}^{S} \gamma_{ij}\widehat{p_j} - \widehat{\kappa p_i} - \widehat{p_i a_{21}}\theta\varepsilon > 0, \ \forall i \in \mathbb{S}. \tag{46}$$

According to the proof of Theorem 3.1, for $\forall\varepsilon \in (0, 1)$, there exist $T > 0$ and $\Omega_\varepsilon \in \mathcal{F}$ such that for any $t > T$ and $\omega \in \Omega_\varepsilon$, $P(\Omega_\varepsilon) \ge 1 - \varepsilon$ and $x(t, \omega) \le \varepsilon$. Applying Itô's formula again, we compute

$$\mathcal{L}\left[\mathrm{e}^{\widehat{\kappa}t}\widehat{p_i}\left(1+\widehat{U}\right)^{\theta}\right] = \mathrm{e}^{\widehat{\kappa}t}\left\{\widehat{\kappa p_i}\left(1+\widehat{U}\right)^{\theta} + \mathcal{L}\left[\widehat{p_i}\left(1+\widehat{U}\right)^{\theta}\right]\right\}, \tag{47}$$

where

$$\begin{aligned} &\widehat{\kappa p_i}\left(1+\widehat{U}\right)^{\theta} + \mathcal{L}\left[\widehat{p_i}\left(1+\widehat{U}\right)^{\theta}\right] \\ =&\widehat{\kappa p_i}\left(1+\widehat{U}\right)^{\theta} + \widehat{p_i}\theta\left(1+\widehat{U}\right)^{\theta-1}\mathcal{L}\left[\widehat{U}\right] + \tfrac{\theta(\theta-1)}{2}\widehat{p_i}\sigma_2^2(\rho(t))\widehat{U}^2\left(1+\widehat{U}\right)^{\theta-2} + \sum_{j=1}^{S}\gamma_{ij}\widehat{p_j}\left(1+\widehat{U}\right)^{\theta} \\ &+ \int_{\mathbb{Z}}\left[\widehat{p_i}\left(1+\tfrac{\widehat{U}}{1+\gamma_2(\mu,\rho(t))}\right)^{\theta} - \widehat{p_i}\left(1+\widehat{U}\right)^{\theta} + \widehat{p_i}\theta\tfrac{\gamma_2(\mu,\rho(t))}{1+\gamma_2(\mu,\rho(t))}\widehat{U}\left(1+\widehat{U}\right)^{\theta-1}\right]\lambda(\mathrm{d}\mu). \end{aligned} \tag{48}$$

Based on (44), we compute

$$\begin{aligned} &\widehat{\kappa p_i}\left(1+\widehat{U}\right)^{\theta} + \mathcal{L}\left[\widehat{p_i}\left(1+\widehat{U}\right)^{\theta}\right] \\ =&\widehat{\kappa p_i}\left(1+\widehat{U}\right)^{\theta} + \widehat{p_i}\theta\widehat{U}\left(1+\widehat{U}\right)^{\theta-1}\left[-B_2(\rho(t)) + \tfrac{\sigma_2^2(\rho(t))}{2} + \int_{\mathbb{Z}}\ln\left(1+\gamma_2(\mu,\rho(t))\right)\lambda(\mathrm{d}\mu) + \tfrac{a_{22}(\rho(t))}{\widehat{U}} + a_{21}(\rho(t))x\right] \\ &+ \tfrac{\theta(\theta-1)}{2}\widehat{p_i}\sigma_2^2(\rho(t))\widehat{U}^2\left(1+\widehat{U}\right)^{\theta-2} + \sum_{j=1}^{S}\gamma_{ij}\widehat{p_j}\left(1+\widehat{U}\right)^{\theta} + \int_{\mathbb{Z}}\left[\widehat{p_i}\left(1+\tfrac{\widehat{U}}{1+\gamma_2(\mu,\rho(t))}\right)^{\theta} - \widehat{p_i}\left(1+\widehat{U}\right)^{\theta}\right]\lambda(\mathrm{d}\mu) \\ =&:\mathcal{O}\left(\widehat{U}^{\theta}\right)\widehat{U}^{\theta} + \widehat{F}\left(\widehat{U}\right), \end{aligned} \tag{49}$$

where $\lim_{\widehat{U}\to+\infty}\frac{\widehat{F}(\widehat{U})}{\widehat{U}^\theta}=0$ and

$$\begin{aligned}
O\left(\widehat{U}^\theta\right) =&\widehat{\kappa p_i}+\widehat{p_i}\theta\left[-B_2(\rho(t))+\frac{\sigma_2^2(\rho(t))}{2}+\int_{\mathbb{Z}}\ln\left(1+\gamma_2(\mu,\rho(t))\right)\lambda(\mathrm{d}\mu)+a_{21}(\rho(t))x\right]\\
&+\widehat{p_i}\frac{\theta(\theta-1)\sigma_2^2(\rho(t))}{2}+\sum_{j=1}^{S}\gamma_{ij}\widehat{p_j}+\int_{\mathbb{Z}}\left[\widehat{p_i}\left(1+\gamma_2(\mu,\rho(t))\right)^{-\theta}-\widehat{p_i}\right]\lambda(\mathrm{d}\mu)\\
=&\widehat{\kappa p_i}+\sum_{j=1}^{S}\gamma_{ij}\widehat{p_j}-\widehat{p_i\nu_i(\theta)}+\widehat{p_i}\theta a_{21}(\rho(t))x\\
\le&\widehat{\kappa p_i}+\sum_{j=1}^{S}\gamma_{ij}\widehat{p_j}-\widehat{p_i\nu_i(\theta)}+\widehat{p_i}\theta\overline{a_{21}}\varepsilon<0.
\end{aligned} \tag{50}$$

In view of (47), (49) and (50), there exists $\widehat{\mathcal{H}(\theta)}>0$ such that

$$\mathcal{L}\left[\mathrm{e}^{\widehat{\kappa}t}\widehat{p_i}\left(1+\widehat{U(t)}\right)^\theta\right]\le\widehat{\mathcal{H}(\theta)}\mathrm{e}^{\widehat{\kappa}t}. \tag{51}$$

Integrating $\mathrm{d}\left[\mathrm{e}^{\widehat{\kappa}t}\widehat{p_i}\left(1+\widehat{U(t)}\right)^\theta\right]$ from 0 to t and then taking expectations yield

$$\mathbb{E}\left[\widehat{p_i}\mathrm{e}^{\widehat{\kappa}t}\left[1+\widehat{U(t)}\right]^\theta\right]-\widehat{p_i}\left[1+\widehat{U(0)}\right]^\theta\le\frac{\widehat{\mathcal{H}(\theta)}}{\widehat{\kappa}}\left(\mathrm{e}^{\widehat{\kappa}t}-1\right). \tag{52}$$

Based on (52), we deduce

$$\mathbb{E}\left[1+\widehat{U(t)}\right]^\theta\le\frac{\widehat{\mathcal{H}(\theta)}}{\widehat{\kappa}\min_{i\in\mathbb{S}}\{\widehat{p_i}\}}+\left(1+\frac{1}{y(0)}\right)^\theta\mathrm{e}^{-\widehat{\kappa}t}. \tag{53}$$

Define $\widehat{H(\theta)}=\frac{\widehat{\mathcal{H}(\theta)}}{\widehat{\kappa}\min_{i\in\mathbb{S}}\{\widehat{p_i}\}}$. Then the required assertion follows from (53). □

In a similar way, we obtain

Lemma 4.10. *Let $X(t)$ be the solution to system (3) with initial value $X(0)\in\mathbb{R}_+^2$. If $\mathcal{B}_2<0$ and there exists a constant $\theta>0$ such that $\overline{G(\theta)}$ is a nonsingular M-matrix, then there exists $\overline{H(\theta)}>0$ such that*

$$\limsup_{t\to+\infty}\mathbb{E}\left[x^{-\theta}(t)\right]\le\overline{H(\theta)}. \tag{54}$$

Theorem 4.11. *Under* ($\mathbf{H_1}$) *and* ($\mathbf{H_3}$). *If $\mathcal{B}>0$, then system (3) is stochastically permanent.*

Proof. Noting that

$$|X(t)|^{-\theta}\le2^{0.5\theta}\left[x(t)+y(t)\right]^{-\theta}, \tag{55}$$

thanks to Lemma 4.8, we deduce

$$\limsup_{t\to+\infty}\mathbb{E}\left[|X(t)|^{-\theta}\right]\le2^{\frac{\theta}{2}}H(\theta). \tag{56}$$

From Chebyshev's inequality, for $\forall\epsilon\in(0,1)$, there exists $\delta_*=\frac{\sqrt{2}}{2}\left(\frac{\epsilon}{H(\theta)}\right)^{\frac{1}{\theta}}>0$ such that

$$\limsup_{t\to+\infty}P\left\{|X(t)|<\delta_*\right\}=\limsup_{t\to+\infty}P\left\{|X(t)|^{-1}>\delta_*^{-1}\right\}\le(\delta_*)^\theta\limsup_{t\to+\infty}\mathbb{E}\left[|X(t)|^{-\theta}\right]\le\epsilon. \tag{57}$$

In other words,

$$\liminf_{t\to+\infty} P\{|X(t)| \geq \delta_*\} \geq 1-\epsilon. \tag{58}$$

The rest of (20) follows from combining Lemma 4.3 with Chebyshev's inequality. Hence, system (3) is stochastically permanent. □

Corollary 4.12. *Under* $(\mathbf{H_2})$. *If* $B(i) > 0$, *then system (4) is stochastically permanent.*

Theorem 4.13. *Under* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$. *If* $\mathcal{B}_1\mathcal{B}_2 < 0$, *then system (3) is stochastically permanent.*

Proof. We only prove that if $\mathcal{B}_1 < 0$ and $\mathcal{B}_2 > 0$, then system (3) is stochastically permanent, the proof of the other case is similar. By Chebyshev's inequality, for $\forall\epsilon \in (0,1)$, there exists $\delta_* = \left(\frac{\epsilon}{\widehat{H(\theta)}}\right)^{\frac{1}{\theta}} > 0$ such that

$$\limsup_{t\to+\infty} P\{|X(t)| < \delta_*\} \leq \limsup_{t\to+\infty} P\left\{y^{-1}(t) > \delta_*^{-1}\right\} \leq (\delta_*)^{\theta} \limsup_{t\to+\infty} \mathbb{E}\left[y^{-\theta}(t)\right] \leq (\delta_*)^{\theta}\, \widehat{H(\theta)} = \epsilon. \tag{59}$$

In view of (59), we obtain

$$\liminf_{t\to+\infty} P\{|X(t)| \geq \delta_*\} \geq 1-\epsilon. \tag{60}$$

The second part of (20) follows from combining Lemma 4.3 with Chebyshev's inequality. So system (3) is stochastically permanent. □

Corollary 4.14. *Under* $(\mathbf{H_2})$. *If* $B_1(i)B_2(i) < 0$, *then system (4) is stochastically permanent.*

Remark 4.15. *Theorem 4.13 implies that in the sense of Definition 4.1, although one species is extinctive, system (3) can still be stochastically permanent. Therefore, it is interesting to study a new definition of stochastic permanence (see [29]) and "stochastic persistence in probability" (see e.g. [30, 31]) of system (3).*

5. Asymptotic properties

Theorem 5.1. *Under* $(\mathbf{H_1})$. *The solution* $(x(t), y(t))^{\mathrm{T}}$ *of system (3) with initial value* $X(0) \in \mathbb{R}_+^2$ *satisfies*

$$\limsup_{t\to+\infty} \frac{\ln[x(t)+y(t)]}{\ln t} \leq 1 \ \text{a.s.} \tag{61}$$

Proof. It is easy to see that

$$\begin{aligned} \mathrm{d}[x(t)+y(t)] \leq{}& [r_1(\rho(t))x(t) + r_2(\rho(t))y(t)]\,\mathrm{d}t + [\sigma_1(\rho(t))x(t) + \sigma_2(\rho(t))y(t)]\,\mathrm{d}W(t) \\ &+ \int_{\mathbb{Z}} [\gamma_1(\mu,\rho(t))x(t) + \gamma_2(\mu,\rho(t))y(t)]\,\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu). \end{aligned} \tag{62}$$

Integrating both sides of (62) from t to u $(u > t)$ yields

$$\begin{aligned} [x(u)+y(u)] - [x(t)+y(t)] \leq{}& \int_t^u [r_1(\rho(s))x(s) + r_2(\rho(s))y(s)]\,\mathrm{d}s + \int_t^u [\sigma_1(\rho(s))x(s) + \sigma_2(\rho(s))y(s)]\,\mathrm{d}W(s) \\ &+ \int_t^u \int_{\mathbb{Z}} [\gamma_1(\mu,\rho(s))x(s) + \gamma_2(\mu,\rho(s))y(s)]\,\widetilde{N}(\mathrm{d}s,\mathrm{d}\mu). \end{aligned} \tag{63}$$

According to (63), we compute

$$\begin{aligned}\mathbb{E}\left[\sup_{t\le u\le t+1}[x(u)+y(u)]\right]\le&\mathbb{E}\left[x(t)+y(t)\right]+r\int_t^{t+1}\mathbb{E}\left[x(s)+y(s)\right]\mathrm{d}s\\&+\mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u\left[\sigma_1(\rho(s))x(s)+\sigma_2(\rho(s))y(s)\right]\mathrm{d}W(s)\right]\\&+\mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u\int_{\mathbb{Z}}\left[\gamma_1(\mu,\rho(s))x(s)+\gamma_2(\mu,\rho(s))y(s)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}\mu)\right].\end{aligned}\tag{64}$$

In view of the Burkholder-Davis-Gundy inequality (see e.g. pp.264-265 in [24]) and Young inequality, we deduce

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u\left[\sigma_1(\rho(s))x(s)+\sigma_2(\rho(s))y(s)\right]\mathrm{d}W(s)\right]\le J\mathbb{E}\left(\int_t^{t+1}\left[\sigma_1(\rho(s))x(s)+\sigma_2(\rho(s))y(s)\right]^2\mathrm{d}s\right)^{0.5}\\\le&J\mathbb{E}\left(\int_t^{t+1}\sigma^2\left[x(s)+y(s)\right]^2\mathrm{d}s\right)^{0.5}\le J\mathbb{E}\left(\sigma^2\sup_{t\le u\le t+1}[x(u)+y(u)]\int_t^{t+1}\left[x(s)+y(s)\right]\mathrm{d}s\right)^{0.5}\\\le&J\mathbb{E}\left(\frac{1}{2J}\sup_{t\le u\le t+1}[x(u)+y(u)]+\frac{\sigma^2J}{2}\int_t^{t+1}[x(s)+y(s)]\mathrm{d}s\right)\\=&\frac{1}{2}\mathbb{E}\left(\sup_{t\le u\le t+1}[x(u)+y(u)]\right)+\frac{\sigma^2J^2}{2}\int_t^{t+1}\mathbb{E}\left[x(s)+y(s)\right]\mathrm{d}s.\end{aligned}\tag{65}$$

Making use of the Burkholder-Davis-Gundy inequality and Hölder's inequality, we obtain

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u\int_{\mathbb{Z}}\left[\gamma_1(\mu,\rho(s))x(s)+\gamma_2(\mu,\rho(s))y(s)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}\mu)\right]\\\le&J\mathbb{E}\left(\int_t^{t+1}\int_{\mathbb{Z}}\left[\gamma_1(\mu,\rho(s))x(s)+\gamma_2(\mu,\rho(s))y(s)\right]^2N(\mathrm{d}s,\mathrm{d}\mu)\right)^{0.5}\\\le&J\mathbb{E}\left(\int_t^{t+1}\int_{\mathbb{Z}}\left[\gamma^\star\left(x(s)+y(s)\right)\right]^2N(\mathrm{d}s,\mathrm{d}\mu)\right)^{0.5}\\\le&J\left(\mathbb{E}\int_t^{t+1}\int_{\mathbb{Z}}\left[\gamma^\star\left(x(s)+y(s)\right)\right]^2N(\mathrm{d}s,\mathrm{d}\mu)\right)^{0.5}\\=&J\left(\int_{\mathbb{Z}}\left[\gamma^\star\right]^2\lambda(\mathrm{d}\mu)\right)^{0.5}\left(\mathbb{E}\int_t^{t+1}\left[x(s)+y(s)\right]^2\mathrm{d}s\right)^{0.5}.\end{aligned}\tag{66}$$

Substituting (65) and (66) into (64), we have

$$\begin{aligned}\mathbb{E}\left(\sup_{t\le u\le t+1}[x(u)+y(u)]\right)\le&2\mathbb{E}\left[x(t)+y(t)\right]+2r\int_t^{t+1}\mathbb{E}\left[x(s)+y(s)\right]\mathrm{d}s+\sigma^2J^2\int_t^{t+1}\mathbb{E}\left[x(s)+y(s)\right]\mathrm{d}s\\&+2J\left(\int_{\mathbb{Z}}\left[\gamma^\star\right]^2\lambda(\mathrm{d}\mu)\right)^{0.5}\left(\int_t^{t+1}\mathbb{E}\left[x(s)+y(s)\right]^2\mathrm{d}s\right)^{0.5}\\\le&2\mathbb{E}\left[x(t)+y(t)\right]+2r\int_t^{t+1}\mathbb{E}\left[x(s)+y(s)\right]\mathrm{d}s+\sigma^2J^2\int_t^{t+1}\mathbb{E}\left[x(s)+y(s)\right]\mathrm{d}s\\&+2J\left(\int_{\mathbb{Z}}\left[\gamma^\star\right]^2\lambda(\mathrm{d}\mu)\right)^{0.5}\left(2\int_t^{t+1}\mathbb{E}\left[x^2(s)+y^2(s)\right]\mathrm{d}s\right)^{0.5}.\end{aligned}\tag{67}$$

Based on Lemma 4.3, there exists $K^*(\theta_1,\theta_2)>0$ such that $\sup_{t\geq 0}\mathbb{E}\left[x^{\theta_1}(t)+y^{\theta_2}(t)\right]\leq K^*(\theta_1,\theta_2)$. Hence,

$$\mathbb{E}\left(\sup_{t\leq u\leq t+1}[x(u)+y(u)]\right)\leq 2K^*(1,1)+2rK^*(1,1)+\sigma^2J^2K^*(1,1)+2J\left(2K^*(2,2)\int_{\mathbb{Z}}[\gamma^\star]^2\lambda(\mathrm{d}\mu)\right)^{0.5}=:\widetilde{K}. \tag{68}$$

Therefore, from (68) we get

$$\mathbb{E}\left(\sup_{k\leq u\leq k+1}[x(u)+y(u)]\right)\leq \widetilde{K},\ k=1,2,... \tag{69}$$

Then by Chebyshev's inequality and (69), we observe that for $\forall\epsilon\in(0,1)$,

$$P\left(\omega:\sup_{k\leq t\leq k+1}[x(t)+y(t)]>k^{1+\epsilon}\right)\leq\frac{\widetilde{K}}{k^{1+\epsilon}},\ k=1,2,... \tag{70}$$

Using Borel-Cantelli's lemma, we obtain that there exists a set $\Omega_o\in\mathcal{F}$ with $P(\Omega_o)=1$ and an integer-valued random variable k_o such that for every $\omega\in\Omega_o$,

$$\sup_{k\leq t\leq k+1}[x(t)+y(t)]\leq k^{1+\epsilon} \tag{71}$$

holds whenever $k\geq k_o(\omega)$. Thus, for almost all $\omega\in\Omega$, if $k\geq k_o$ and $k\leq t\leq k+1$,

$$\frac{\ln[x(t)+y(t)]}{\ln t}\leq\frac{\ln\left(\sup_{k\leq t\leq k+1}[x(t)+y(t)]\right)}{\ln t}\leq\frac{\ln k^{1+\epsilon}}{\ln t}\leq 1+\epsilon. \tag{72}$$

In view of (72), we deduce

$$\limsup_{t\to+\infty}\frac{\ln[x(t)+y(t)]}{\ln t}\leq 1+\epsilon\ a.s. \tag{73}$$

So the desired assertion (61) follows from letting $\epsilon\to 0^+$ in (73). □

6. Conclusions and an example

This paper concerns permanence and asymptotic behaviors of a stochastic competitive Lotka-Volterra system with Markov switching and Lévy noise. Corollaries 4.12 and 4.14 tell us that for some $i\in\mathbb{S}$, if $B(i)>0$ or $B_1(i)B_2(i)<0$, then system (4) is stochastically permanent. Theorems 4.11 and 4.13 tell us that if for $\forall i\in\mathbb{S}$, system (4) is stochastically permanent, then as the result of Markov switching, system (3) remains stochastically permanent. On the other hand, Corollary 3.2 indicates that for some $i\in\mathbb{S}$, if $\max_{j=1,2}\{B_j(i)\}<0$, then system (4) is extinctive. Theorem 3.1 tells us that if for $\forall i\in\mathbb{S}$, system (4) is extinctive, then as the result of Markov switching, system (3) remains extinctive. However, Theorems 4.11, 4.13 and 3.1 provide a more interesting result that if some subsystems are stochastically permanent while some are extinctive, again as the result of Markov switching, system (3) may be stochastically permanent or extinctive, depending on the signs of $\mathcal{B}_1$ and $\mathcal{B}_2$. That is,

Theorem 6.1. *Under* ($\mathbf{H_1}$) *and* ($\mathbf{H_3}$). *If $\mathcal{B}_1\mathcal{B}_2\neq 0$, then system (3) is either stochastically permanent or extinctive. That is, it is extinctive if and only if $\mathcal{B}_1<0$ and $\mathcal{B}_2<0$. Otherwise, it is stochastically permanent.*

Next, consider the following stochastic logistic model with Markov switching and Lévy noise:

$$\mathrm{d}x(t)=x(t^-)\left\{[r_1(\rho(t))-a_{11}(\rho(t))x(t^-)]\,\mathrm{d}t+\sigma_1(\rho(t))\mathrm{d}W(t)+\int_{\mathbb{Z}}\gamma_1(\mu,\rho(t))\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)\right\}. \tag{74}$$

Theorem 6.2. *Under* $(\mathbf{H_1})$ *with* $j = k = 1$ *and* $(\mathbf{H_3})$*. If* $\mathcal{B}_1 \neq 0$*, then system (74) is either stochastically permanent or extinctive. That is, it is stochastically permanent if and only if* $\mathcal{B}_1 > 0$*, while it is extinctive if and only if* $\mathcal{B}_1 < 0$*.*

Corollary 6.3. *Under* $(\mathbf{H_3})$ *and* $(\mathbf{H_4})$*. If* $\mathcal{B}_1^* \neq 0$*, then the following stochastic logistic model with Markov switching*

$$\mathrm{d}x(t) = x(t)\left\{[r_1(\rho(t)) - a_{11}(\rho(t))x(t)]\,\mathrm{d}t + \sigma_1(\rho(t))\mathrm{d}W(t)\right\} \tag{75}$$

is either stochastically permanent or extinctive. That is, it is stochastically permanent if and only if $\mathcal{B}_1^* > 0$*, while it is extinctive if and only if* $\mathcal{B}_1^* < 0$*.*

Remark 6.4. *Corollary 6.3 implies that Theorem 6.2 contains Theorem 6.1 in [22] as a special case.*

Corollary 6.5. *Under* $(\mathbf{H_2})$ *with* $j = k = 1$*. If* $B_1(i) \neq 0$*, then the following stochastic logistic model with Lévy noise*

$$\mathrm{d}x(t) = x(t^-)\left\{[r_1(i) - a_{11}(i)x(t^-)]\,\mathrm{d}t + \sigma_1(i)\mathrm{d}W(t) + \int_{\mathbb{Z}} \gamma_1(\mu, i)\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\} \tag{76}$$

is either stochastically permanent or extinctive. That is, it is stochastically permanent if and only if $B_1(i) > 0$*, while it is extinctive if and only if* $B_1(i) < 0$*.*

Remark 6.6. *Corollary 6.5 implies that Theorem 6.2 contains Remark 1 in [15] as a special case.*

Our results are illustrated by considering the following stochastic competitive Lotka-Volterra system with Markov switching and Lévy noise:

$$\begin{cases} \mathrm{d}x(t) = x(t^-)\left\{[r_1(\rho(t)) - a_{11}(\rho(t))x(t^-) - a_{12}(\rho(t))y(t^-)]\,\mathrm{d}t + \sigma_1(\rho(t))\mathrm{d}W(t) + \int_{\mathbb{Z}} \gamma_1(\mu, \rho(t))\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \\ \mathrm{d}y(t) = y(t^-)\left\{[r_2(\rho(t)) - a_{21}(\rho(t))x(t^-) - a_{22}(\rho(t))y(t^-)]\,\mathrm{d}t + \sigma_2(\rho(t))\mathrm{d}W(t) + \int_{\mathbb{Z}} \gamma_2(\mu, \rho(t))\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \end{cases} \tag{77}$$

where $\rho(t)$ is a right-continuous Markov chain taking values in $\mathbb{S} = \{1, 2\}$. System (77) may be regarded as the result of regime switching, which switches between the following two subsystems:

$$\begin{cases} \mathrm{d}x(t) = x(t^-)\left\{\left[\frac{2}{9} - \frac{x(t^-)}{9} - \frac{2y(t^-)}{9}\right]\mathrm{d}t + \mathrm{d}W(t) + \int_{\mathbb{Z}} \widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \\ \mathrm{d}y(t) = y(t^-)\left\{\left[\frac{1}{9} - \frac{x(t^-)}{9} - \frac{2y(t^-)}{9}\right]\mathrm{d}t + \frac{2}{3}\mathrm{d}W(t) + \int_{\mathbb{Z}} \widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \end{cases} \tag{78}$$

and

$$\begin{cases} \mathrm{d}x(t) = x(t^-)\left\{\left[\frac{5}{9} - \frac{2x(t^-)}{9} - \frac{y(t^-)}{3}\right]\mathrm{d}t + \frac{1}{3}\mathrm{d}W(t) + \int_{\mathbb{Z}} \widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}, \\ \mathrm{d}y(t) = y(t^-)\left\{\left[\frac{4}{9} - \frac{x(t^-)}{9} - \frac{y(t^-)}{3}\right]\mathrm{d}t + \frac{1}{3}\mathrm{d}W(t) + \int_{\mathbb{Z}} \widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)\right\}. \end{cases} \tag{79}$$

Here, $\lambda(\mathbb{Z}) = \frac{1}{9}$ and

$$\begin{cases} r_1(1) = \frac{2}{9}, & r_2(1) = \frac{1}{9}, & r_1(2) = \frac{5}{9}, & r_2(2) = \frac{4}{9}, \\ a_{11}(1) = \frac{1}{9}, & a_{21}(1) = \frac{1}{9}, & a_{11}(2) = \frac{2}{9}, & a_{21}(2) = \frac{1}{9}, \\ a_{12}(1) = \frac{2}{9}, & a_{22}(1) = \frac{2}{9}, & a_{12}(2) = \frac{1}{3}, & a_{22}(2) = \frac{1}{3}, \\ \sigma_1(1) = 1, & \sigma_2(1) = \frac{2}{3}, & \sigma_1(2) = \frac{1}{3}, & \sigma_2(2) = \frac{1}{3}, \\ \gamma_1(\mu, 1) = 1, & \gamma_2(\mu, 1) = 1, & \gamma_1(\mu, 2) = 1, & \gamma_2(\mu, 2) = 1. \end{cases} \tag{80}$$

Based on (80), we compute

$$\begin{cases} B_1(1) = \dfrac{2\ln 2 - 7}{18}, \quad B_2(1) = \dfrac{\ln 2 - 2}{9}, \quad B(1) = \dfrac{2\ln 2 - 7}{18}, \\ B_1(2) = \dfrac{2\ln 2 + 7}{18}, \quad B_2(2) = \dfrac{2\ln 2 + 5}{18}, \quad B(2) = \dfrac{2\ln 2 + 5}{18}. \end{cases} \tag{81}$$

From Corollary 3.2, system (78) is extinctive. By Corollary 4.12, system (79) is stochastically permanent.

Case 1. Let the generator of the Markov chain $\rho(t)$ be

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -5 & 5 \\ 1 & -1 \end{pmatrix}. \tag{82}$$

Solving equation (6) yields the unique stationary probability distribution

$$\pi = (\pi_1, \pi_2) = \left(\frac{1}{6}, \frac{5}{6}\right). \tag{83}$$

Thus, we have

$$\mathcal{B} = \sum_{i=1}^{2} \pi_i B(i) = \frac{2\ln 2 + 3}{18} > 0. \tag{84}$$

Therefore, according to Theorem 4.11, system (77) is stochastically permanent.

Case 2. Let the generator of the Markov chain $\rho(t)$ be

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}. \tag{85}$$

Solving equation (6) yields the unique stationary probability distribution

$$\pi = (\pi_1, \pi_2) = \left(\frac{3}{4}, \frac{1}{4}\right). \tag{86}$$

Then, we get

$$\mathcal{B}_1 = \sum_{i=1}^{2} \pi_i B_1(i) = \frac{4\ln 2 - 7}{36} < 0, \quad \mathcal{B}_2 = \sum_{i=1}^{2} \pi_i B_2(i) = \frac{8\ln 2 - 7}{72} < 0. \tag{87}$$

So based on Theorem 3.1, system (77) is extinctive.

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