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Strong Consistency Rates of Estimators in Semi-Parametric Errors-In-Variables Model with Missing Responses

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Abstract. In this article, we focus on the semi-parametric error-in-variables model with missing responses: $y_i = \xi_i\beta + g(t_i) + \epsilon_i$, $x_i = \xi_i + \mu_i$, where y_i are the response variables missing at random, (ξ_i, t_i) are design points, ξ_i are the potential variables observed with measurement errors μ_i, the unknown slope parameter β and nonparametric component $g(\cdot)$ need to be estimate. Here we choose three different approaches to estimate β and $g(\cdot)$. Under appropriate conditions, we study the strong consistency rates for the proposed estimators. In general, we concluded that the strong consistency rates for all estimators can achieve $o(n^{-1/4})$.

1. Introduction

Consider the following semi-parametric error-in-variables(EV) model

$$\begin{cases} y_i =\xi_i\beta + g(t_i) + \epsilon_i, \\ x_i =\xi_i + \mu_i, \end{cases} \tag{1.1}$$

where y_i are the response variables, (ξ_i, t_i) are design points, ξ_i are the potential variables observed with measurement errors μ_i, $E\mu_i = 0$, ϵ_i are random errors with $E\epsilon_i = 0$. $\beta \in \mathcal{R}$ is an unknown parameter that needs to be estimated. $g(\cdot)$ is a unknown function defined on close interval $[0, 1]$, $h(\cdot)$ is a known function defined on $[0, 1]$ satisfying

$$\xi_i = h(t_i) + v_i, \tag{1.2}$$

where v_i are also design points.

Model (1.1) and its special forms have gained much attention in recent years. When $\mu_i \equiv 0$, ξ_i are observed exactly, the model (1.1) reduces to the general semi-parametric model, which was first introduced by Engle et al.[6] to study the effect of weather on electricity demand. However, in many applications, there are often covariates measurement errors. For example, it has been well documented in the literature that covariates such as blood pressure, urinary sodium chloride level, and exposure to pollutants are often subject to measurement errors, which may cause difficulties and complications in conducting statistical analysis. So the EV models are somewhat more practical than the ordinary regression model. In addition,

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when y_i are complete observed and $g(\cdot) \equiv 0$, the model (1.1) reduces to the usual linear EV model, which has been studied by Liu and Chen[12], Miao et al.[14], Miao and Liu[13], Fan et al.[7] and so on. For complete data, the model (1.1) itself also has been studied by many authors: Cui and Li[5] presented the asymptotic distributions for estimators of β, $g(\cdot)$ and error variance; Liang et al.[11] derived a consistent estimator of β and its asymptotic distribution; Zhou et al.[18] considered estimation and inference procedures for fixed-effects model (1.1). The EV models are widely applied in economy, biology and forestry. In recent years, the semi-parametric EV models have been widely concerned.

On the other hand, we often encounter incomplete data in the practical application of the models. In particular, some response variables may be missing, by design or by happenstance. For example, the responses y_i may be very expensive to measure and only part of y_i are available. Actually, missing of responses is very common in opinion polls, social-economic investigations, market research surveys, mail enquiries, medical studies and other scientific experiments. Therefore, we focus our attention on the case that missing data occur only in the response variables. When ξ_i can fully be observed, the model (1.1) reduces to the usual reduces to the usual semi-parametric model which has been studied by many scholars in the literature. For examples: Wang et al.[16] considered regression imputation of missing responses in order to make inference on the mean of $\{y_i\}$. Wang and Sun[17] studied estimators of the regression coefficients and the nonparametric function using either imputation, semi-parametric regression surrogate or an inverse marginal probability weighted approach. Since these estimators are based on weighted means of the response variables, they are highly sensitive to outliers. The lack of robustness of weighted means procedures pushed on the search of procedures resistant to outliers as those given in Bianco et al.[2], who introduced robust estimators based on bounded score functions together with algorithms to compute them.

To deal with missing data, one method is to impute a plausible value for each missing datum and then analyze the results as if they are complete. In regression problems, commonly used imputation approaches include linear regression imputation by Healy and Westmacott[10], nonparametric kernel regression imputation by Cheng[4], semi-parametric regression imputation by Wang et al.[16], Wang and Sun[17], among others. We here extend the methods to the estimation of β and $g(\cdot)$ under the semi-parametric EV model (1.1). We obtain three approaches to estimate β and $g(\cdot)$ with missing responses and study the strong consistency rates for the estimators.

In this paper, suppose we obtain a random sample of incomplete data $\{(y_i, \delta_i, x_i, t_i)\}$ from the model (1.1), where $\delta_i = 0$ if y_i is missing, otherwise $\delta_i = 1$. Throughout this paper, we assume that y_i is missing at random. The assumption implies that δ_i and y_i are independent. That is, $P(\delta_i = 1|y_i) = P(\delta_i = 1)$. This assumption is a common assumption for statistical analysis with missing data and is reasonable in many practical situations.

The paper is organized as follows. In Section 2, we list some assumptions. The main results are given in Section 3. A simulation study is presented in section 4. Some preliminary lemmas are stated in Section 5. Proofs of the main results and Lemmas are provided in Sections 6.

2. Assumptions

In this section, we list some assumptions which will be used in the main results. Here $a_n = O(b_n)$ means $|a_n| \le C|b_n|$, $a_n = o(b_n)$ means $a_n/b_n \to 0$, while a.s. is stand for almost sure.

(A0) Let $\{\epsilon_i, 1 \le i \le n\}$, $\{\mu_i,\ 1 \le i \le n\}$ and $\{\delta_i, 1 \le i \le n\}$ be independent random variables satisfying

(i) $E\epsilon_i = 0$, $E\mu_i = 0$, $E\epsilon_i^2 = 1$, $E\mu_i^2 = \Xi_\mu^2 > 0$ is known.

(ii) $\sup_i E|\epsilon_i|^p < \infty$, $\sup_i E|\mu_i|^p < \infty$ for some $p > 4$.

(iii) $\{\epsilon_i, 1 \le i \le n\}$, $\{\mu_i, 1 \le i \le n\}$, $\{\delta_i, 1 \le i \le n\}$ are independent of each other.

(A1) Let $\{v_i, 1 \le i \le n\}$ in (1.2) be a sequence satisfying

(i) $\lim_{n\to\infty} n^{-1} \sum_{i=1}^n v_i^2 = \Sigma_0$, $\lim_{n\to\infty} n^{-1} \sum_{i=1}^n \delta_i v_i^2 = \Sigma_1$ a.s. $(0 < \Sigma_0, \Sigma_1 < \infty)$.

(ii) $\lim_{n\to\infty} \sup_n (\sqrt{n}\log n)^{-1} \cdot \max_{1\le m\le n} |\sum_{i=1}^m v_{j_i}| < \infty$, where $\{j_1, j_2, ..., j_n\}$ is a permutation of $(1, 2, ..., n)$.

(iii) $\max_{1\le i\le n}|v_i| = O(1)$.

(A2) $g(\cdot)$ and $h(\cdot)$ are continuous functions satisfying the first-order Lipschitz condition on the close interval $[0,1]$.

(A3) Let $W^c_{nj}(t_i)$ $(1 \le i, j \le n)$ be weight functions defined on [0, 1] and satisfy

(i) $\max_{1\le j\le n}\sum_{i=1}^n \delta_j W^c_{nj}(t_i) = O(1)$ a.s.

(ii) $\max_{1\le i\le n}\sum_{j=1}^n \delta_j W^c_{nj}(t_i)I(\left|t_i - t_j\right| > a\cdot n^{-1/4}) = o(n^{-1/4})$ a.s. for any $a > 0$.

(iii) $\max_{1\le i,j\le n} W^c_{nj}(t_i) = o(n^{-1/2}\log^{-1} n)$ a.s.

(A4) The probability weight functions $W_{nj}(t_i)$ $(1 \le i, j \le n)$ are defined on $[0,1]$ and satisfy

(i) $\max_{1\le j\le n}\sum_{i=1}^n W_{nj}(t_i) = O(1)$.

(ii) $\max_{1\le i\le n}\sum_{j=1}^n W_{nj}(t_i)I(\left|t_i - t_j\right| > a\cdot n^{-1/4}) = o(n^{-1/4})$, for any $a > 0$.

(iii) $\max_{1\le i,j\le n} W_{nj}(t_i) = o(n^{-1/2}\log^{-1} n)$.

Remark 2.1. *Conditions (A0)-(A4) are standard regularity conditions and used commonly in the literature, see Härdle et al.[9], Gao et al.[8] and Chen[3];*

Remark 2.2. *Under some mild conditions, the following two weight functions satisfy hypothesis (A4):*

$$W^{(1)}_{ni}(t) = \frac{1}{b_n}\int_{s_i-1}^{s_i} M\Big(\frac{t-s}{b_n}\Big)ds,$$

$$W^{(2)}_{ni}(t) = M\Big(\frac{t-t_i}{b_n}\Big)\Big[\sum_{j=1}^n M\Big(\frac{t-t_i}{b_n}\Big)\Big]^{-1},$$

where $s_i = (t_i + t_{i-1})/2,\ i = 1, 2, ..., n-1, s_0 = 0, s_n = 1,\ M(\cdot)$ is the Parzen-Rosenblatt kernel function, which we can see in Parzen[15] or Härdle et al.[9], and b_n are bandwidth parameters. Similarly, under some mild conditions, the following weight functions satisfy hypothesis (A3):

$$W^{(3c)}_{ni}(t) = K\Big(\frac{t-t_i}{h_n}\Big)\Big[\sum_{j=1}^n \delta_j K\Big(\frac{t-t_i}{h_n}\Big)\Big]^{-1},$$

where $K(\cdot)$ is the Parzen-Rosenblatt kernel function, and h_n are bandwidth parameters.

3. Main Results

For model (1.1), we want to seek the estimators of β and $g(\cdot)$. The most natural idea is to delete all the missing data. Therefore, one can get model $\delta_i y_i = \delta_i\xi_i\beta + \delta_i g(t_i) + \delta_i\epsilon_i$. If ξ_i can be observed, we can apply the least squares estimation method to estimate the parameter β. If the parameter β is known, using the complete data $\{(\delta_i y_i, \delta_i x_i, \delta_i t_i), 1 \le i \le n\}$, we can define the estimator of $g(\cdot)$ to be

$$g^*_n(t,\beta) = \sum_{j=1}^n W^c_{nj}(t)(\delta_j y_j - \delta_j x_j\beta),$$

where $W^c_{nj}(t)$ are weight functions satisfying (A3). On the other hand, under this condition of the semi-parametric EV model, Liang et al.[11] improved the least squares estimator(LSE) on the basis of the usual partially linear model, and employ the estimator of parameter β to minimize the following formula:

$$SS(\beta) = \sum_{i=1}^n \delta_i\Big\{\Big[y_i - x_i\beta - g^*_n(t_i,\beta)\Big]^2 - \Xi^2_\mu\beta^2\Big\} = min!$$

Therefore, we can achieve the modified LSE of β as follow:

$$\hat{\beta}_c = \Big[\sum_{i=1}^{n}(\delta_i \tilde{x}_i^{c2} - \delta_i \Xi_\mu^2)\Big]^{-1} \sum_{i=1}^{n} \delta_i \tilde{x}_i^c \tilde{y}_i^c, \tag{3.1}$$

where $\tilde{x}_i^c = x_i - \sum_{j=1}^n \delta_j W_{nj}^c(t_i)x_j$, $\tilde{y}_i^c = y_i - \sum_{j=1}^n \delta_j W_{nj}^c(t_i)y_j$. We substitute (3.1) into $g_n^*(t,\beta)$, then

$$\hat{g}_n^c(t) = \sum_{j=1}^{n} \delta_j W_{nj}^c(t)(y_j - x_j\hat{\beta}_c). \tag{3.2}$$

Apparently, the estimators $\hat{\beta}_c$ and $\hat{g}_n^c(t)$ are formed without taking all sample information into consideration. Hence, in order to make up for the missing data, we imply an imputation method from Wang and Sun[17], and let

$$U_i^{[I]} = \delta_i y_i + (1-\delta_i)[x_i\hat{\beta}_c + \hat{g}_n^c(t_i)]. \tag{3.3}$$

Therefore, Using complete data $\{(U_i^{[I]}, x_i, t_i), 1 \le i \le n\}$, similar to (3.1)-(3.2), one can get another estimators for β and $g(\cdot)$, that is

$$\hat{\beta}_I = \Big[\sum_{i=1}^{n}(\tilde{x}_i^2 - \delta_i \Xi_\mu^2)\Big]^{-1} \sum_{i=1}^{n} \tilde{x}_i \tilde{U}_i^{[I]}, \tag{3.4}$$

$$\hat{g}_n^{[I]}(t) = \sum_{j=1}^{n} W_{nj}(t)(U_j^{[I]} - x_j\hat{\beta}_I). \tag{3.5}$$

where $\tilde{U}_i^{[I]} = U_i^{[I]} - \sum_{j=1}^n W_{nj}(t_i)U_j^{[I]}$, $\tilde{x}_i = x_i - \sum_{j=1}^n W_{nj}(t_i)x_j$, $W_{nj}(t)$ are weight functions satisfying (A4).

Thirdly, Wang and Sun[17] developed a so-call semi-parametric regression surrogate approach. This method uses estimated semi-parametric regression values instead of the corresponding response values to define estimators, whenever the responses are observed or not. Let

$$U_i^{[R]} = x_i\hat{\beta}_c + \hat{g}_n^c(t_i). \tag{3.6}$$

Therefore, Using complete data $\{(U_i^{[R]}, x_i, t_i), 1 \le i \le n\}$, similar to (3.1)-(3.2), one can get the third estimators for β and $g(\cdot)$, that is

$$\hat{\beta}_R = \Big(\sum_{i=1}^{n}\tilde{x}_i^2\Big)^{-1} \sum_{i=1}^{n} \tilde{x}_i \tilde{U}_i^{[R]}, \tag{3.7}$$

$$\hat{g}_n^{[R]}(t) = \sum_{j=1}^{n} W_{nj}(t)(U_j^{[R]} - x_j\hat{\beta}_R), \tag{3.8}$$

where $\tilde{U}_i^{[R]} = U_i^{[R]} - \sum_{j=1}^n W_{nj}(t_i)U_j^{[R]}$, $W_{nj}(t)$ are weight functions satisfying (A4).

Based on the three estimators for β and $g(\cdot)$, we have the following results.

Theorem 3.1. *Suppose that (A0)-(A3) are satisfied. For every $t \in [0,1]$, we have*

(a) $\hat{\beta}_c - \beta = o(n^{-\frac{1}{4}})$ a.s.

(b) $\hat{g}_n^c(t) - g(t) = o(n^{-\frac{1}{4}})$ a.s.

Theorem 3.2. *Suppose that (A0)-(A4) are satisfied. For every $t \in [0,1]$, we have* *(a)* $\hat{\beta}_I - \beta = o(n^{-\frac{1}{4}})$ a.s.

(b) $\hat{g}_n^{[I]}(t) - g(t) = o(n^{-\frac{1}{4}})$ a.s.

Theorem 3.3. *Suppose that (A0)-(A4) are satisfied. For every $t \in [0, 1]$, we have*

(a) $\hat{\beta}_R - \beta = o(n^{-\frac{1}{4}})$ a.s.

(b) $\hat{g}_n^{[R]}(t) - g(t) = o(n^{-\frac{1}{4}})$ a.s.

4. Simulation Study

In this section, we carry out a simulation to study the finite sample performance of the proposed estimators. In particular:

(i) we compare the performance of the estimators among $\hat{\beta}_c$, $\hat{\beta}_I$ and $\hat{\beta}_R$ by their mean squared errors (MSE).

(ii) we give the boxplots for the estimators of β and $g(\cdot)$.

Observations are generated from

$$\begin{cases} y_i = \xi_i\beta + g(t_i) + \epsilon_i, \\ x_i = \xi_i + \mu_i, \quad i = 1, 2, \cdots, n, \end{cases}$$

where $\beta = 1$, $g(t) = \sin(2\pi t)$, $t_i = (i - 0.5)/n$, $\xi_i = t_i^2 + v_i$ with $v_i = \sin(i)/(n^{1/3})$ for $1 \leq i \leq n$. $\{\mu_i,\ 1 \leq i \leq n\}$ and $\{\epsilon_i,\ 1 \leq i \leq n\}$ are i.i.d. $N(0, 0.2^2)$ sequences. For the proposed estimators, the weight functions are taken as

$$W_{nj}^c(t) = \frac{K((t - t_i)/h_n)}{\sum_{j=1}^n K((t - t_j)/h_n)}, \qquad W_{nj}(t) = \frac{M((t - t_i)/b_n)}{\sum_{j=1}^n M((t - t_j)/b_n)}.$$

where $K(\cdot)$ and $M(\cdot)$ are Gaussian kernel function, h_n and b_n are two bandwidth sequences.

4.1. *The MSE for three estimators of β*

In this subsection, we generate the observed data with sample sizes $n = 100,\ 300$, 500 and the missing probability of the response variables is $\delta = 0.1,\ 0.25$, 0.5 from the model above. The MSE of the estimators for β based on $M = 500$ replications are defined as

$$\text{MSE}(\hat{\beta}) = \frac{1}{M}\sum_{l=1}^{M}\left[\hat{\beta}(l) - \beta_0\right]^2.$$

We compute the MSE for each estimators based on $M = 500$ replications and a grid of bandwidths from $0.01 - 0.99$. Choose the optimal bandwidths to minimize the MSE. The minimum MSE and the corresponding optimal bandwidths for the estimators are reported in Tables 1.

From Tables 1, it can be seen that:

(i) The strong consistency of all three estimators for β is significant.

(ii) For every fixed n, the MSE of all estimators increase as the increasing of the missing probability.

(iii) For every fixed missing probability, the MSE of all estimators decrease as the increasing of sample size n.

(iv) Compared to $\hat{\beta}_c$, the estimated value $\hat{\beta}_I$ and $\hat{\beta}_R$ are closer to the true value. Therefore, the compensation for missing data is meaningful.

(v) The simulation results are consistent with the theoretical results.

Table 1: The MSE and corresponding optimal bandwidths for three estimators of β

n	δ	$\hat{\beta}_c$	h_1	$\hat{\beta}_I$	h_1	h_2	$\hat{\beta}_R$	h_1	h_2
100	0.1	$1.46 * 10^{-5}$	0.4400	$1.65 * 10^{-5}$	0.8200	0.3800	$1.38 * 10^{-5}$	0.4700	0.2900
300	0.1	$5.54 * 10^{-6}$	0.3100	$4.37 * 10^{-6}$	0.8400	0.2000	$4.2 * 10^{-5}$	0.2100	0.1300
500	0.1	$3.82 * 10^{-6}$	0.3700	$2.85 * 10^{-6}$	0.4900	0.3200	$2.86 * 10^{-6}$	0.0100	0.3100
100	0.25	$2.67 * 10^{-5}$	0.3500	$1.84 * 10^{-5}$	0.2900	0.0200	$1.82 * 10^{-5}$	0.0600	0.2200
300	0.25	$5.90 * 10^{-6}$	0.3300	$5.33 * 10^{-6}$	0.2100	0.3000	$5.86 * 10^{-6}$	0.0700	0.2700
500	0.25	$3.68 * 10^{-6}$	0.5100	$2.98 * 10^{-6}$	0.4600	0.4200	$2.76 * 10^{-6}$	0.5000	0.0700
100	0.5	$5.76 * 10^{-4}$	0.3400	$3.92 * 10^{-5}$	0.3800	0.0400	$3.69 * 10^{-5}$	0.3400	0.0200
300	0.5	$8.16 * 10^{-6}$	0.6000	$7.62 * 10^{-6}$	0.1500	0.8200	$7.54 * 10^{-6}$	0.6300	0.1700
500	0.5	$3.45 * 10^{-6}$	0.4200	$3.07 * 10^{-6}$	0.8400	0.3000	$3.09 * 10^{-6}$	0.2200	0.2100

4.2. *Boxplots*

In this subsection, we give the boxplots for the estimators of β and $g(\cdot)$. We consider all estimators of β and $g(\cdot)$ under the different missing probability. In Figures 1-3, we give the boxplots for $\hat{\beta}_c$, $\hat{\beta}_I$ and $\hat{\beta}_R$ with $n = 100, 300$ and 500, respectively. In Figures 4-6, we provide the boxplots for $\hat{g}_n^c(0.5)$, $\hat{g}_n^{[I]}(0.5)$ and $\hat{g}_n^{[R]}(0.5)$ with $n = 100, 300$ and 500, respectively.

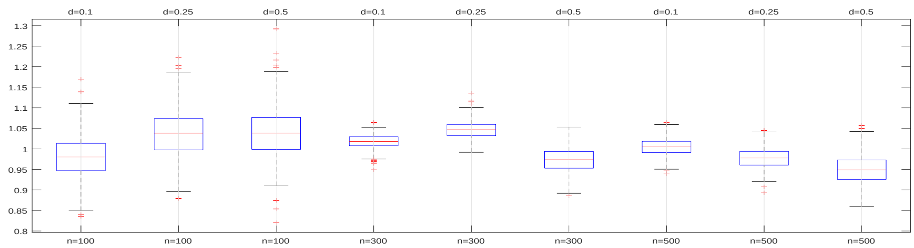

Figure 1: The boxplots for $\hat{\beta}_c$ with M=500, n=100, 300 and 500, respectively.

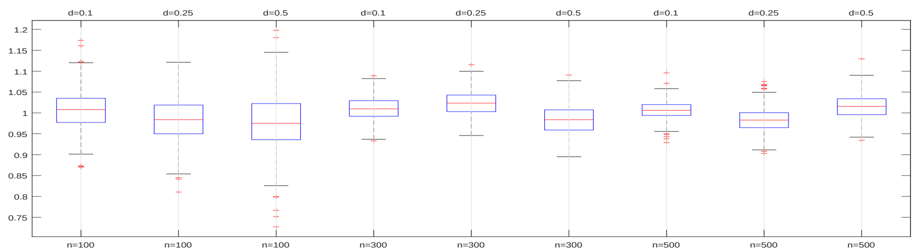

Figure 2: The boxplots for $\hat{\beta}_I$ with M=500, n=100, 300 and 500, respectively.

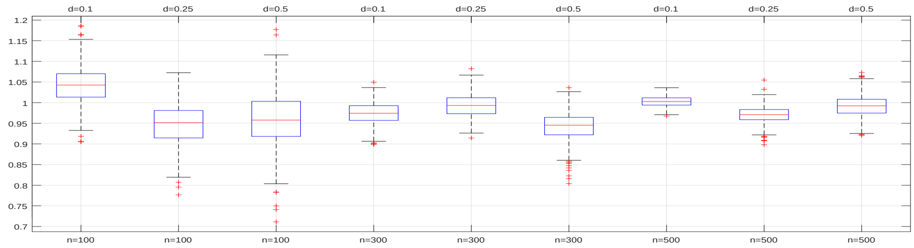

Figure 3: The boxplots for $\hat{\beta}_R$ with M=500, n=100, 300 and 500, respectively.

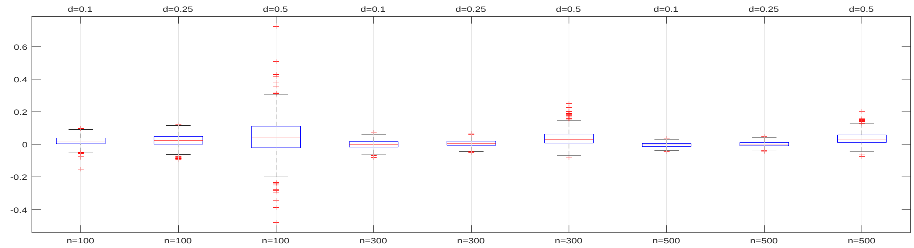

Figure 4: The boxplots for $\hat{g}_n^c(0.5)$ with M=500, n=100, 300 and 500, respectively.

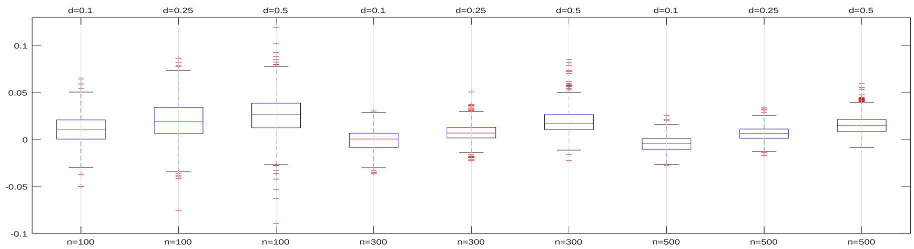

Figure 5: The boxplots for $\hat{g}_n^{[I]}(0.5)$ with M=500, n=100, 300 and 500, respectively.

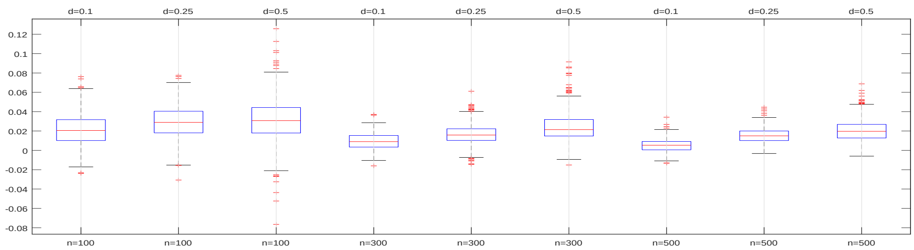

Figure 6: The boxplots for $\hat{g}_n^{[R]}(0.5)$ with M=500, n=100, 300 and 500, respectively.

From Figures 1-6, one can see that:

(i) the estimators $\hat{\beta}_I$ and $\hat{\beta}_R$ have better performance than $\hat{\beta}_c$.

(ii) The estimators $\hat{g}_n^{[I]}(\cdot)$ and $\hat{g}_n^{[R]}(\cdot)$ have better performance than $\hat{g}_n^c(\cdot)$.

(iii) For every estimator, the variances of the estimators decrease as the increasing of sample size n.

(iv) The simulation results are consistent with the theoretical results.

5. Preliminary Lemmas

In the sequel, let $C, C_1, \cdots$ be some finite positive constants, whose values are unimportant and may change. Now, we introduce several lemmas, which will be used in the proof of the main results.

Lemma 5.1 (Baek ang Liang[1], Lemma 3.1). *Let $\alpha > 2, e_1, \cdots, e_n$ be independent random variables with $Ee_i = 0$. Assume that $\{a_{ni}, 1 \le i \le n\}$ is a triangular array of numbers with $\max_{1\le i\le n} |a_{ni}| = O(n^{-1/2})$ and $\sum_{i=1}^n a_{ni}^2 = o(n^{-2/\alpha}\log^{-1} n)$. If $\sup_i E|e_i|^p < \infty$ for some $p > 2\alpha/(\alpha-1)$. Then*

$$\sum_{i=1}^n a_{ni}e_i = o(n^{-1/\alpha}) \text{ a.s.}$$

Lemma 5.2 (Härdle et al[9], Lemma A.3). *Let $V_1, \cdots, V_n$ be independent random variables with $EV_i = 0$, finite variances and $\sup_{1\le j\le n} E|V_j|^r \le C < \infty$ $(r > 2)$. Assume that $\{a_{ki}, k, i = 1, \cdots, n\}$ is a sequence of numbers such that $\sup_{1\le i,k\le n} |a_{ki}| = O(n^{-p_1})$ for some $0 < p_1 < 1$ and $\sum_{j=1}^n a_{ji} = O(n^{p_2})$ for $p_2 \ge \max(0, 2/r - p_1)$. Then*

$$\max_{1\le i\le n}\left|\sum_{k=1}^n a_{ki}V_k\right| = O(n^{-s}\log n) \text{ a.s. } \textit{for } \mathrm{s} = (\mathrm{p}_1 - \mathrm{p}_2)/2.$$

Lemma 5.3. *(a) Let $\tilde{A}_i = A(t_i) - \sum_{j=1}^n W_{nj}(t_i)A(t_j)$, where $A(\cdot) = g(\cdot)$ or $h(\cdot)$. Let $\tilde{A}_i^c = A(t_i) - \sum_{j=1}^n \delta_j W_{nj}^c(t_i)A(t_j)$, where $A(\cdot) = g(\cdot)$ or $h(\cdot)$. Then, (A0)-(A4) imply that $\max_{1\le i\le n} |\tilde{A}_i| = o(n^{-1/4})$ and $\max_{1\le i\le n} |\tilde{A}_i^c| = o(n^{-1/4})$ a.s.*

(b) (A0)-(A4) imply that $n^{-1}\sum_{i=1}^n \tilde{\xi}_i^2 \to \Sigma_0$, $\sum_{i=1}^n |\tilde{\xi}_i| \le C_1 n$, $n^{-1}\sum_{i=1}^n \delta_i(\tilde{\xi}_i^c)^2 \to \Sigma_1$ a.s. and $\sum_{i=1}^n |\delta_i\tilde{\xi}_i^c| \le C_2 n$ a.s.

(c) (A0)-(A4) imply that $\max_{1\le i\le n} |\tilde{\xi}_i| = O(1)$ and $\max_{1\le i\le n} |\tilde{\xi}_i^c| = O(1)$ a.s.

Lemma 5.4. *Suppose that (A0)-(A4) are satisfied. Then one can deduce that*

$$\max_{1\le i\le n}\left|\hat{g}_n^c(t_i) - g(t_i)\right| = o(n^{-\frac{1}{4}}) \text{ a.s.}$$

The proof Lemma 5.3 will be list in section 6. The proof Lemma 5.4 is analogous to the proof of Theorem 3.1*(b)*.

6. Proof of Main Results and Lemmas

Firstly, we introduce some notations, which will be used in the proofs below.

$$\tilde{\xi}_i^c = \xi_i - \sum_{j=1}^n \delta_j W_{nj}^c(t_i)\xi_j,\ \ \tilde{\mu}_i^c = \mu_i - \sum_{j=1}^n \delta_j W_{nj}^c(t_i)\mu_j,\ \ \tilde{g}_i^c = g(t_i) - \sum_{j=1}^n \delta_j W_{nj}^c(t_i)g(t_j),$$

$$\tilde{\epsilon}_i^c = \epsilon_i - \sum_{j=1}^n \delta_j W_{nj}^c(t_i)\epsilon_j,\ \ \tilde{\xi}_i = \xi_i - \sum_{j=1}^n W_{nj}(t_i)\xi_j,\ \ \tilde{\mu}_i = \mu_i - \sum_{j=1}^n W_{nj}(t_i)\mu_j,$$

$$\tilde{g}_i = g(t_i) - \sum_{j=1}^n W_{nj}(t_i)g(t_j),\ \ \tilde{\epsilon}_i = \epsilon_i - \sum_{j=1}^n W_{nj}(t_i)\epsilon_j,\ \ \eta_i = \epsilon_i - \mu_i\beta,\ \ B_{1n}^2 = \sum_{i=1}^n \delta_i\tilde{\xi}_i^2,$$

$$S_n^2 = \sum_{i=1}^n \tilde{\xi}_i^2,\ \ S_{1n}^2 = \sum_{i=1}^n(\delta_i\tilde{x}_i^2 - \delta_i\Xi_\mu^2),\ \ S_{2n}^2 = \sum_{i=1}^n(\tilde{x}_i^2 - \Xi_\mu^2),\ \ S_{3n}^2 = \sum_{i=1}^n \tilde{x}_i^2.$$

Proof of Theorem 3.1(a). From (3.1), one can write that

$$\begin{aligned}
\hat{\beta}_c - \beta &= S_{1n}^{-2}\Big[\sum_{i=1}^n \delta_i(\tilde{\xi}_i^c + \tilde{\mu}_i^c)(\tilde{y}_i^c - \tilde{\xi}_i^c\beta - \tilde{\mu}_i^c\beta) + \sum_{i=1}^n \delta_i \Xi_\mu^2 \beta\Big] \\
&= S_{1n}^{-2}\Big\{\sum_{i=1}^n \Big[\delta_i(\tilde{\xi}_i^c + \tilde{\mu}_i^c)(\tilde{\epsilon}_i^c - \tilde{\mu}_i^c\beta) + \delta_i\Xi_\mu^2\beta\Big] + \sum_{i=1}^n \delta_i\tilde{\xi}_i^c\tilde{g}_i^c + \sum_{i=1}^n \delta_i\tilde{\mu}_i^c\tilde{g}_i^c\Big\} \\
&= S_{1n}^{-2}\Big\{\sum_{i=1}^n \delta_i\tilde{\xi}_i^c(\epsilon_i - \mu_i\beta) + \sum_{i=1}^n \delta_i\mu_i\epsilon_i - \sum_{i=1}^n \delta_i(\mu^2 - \Xi_\mu^2)\beta + \sum_{i=1}^n \delta_i\tilde{\xi}_i^c\tilde{g}_i^c \\
&\quad + \sum_{i=1}^n \delta_i\tilde{\mu}_i^c\tilde{g}_i^c - \sum_{i=1}^n\sum_{j=1}^n \delta_i\delta_j W_{nj}^c(t_i)\tilde{\xi}_i^c\epsilon_j - \sum_{i=1}^n\sum_{j=1}^n \delta_i\delta_j W_{nj}^c(t_i)\epsilon_i\mu_j \\
&\quad - \sum_{i=1}^n\sum_{j=1}^n \delta_i\delta_j W_{nj}^c(t_i)\mu_i\epsilon_j + \sum_{i=1}^n\sum_{j=1}^n \delta_i\delta_j W_{nj}^c(t_i)\tilde{\xi}_i^c\mu_j\beta + 2\sum_{i=1}^n\sum_{j=1}^n \delta_i\delta_j W_{nj}^c(t_i)\mu_i\mu_j\beta \\
&\quad + \sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n \delta_i\delta_j\delta_k W_{nj}^c(t_i)W_{nk}^c(t_i)\mu_j\epsilon_k - \sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n \delta_i\delta_j\delta_k W_{nj}^c(t_i)W_{nk}^c(t_i)\mu_j\mu_k\beta\Big\} \\
&:= S_{1n}^{-2}\sum_{k=1}^{12} A_{kn}. \qquad (6.1)
\end{aligned}$$

Thus, to prove $\hat{\beta}_c - \beta = o(n^{-1/4})$ a.s., we only need to verify that $S_{1n}^{-2} \le Cn^{-1}$ a.s. and $n^{-1}A_{kn} = o(n^{-1/4})$ a.s. for $k = 1, 2, \cdots, 12$.

Step 1. We prove $S_{1n}^{-2} \le Cn^{-1}$ a.s. Note that

$$\begin{aligned}
S_{1n}^2 &= \sum_{i=1}^n \Big[\delta_i(\tilde{\xi}_i^c + \tilde{\mu}_i^c)^2 - \delta_i\Xi_\mu^2\Big] \\
&= \sum_{i=1}^n \delta_i(\tilde{\xi}_i^c)^2 + \sum_{i=1}^n \delta_i(\mu_i^2 - \Xi_\mu^2) + \sum_{i=1}^n \delta_i\Big[\sum_{j=1}^n \delta_j W_{nj}^c(t_i)\mu_j\Big]^2 \\
&\quad + 2\sum_{i=1}^n \delta_i\tilde{\xi}_i^c\mu_i - 2\sum_{i=1}^n \delta_i\tilde{\xi}_i^c\sum_{j=1}^n \delta_j W_{nj}^c(t_i)\mu_j - 2\sum_{i=1}^n \delta_i\mu_i\sum_{j=1}^n \delta_j W_{nj}^c(t_i)\mu_j \\
&:= B_{1n} + B_{2n} + B_{3n} + B_{4n} + B_{5n} + B_{6n}.
\end{aligned}$$

By Lemma 5.3*(a)*, we have $n^{-1}B_{1n} \to \Sigma_1$ a.s. Hence, it suffices to verify that $B_{kn} = o(B_{1n}) = o(n)$ a.s. for $k = 2, 3, \cdots, 6$. Applying (A0), taking $r = p/2 > 2$, $p_1 = 1/2$, $p_2 = 1/2$ in Lemma 5.2, we can verity that

$$\sum_{i=1}^n \Big(\zeta_i - E\zeta_i\Big) = n^{\frac{1}{2}} \cdot \sum_{i=1}^n n^{-\frac{1}{2}}\Big(\zeta_i - E\zeta_i\Big) = O(n^{\frac{1}{2}}\log n) \text{ a.s.} \qquad (6.2)$$

where ζ_i are independent random variables satisfying $E\zeta_i = 0$ and $\sup_{1\le i\le n} E|\zeta_i|^{p/2} < \infty$. Therefore, we obtain $B_{2n} = O(n^{1/2}\log n) = o(n)$ a.s. from (A0) and (6.2). On the other hand, taking $\alpha = 4$, $p > 4$ in Lemma 5.1, we have

$$\max_{1\le i\le n}\Big|\sum_{j=1}^n \delta_j W_{nj}^c(t_i)\zeta_j\Big| = o(n^{-\frac{1}{4}}) \ a.s., \quad \max_{1\le i\le n}\Big|\sum_{j=1}^n W_{nj}(t_i)\zeta_j\Big| = o(n^{-\frac{1}{4}}) \text{ a.s.} \qquad (6.3)$$

where ζ_i are independent random variables satisfying $E\zeta_i = 0$ and $\sup_{1\le i\le n} E|\zeta_i|^p < \infty$. By (A0) and Lemma 5.3, taking $r = 4$, $p_1 = 1/4$, $p_2 = 3/4$ in Lemma 5.2, one can also deduce that

$$\Big|B_{4n}\Big| = 2n^{\frac{1}{4}} \cdot \Big|\sum_{i=1}^n n^{-\frac{1}{4}}\delta_i\tilde{\xi}_i^c\mu_i\Big| = O(n^{\frac{1}{2}}\log n) = o(n) \text{ a.s.} \qquad (6.4)$$

Note that, from Lemma 5.3*(a)*, (6.2) and (6.3), we have

$$\left|B_{3n}\right| \leq \sum_{i=1}^{n}|\delta_i|\cdot\max_{1\leq i\leq n}\Big|\sum_{j=1}^{n}\delta_j W_{nj}^c(t_i)\mu_j\Big|^2 = o(n^{\frac{1}{2}}) \text{ a.s.} \tag{6.5}$$

$$\left|B_{5n}\right| \leq 2\sum_{i=1}^{n}|\delta_i\tilde{\xi}_i^c|\cdot\max_{1\leq i\leq n}\Big|\sum_{j=1}^{n}\delta_j W_{nj}^c(t_i)\mu_j\Big| = o(n^{\frac{3}{4}}) \text{ a.s.} \tag{6.6}$$

$$\left|B_{6n}\right| \leq 2\Big[\sum_{i=1}^{n}(|\delta_i\mu_i| - E|\delta_i\mu_i|) + \sum_{i=1}^{n}E|\delta_i\mu_i|\Big]\cdot\max_{1\leq i\leq n}\Big|\sum_{j=1}^{n}\delta_j W_{nj}^c(t_i)\mu_j\Big| = o(n^{\frac{3}{4}}) \text{ a.s.} \tag{6.7}$$

Therefore, for (6.2)-(6.7), one can deduce that $S_{1n}^2 = B_{1n} + o(n) = B_{1n} + o(B_{1n})$ a.s. , which yields that

$$\lim_{n\to\infty}\frac{B_{1n}}{S_{1n}^2} = \lim_{n\to\infty}\frac{B_{1n}}{B_{1n}+o(B_{1n})} = 1 \text{ a.s.}$$

Therefore, by the Lemma 5.3*(b)*, we can get that $S_{1n}^{-2} \leq Cn^{-1}$ a.s.

Step 2. We verify that $n^{-1}A_{kn} = o(n^{-1/4})$ a.s. for $k = 1, 2, \cdots, 12$. From (A0), we find out $\{\eta_i = \epsilon_i - \mu_i\beta, 1 \leq i \leq n\}$ are sequences of independent random variables with $E\eta_i = 0$, $\sup_i E\left|\eta_i\right|^p \leq C\sup_i E\left|e_i\right|^p + C\sup_i E\left|\mu_i\right|^p < \infty$, for some $p > 4$. Similar to (6.4), we deduce that

$$n^{-1}A_{1n} = n^{-1}\sum_{i=1}^{n}\delta_i\tilde{\xi}_i^c\eta_i = O(n^{-\frac{1}{2}}\log n) \text{ a.s.}$$

Meanwhile, from (A0)-(A3), Lemma 5.3, (6.2) and (6.3), one can achieve that

$$\begin{aligned}
S_n^{-2}A_{2n} &\leq \frac{C}{n}\Big|\sum_{i=1}^{n}\delta_i\mu_i\epsilon_i\Big| = O(n^{-\frac{1}{2}}\log n) \text{ a.s.}\\
S_n^{-2}A_{3n} &\leq \frac{C}{n}\Big|\sum_{i=1}^{n}\delta_i\big(\mu_i^2 - \Xi_\mu^2\big)\beta\Big| = O(n^{-\frac{1}{2}}\log n) \text{ a.s.}\\
S_n^{-2}A_{4n} &\leq \frac{C}{n}\Big|\sum_{i=1}^{n}\delta_i\tilde{\xi}_i^c\tilde{g}_i^c\Big| \leq \frac{C}{n}\Big[\max_{1\leq i\leq n}\sum_{i=1}^{n}\left|\delta_i\tilde{\xi}_i^c\right|\max_{1\leq i\leq n}\left|\tilde{g}_i^c\right|\Big] = o(n^{-\frac{1}{4}}) \text{ a.s.}\\
S_n^{-2}A_{5n} &\leq \frac{C}{n}\Big|\sum_{i=1}^{n}\delta_i\tilde{\mu}_i^c\tilde{g}_i^c\Big| \leq \frac{C}{n}\Big[\sum_{i=1}^{n}\left|\mu_i\right|\cdot\left|\delta_i\tilde{g}_i^c\right| + \sum_{i=1}^{n}\left|\delta_i\tilde{g}_i^c\right|\cdot\Big|\sum_{j=1}^{n}\delta_j W_{nj}^c(t_i)\mu_j\Big|\Big]\\
&\leq \frac{C}{n}\cdot\max_{1\leq i\leq n}\Big(\sum_{i=1}^{n}\big(\left|\mu_i\right| - E\left|\mu_i\right|\big) + \sum_{i=1}^{n}E\left|\mu_i\right|\Big)\cdot\max_{1\leq i\leq n}\left|\delta_i\tilde{g}_i^c\right| + o(n^{-\frac{1}{2}}) = o(n^{-\frac{1}{4}}) \text{ a.s.}
\end{aligned}$$

The proof of $n^{-1}A_{kn} = o(n^{-1/4})$ a.s. for $k = 6, \cdots, 12$ is analogous. Thus, the proof of Theorem 3.1*(a)* is completed. ■

Proof of Theorem 3.1(b). From (3.2), for every $t \in [0, 1]$, one can write that

$$\begin{aligned}
\hat{g}_n^c(t) - g(t) &= \sum_{j=1}^{n}W_{nj}^c(t)\delta_j(y_j - x_j\hat{\beta}_c) - g(t)\\
&= \sum_{j=1}^{n}W_{nj}^c(t)\delta_j\Big[\xi_j\beta + g(t_j) + \epsilon_j - (\xi_j + \mu_j)\hat{\beta}_c\Big] - g(t)\\
&= \sum_{j=1}^{n}W_{nj}^c(t)\delta_j\xi_j(\beta - \hat{\beta}_c) + \sum_{j=1}^{n}W_{nj}^c(t_i)\delta_j[g(t_j) - g(t)]
\end{aligned}$$

$$+\sum_{j=1}^{n} W_{nj}^{c}(t_i)\delta_j\epsilon_j + \sum_{j=1}^{n} W_{nj}^{c}(t_i)\delta_j\mu_j\beta + \sum_{j=1}^{n} W_{nj}^{c}(t_i)\delta_j\mu_j(\hat{\beta}_c - \beta)$$
$$:= \quad F_{1n}(t) + F_{2n}(t) + F_{3n}(t) + F_{4n}(t) + F_{5n}(t).$$

Therefore, we only need to prove that $F_{kn}(t) = o(n^{-1/4})$ a.s. for $k = 1, 2, \cdots, 5$. From (A0)-(A3), Theorem 3.1*(a)*, Lemma 5.3, (6.2) and (6.3), for every $t \in [0,1]$ and any $a > 0$, one can get

$$\begin{aligned}
F_{1n}(t) &\le \left|\beta - \hat{\beta}_c\right| \cdot \max_{1\le j\le n} |h(t_j) + v_j| \cdot \sum_{j=1}^{n} \delta_j W_{nj}^{c}(t) = o(n^{-\frac{1}{4}}) \text{ a.s.} \\
F_{2n}(t) &\le \sum_{j=1}^{n} \delta_j W_{nj}^{c}(t) \cdot \left[g(t_j) - g(t)\right] \cdot I(|t_j - t| > a \cdot n^{-\frac{1}{4}}) \\
&\quad + \sum_{j=1}^{n} \delta_j W_{nj}^{c}(t) \cdot \left[g(t_j) - g(t)\right] \cdot I(|t_j - t| < a \cdot n^{-\frac{1}{4}}) \\
&\le C \cdot a \cdot n^{-\frac{1}{4}} = o(n^{-\frac{1}{4}}) \text{ a.s.} \\
F_{3n}(t) &\le \Big|\sum_{j=1}^{n} W_{nj}^{c}(t)\delta_j\epsilon_j\Big| = o(n^{-\frac{1}{4}}) \text{ a.s.} \\
F_{4n}(t) &\le \Big|\sum_{j=1}^{n} W_{nj}^{c}(t)\delta_j\mu_j\beta\Big| = o(n^{-\frac{1}{4}}) \text{ a.s.} \\
F_{5n}(t) &\le \left|\beta - \hat{\beta}_c\right| \cdot \Big|\sum_{j=1}^{n} W_{nj}^{c}(t)\delta_j\mu_j\Big| = o(n^{-\frac{1}{2}}) \text{ a.s.}
\end{aligned}$$

Thus, the proof of Theorem 3.1*(b)* is completed. ∎

Proof of Theorem 3.2(a). From (3.3)-(3.4), write that

$$\begin{aligned}
\hat{\beta}_I - \beta &= S_{2n}^{-2}\Big\{\sum_{i=1}^{n}(\tilde{\xi}_i + \tilde{\mu}_i)\Big[\delta_i(y_i - \xi_i\beta - \mu_i\beta) + (1-\delta_i)(\xi_i + \mu_i)(\hat{\beta}_c - \beta) \\
&\quad + (1-\delta_i)\hat{g}_n^{c}(t_i) - \sum_{j=1}^{n} W_{nj}(t_i)\Big(\delta_j(y_j - \xi_j\beta - \mu_j\beta) + (1-\delta_j)(\xi_j + \mu_j)(\hat{\beta}_c - \beta) \\
&\quad + (1-\delta_j)\hat{g}_n^{c}(t_j)\Big)\Big] + \sum_{i=1}^{n}\delta_i\Xi_\mu^2\beta\Big\} \\
&= S_{2n}^{-2}\Big\{\sum_{i=1}^{n}(\tilde{\xi}_i + \tilde{\mu}_i)\Big[\delta_i(\epsilon_i - \mu_i\beta) + \delta_i\Big(g(t_i) - \hat{g}_n^{c}(t_i)\Big) + (1-\delta_i)(\xi_i + \mu_i)(\hat{\beta}_c - \beta) \\
&\quad + \hat{g}_n^{c}(t_i) - \sum_{j=1}^{n} W_{nj}(t_i)\Big(\delta_j(\epsilon_j - \mu_j\beta) + \delta_j\Big(g(t_j) - \hat{g}_n^{c}(t_j)\Big) \\
&\quad + (1-\delta_j)(\xi_j + \mu_j)(\hat{\beta}_c - \beta) + \hat{g}_n^{c}(t_j)\Big)\Big] + \sum_{i=1}^{n}\delta_i\Xi_\mu^2\beta\Big\} \\
&= S_{2n}^{-2}\Big\{\sum_{i=1}^{n}\delta_i\tilde{\xi}_i(\epsilon_i - \mu_i\beta) + \sum_{i=1}^{n}\delta_i\tilde{\xi}_i(g(t_i) - \hat{g}_n^{c}(t_i)) + \sum_{i=1}^{n}\tilde{\xi}_i(\xi_i + \mu_i)(1-\delta_i)(\hat{\beta}_c - \beta) \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_j W_{nj}(t_i)\tilde{\xi}_i\epsilon_j + \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_j W_{nj}(t_i)\tilde{\xi}_i\mu_j\beta - \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_j W_{nj}(t_i)\tilde{\xi}_i(g(t_j) - \hat{g}_n^{c}(t_j))
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{i=1}^{n}\sum_{j=1}^{n} W_{nj}(t_i)\tilde{\xi}_i(1-\delta_j)(\xi_j+\mu_j)(\hat{\beta}_c-\beta)+\sum_{i=1}^{n}\tilde{\xi}_i\tilde{g}_i^c+\sum_{i=1}^{n}\delta_i\mu_i\epsilon_i \\
& -\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i W_{nj}(t_i)\epsilon_i\mu_j-\sum_{i=1}^{n}\delta_i(\mu_i^2-\Xi_\mu^2)\beta+\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i W_{nj}(t_i)\mu_i\mu_j\beta \\
& +\sum_{i=1}^{n}\delta_i\tilde{\mu}_i(g(t_i)-\hat{g}_n^c(t_i))+\sum_{i=1}^{n}\tilde{\mu}_i(1-\delta_i)(\xi_i+\mu_i)(\hat{\beta}_c-\beta)-\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_j W_{nj}(t_i)\mu_i\epsilon_j \\
& +\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\delta_j W_{nj}(t_i)W_{nk}(t_i)\mu_k\epsilon_j+\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_j W_{nj}(t_i)\mu_i\mu_j\beta \\
& -\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\delta_j W_{nj}(t_i)W_{nk}(t_i)\mu_j\mu_k\beta-\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_j W_{nj}(t_i)\tilde{\mu}_i(g(t_j)-\hat{g}_n^c(t_j)) \\
& -\sum_{i=1}^{n}\sum_{j=1}^{n} W_{nj}(t_i)(1-\delta_j)\tilde{\mu}_i(\xi_j+\mu_j)(\hat{\beta}_c-\beta)+\sum_{i=1}^{n}\tilde{\mu}_i\tilde{g}_i^c \\
:= \ & S_{2n}^{-2}\sum_{k=1}^{21} D_{kn}.
\end{aligned}
$$

Using a similar approach as step 1 in the proof of Theorem 3.1*(a)*, one can get $S_{2n}^{-2}\le Cn^{-1}$ a.s. Therefore, we only need to verify that $n^{-1}D_{kn}=o(n^{-1/4})$ a.s. for $k=1,2,\cdots,21$. From (A0)-(A4), Lemmas 5.2-5.4, Theorem 3.1*(a)*, (6.2)-(6.4), we have

$$
\begin{aligned}
n^{-1}D_{1n} &= n^{-1}\sum_{i=1}^{n}\delta_i\tilde{\xi}_i(\epsilon_i-\mu_i\beta)=n^{-1}\cdot O(n^{\frac{1}{2}}\log n)=o(n^{-\frac{1}{4}}) \text{ a.s.} \\
n^{-1}D_{2n} &\le n^{-1}\Big[\sum_{i=1}^{n}\big|\delta_i\tilde{\xi}_i\big|\cdot\max_{1\le i\le n}\big|g(t_i)-\hat{g}_n^c(t_i)\big|\Big]=o(n^{-\frac{1}{4}}) \text{ a.s.} \\
n^{-1}D_{3n} &= n^{-1}\sum_{i=1}^{n}\tilde{\xi}_i^2(1-\delta_i)(\hat{\beta}_c-\beta)+n^{-1}\sum_{i=1}^{n}\tilde{\xi}_i(1-\delta_i)\sum_{j=1}^{n}W_{nj}(t_i)\xi_j(\hat{\beta}_c-\beta) \\
&\quad +n^{-1}\sum_{i=1}^{n}\tilde{\xi}_i\mu_i(1-\delta_i)(\hat{\beta}_c-\beta)=o(n^{-\frac{1}{4}}) \text{ a.s.} \\
n^{-1}D_{4n} &\le n^{-1}\sum_{i=1}^{n}\big|\tilde{\xi}_i\big|\cdot\max_{1\le i\le n}\Big|\sum_{j=1}^{n}\delta_j W_{nj}(t_i)\epsilon_j\Big|=o(n^{-\frac{1}{4}}) \text{ a.s.} \\
n^{-1}D_{7n} &= n^{-1}\sum_{i=1}^{n}|\tilde{\xi}_i|\cdot\max_{1\le i\le n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\Big|\cdot\max_{1\le j\le n}|\xi_j|\cdot|\hat{\beta}_c-\beta| \\
&\quad +n^{-1}\sum_{i=1}^{n}|\tilde{\xi}_i|\cdot\max_{1\le i\le n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)(1-\delta_j)\mu_j\Big|\cdot\big|\hat{\beta}_c-\beta\big|=o(n^{-\frac{1}{4}}) \text{ a.s.} \\
n^{-1}D_{13n} &\le n^{-1}\Big|\sum_{i=1}^{n}\delta_i\tilde{\mu}_i\big(g(t_i)-\hat{g}_n^c(t_i)\big)\Big| \\
&\le n^{-1}\Big[\sum_{i=1}^{n}\big|\mu_i\big|\cdot\big|g(t_i)-\hat{g}_n^c(t_i)\big|+\sum_{i=1}^{n}\big|g(t_i)-\hat{g}_n^c(t_i)\big|\cdot\Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big|\Big]
\end{aligned}
$$

$$\begin{aligned}
&\leq n^{-1}\Big[\Big(\sum_{i=1}^{n}\big(|\mu_i| - E|\mu_i|\big) + \sum_{i=1}^{n} E|\mu_i|\Big)\max_{1\leq i\leq n}\big|g(t_i) - \hat{g}_n^c(t_i)\big| \\
&\quad + n\cdot\max_{1\leq i\leq n}\big|g(t_i) - \hat{g}_n^c(t_i)\big|\max_{1\leq i\leq n}\Big|\sum_{j=1}^{n} W_{nj}(t_i)\mu_j\Big|\Big] = o(n^{-\frac{1}{4}}) \text{ a.s.}
\end{aligned}$$

$$\begin{aligned}
n^{-1}D_{14n} &= n^{-1}\sum_{i=1}^{n}\tilde{\mu}_i(1-\delta_i)(\xi_i+\mu_i)(\hat{\beta}_c-\beta) \leq n^{-1}\Big|\sum_{i=1}^{n}\xi_i\mu_i\Big|\cdot\big|\hat{\beta}_c-\beta\big| \\
&\quad + n^{-1}\Big|\sum_{i=1}^{n}\mu_i^2\Big|\cdot\big|\hat{\beta}_c-\beta\big| + n^{-1}\cdot\sum_{i=1}^{n}|\xi_i|\cdot\max_{1\leq i\leq n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big|\cdot\big|\hat{\beta}_c-\beta\big| \\
&\quad + n^{-1}\cdot\sum_{i=1}^{n}|\mu_i|\cdot\max_{1\leq i\leq n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big|\cdot\big|\hat{\beta}_c-\beta\big| = o(n^{-\frac{1}{4}}) \text{ a.s.}
\end{aligned}$$

$$\begin{aligned}
n^{-1}D_{21n} &\leq n^{-1}\sum_{i=1}^{n}|\tilde{\mu}_i\tilde{g}_i^c| \leq n^{-1}\Big[\sum_{i=1}^{n}|\mu_i|\cdot|\tilde{g}_i^c| + \sum_{i=1}^{n}|\tilde{g}_i^c|\cdot\max_{1\leq i\leq n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big|\Big] \\
&\leq n^{-1}\Big[\max_{1\leq i\leq n}\Big(\sum_{i=1}^{n}\big(|\mu_i| - E|\mu_i|\big) + \sum_{i=1}^{n}E|\mu_i|\Big)\cdot\max_{1\leq i\leq n}|\tilde{g}_i^c| \\
&\quad + n\cdot\max_{1\leq i\leq n}|\tilde{g}_i^c|\cdot\max_{1\leq i\leq n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big|\Big] = o(n^{-\frac{1}{4}}) \text{ a.s.}
\end{aligned}$$

In the same way, from (A0)-(A4), Lemmas 5.2-5.4, (6.2) and (6.3), one can similarly deduce that $n^{-1}D_{kn} = o(n^{-1/4})$ a.s. for $k = 5, 6, 8, 9, 10, 11, 12, 15, \cdots, 20$. Thus, the proof of Theorem 3.2*(a)* is completed. ■

Proof of Theorem 3.2(b). From (3.4), write that

$$\begin{aligned}
\hat{g}_n^{[I]}(t) - g(t) &= \sum_{j=1}^{n}W_{nj}(t)\Big\{\delta_j y_j + (1-\delta_j)[(\xi_j+\mu_j)\hat{\beta}_c + \hat{g}_n^c(t_j)] - (\xi_i+\mu_j)\hat{\beta}_I\Big\} - g(t) \\
&= \sum_{j=1}^{n}W_{nj}(t)\Big\{\delta_j\big[\xi_j\beta + g(t_j) + \epsilon_j\big] + (1-\delta_j)\big[(\xi_j+\mu_j)\hat{\beta}_c + \hat{g}_n^c(t_j)\big] \\
&\quad -(\xi_j+\mu_j)\hat{\beta}_I\Big\} - g(t) \\
&= \sum_{j=1}^{n}W_{nj}(t)\Big\{\delta_j\xi_j\beta + \delta_j g(t_j) + \delta_j\epsilon_j + \xi_j\hat{\beta}_c + \mu_j\hat{\beta}_c - \delta_j\xi_j\hat{\beta}_c - \delta_j\mu_j\hat{\beta}_c \\
&\quad + \hat{g}_n^c(t_j) - \delta_j\hat{g}_n^c(t_j) - \xi_j\hat{\beta}_I - \mu_j\hat{\beta}_I\Big\} - g(t) \\
&= \sum_{j=1}^{n}W_{nj}(t)\delta_j\xi_j(\beta-\hat{\beta}_c) + \sum_{j=1}^{n}W_{nj}(t)\delta_j\big[g(t_j) - \hat{g}_n^c(t_j)\big] + \sum_{j=1}^{n}W_{nj}(t)\delta_j\epsilon_j \\
&\quad + \sum_{j=1}^{n}W_{nj}(t)\xi_j(\hat{\beta}_c-\beta) + \sum_{j=1}^{n}W_{nj}(t)\mu_j(\hat{\beta}_c-\beta) - \sum_{j=1}^{n}W_{nj}(t)\delta_j\mu_j\hat{\beta}_c \\
&\quad + \sum_{j=1}^{n}W_{nj}(t)\big[\hat{g}_n^c(t_j) - g(t_j)\big] + \sum_{j=1}^{n}W_{nj}(t)\big[g(t_j) - g(t)\big] \\
&\quad + \sum_{j=1}^{n}W_{nj}(t)\xi_j(\beta-\hat{\beta}_I) + \sum_{j=1}^{n}W_{nj}(t)\mu_j(\beta-\hat{\beta}_I)
\end{aligned}$$

$$:= \sum_{k=1}^{10} G_{kn}.$$

Therefore, we only need to prove that $G_{kn}(t) = o(n^{-1/4})$ a.s. for $k = 1, 2, \cdots, 10$. From (A0)-(A4), Lemma 5.3-5.4, (6.2), (6.3), one can get

$$\begin{aligned}
G_{1n}(t) &\leq \left|\beta - \hat{\beta}_c\right| \cdot \max_{1\leq j\leq n} \left|\delta_j \xi_j\right| \cdot \sum_{j=1}^{n} W_{nj}(t) \\
&\leq \left|\beta - \hat{\beta}_c\right| \cdot \max_{1\leq j\leq n} \left|\xi_j\right| \cdot \sum_{j=1}^{n} W_{nj}(t) = o(n^{-\frac{1}{4}}) \text{ a.s.} \\
G_{2n}(t) &\leq \max_{1\leq j\leq n} \left|g(t_j) - \hat{g}_n^c(t_j)\right| \cdot \left|\sum_{j=1}^{n} W_{nj}(t)\delta_j\right| = o(n^{-\frac{1}{4}}) \text{ a.s.}
\end{aligned}$$

Meanwhile, the proof of $G_{kn}(t) = o(n^{-1/4})$ a.s. for $k = 3, \cdots, 10$ is analogous. Thus, the proof of Theorem 3.2*(b)* is completed. ■

Proof of Theorem 3.3(a). From (3.6)-(3.7), write that

$$\begin{aligned}
\hat{\beta}_R - \beta &= S_{3n}^{-2} \sum_{i=1}^{n} \left(\tilde{x}_i \tilde{U}_i^{[R]} - \tilde{x}_i^2 \beta\right) \\
&= S_{3n}^{-2} \sum_{i=1}^{n} \left\{\tilde{x}_i \Big[\tilde{x}_i \hat{\beta}_c + \hat{g}_n^c(t_i) - \sum_{j=1}^{n} W_{nj}(t_i)\hat{g}_n^c(t_j)\Big] - \tilde{x}_i^2 \beta\right\} \\
&= S_{3n}^{-2} \Big\{ \sum_{i=1}^{n} \tilde{x}_i^2 (\hat{\beta}_c - \beta) + \sum_{i=1}^{n} (\tilde{\xi}_i + \tilde{\mu}_i)[\hat{g}_n^c(t_i) - g(t_i)] \\
&\quad - \sum_{i=1}^{n} (\tilde{\xi}_i + \tilde{\mu}_i) \sum_{j=1}^{n} W_{nj}(t_i)[\hat{g}_n^c(t_j) - g(t_j)] + \sum_{i=1}^{n} (\tilde{\xi}_i + \tilde{\mu}_i) \sum_{j=1}^{n} W_{nj}(t_i)[g(t_i) - g(t_j)] \Big\} \\
&= S_{3n}^{-2} \Big\{ \sum_{i=1}^{n} \tilde{x}_i^2 (\hat{\beta}_c - \beta) + \sum_{i=1}^{n} \tilde{\xi}_i [\hat{g}_n^c(t_i) - g(t_i)] + \sum_{i=1}^{n} \tilde{\mu}_i [\hat{g}_n^c(t_i) - g(t_i)] \\
&\quad - \sum_{i=1}^{n} \tilde{\xi}_i \sum_{j=1}^{n} W_{nj}(t_i)[\hat{g}_n^c(t_j) - g(t_j)] - \sum_{i=1}^{n} \tilde{\mu}_i \sum_{j=1}^{n} W_{nj}(t_i)[\hat{g}_n^c(t_j) - g(t_j)] \\
&\quad + \sum_{i=1}^{n} \tilde{\xi}_i \sum_{j=1}^{n} W_{nj}(t_i)[g(t_i) - g(t_j)] + \sum_{i=1}^{n} \tilde{\mu}_i \sum_{j=1}^{n} W_{nj}(t_i)[g(t_i) - g(t_j)] \\
&:= S_{3n}^{-2} \sum_{k=1}^{7} H_{kn}.
\end{aligned}$$

Using a similar approach as step 1 in the proof of Theorem 3.1*(a)*, one can get $S_{3n}^{-2} \leq Cn^{-1}$ a.s. Therefore, we only need to verify that $n^{-1}H_{kn} = o(n^{-1/4})$ a.s. for $k = 1, 2, \cdots, 7$. From (A0)-(A4), Theorem 3.1, Lemmas 5.2-5.4, (6.2) and (6.3), we have

$$\begin{aligned}
n^{-1}H_{1n} &\leq C \cdot \left|\hat{\beta}_c - \beta\right| = o(n^{-\frac{1}{4}}) \text{ a.s.} \\
n^{-1}H_{2n} &= n^{-1} \cdot \max_{1\leq i\leq n} \sum_{i=1}^{n} \left|\tilde{\xi}_i\right| \cdot \max_{1\leq i\leq n} \left|\hat{g}_n^c(t_i) - g(t_i)\right| = o(n^{-\frac{1}{4}}) \text{ a.s.}
\end{aligned}$$

$$
\begin{aligned}
n^{-1}H_{3n} &\le n^{-1}\Big|\sum_{i=1}^{n}\tilde{\mu}_i[\hat{g}_n^c(t_i)-g(t_i)]\Big| \le n^{-1}\Big[\sum_{i=1}^{n}\big|\mu_i\big|\cdot\big|\hat{g}_n^c(t_i)-g(t_i)\big| \\
&\quad + \sum_{i=1}^{n}\big|\hat{g}_n^c(t_i)-g(t_i)\big|\cdot\Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big|\Big] \\
&\le n^{-1}\Big[\max_{1\le i\le n}\Big(\sum_{i=1}^{n}\Big(\big|\mu_i\big|-E\big|\mu_i\big|\Big)+\sum_{i=1}^{n}E\big|\mu_i\big|\Big)\cdot\max_{i\le 1\le n}\big|\hat{g}_n^c(t_i)-g(t_i)\big| \\
&\quad + n\cdot\max_{1\le i\le n}\big|\hat{g}_n^c(t_i)-g(t_i)\big|\cdot\max_{1\le i\le n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big|\Big] = o(n^{-\frac{1}{4}}) \text{ a.s.} \\
n^{-1}H_{4n} &\le n^{-1}\Big[\max_{1\le i\le n}\big|\hat{g}_n^c(t_j)-g(t_j)\big|\cdot\max_{1\le i\le n}\sum_{i=1}^{n}\big|\tilde{\xi}_i\big|\cdot\max_{1\le i\le n}\sum_{j=1}^{n}W_{nj}(t_i)\Big] = o(n^{-\frac{1}{4}}) \text{ a.s.} \\
n^{-1}H_{5n} &\le n^{-1}\Big|\sum_{i=1}^{n}\tilde{\mu}_i\sum_{j=1}^{n}W_{nj}(t_i)[\hat{g}_n^c(t_j)-g(t_j)]\Big| \\
&\le n^{-1}\Big[\sum_{i=1}^{n}\big|\mu_i\big|\cdot\max_{1\le i\le n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)[\hat{g}_n^c(t_j)-g(t_j)]\Big| \\
&\quad + \sum_{i=1}^{n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)[\hat{g}_n^c(t_j)-g(t_j)]\Big|\cdot\max_{1\le i\le n}\Big|\sum_{k=1}^{n}W_{nk}(t_i)\mu_k\Big|\Big] \\
&\le n^{-1}\Big[\max_{1\le i\le n}\Big(\sum_{i=1}^{n}\Big(\big|\mu_i\big|-E\big|\mu_i\big|\Big)+\sum_{i=1}^{n}E\big|\mu_i\big|\Big)\cdot\max_{1\le i\le n}\big|\hat{g}_n^c(t_j)-g(t_j)\big| \\
&\quad + n\cdot\max_{1\le i\le n}\big|\hat{g}_n^c(t_j)-g(t_j)\big|\cdot\Big|\sum_{j=1}^{n}W_{nj}(t_i)\Big|\cdot\max_{1\le i\le n}\Big|\sum_{k=1}^{n}W_{nk}(t_i)\mu_k\Big|\Big] = o(n^{-\frac{1}{4}}) \text{ a.s.}
\end{aligned}
$$

The proof of $n^{-1}H_{kn} = o(n^{-1/4})$ a.s. for $k = 6, 7$ is analogous. Thus, the proof of Theorem 3.3*(a)* is completed.■

Proof of Theorem 3.3(b). From (3.7), write that

$$
\begin{aligned}
\hat{g}_n^{[R]}(t)-g(t) &= \sum_{j=1}^{n}W_{nj}(t)\big[U_j^{[R]}-x_j\hat{\beta}_R\big]-g(t) \\
&= \sum_{j=1}^{n}W_{nj}(t)\big[x_j\hat{\beta}_c-x_j\hat{\beta}_R+g_n^c(t_j)\big]-g(t) \\
&\le \Big|\sum_{j=1}^{n}W_{nj}(t)\xi_j(\hat{\beta}_c-\beta)\Big|+\Big|\sum_{j=1}^{n}W_{nj}(t)\xi_j(\hat{\beta}_R-\beta)\Big| \\
&\quad +\Big|\sum_{j=1}^{n}W_{nj}(t)\mu_j(\hat{\beta}_c-\beta)\Big|+\Big|\sum_{j=1}^{n}W_{nj}(t)\mu_j(\hat{\beta}_R-\beta)\Big| \\
&\quad +\Big|\sum_{j=1}^{n}W_{nj}(t)[\hat{g}_n^c(t_j)-g(t_j)]\Big|+\Big|\sum_{j=1}^{n}W_{nj}(t)[g(t_j)-g(t)]\Big| \\
&:= Z_{1n}(t)+Z_{2n}(t)+Z_{3n}(t)+Z_{4n}(t)+Z_{5n}(t)+Z_{6n}(t).
\end{aligned}
$$

Therefore, we only need to prove that $Z_{kn}(t) = o(n^{-1/4})$ a.s. for $k = 1, 2, \cdots, 5$. From (A0)-(A4), Lemma 5.3, (6.2)-(6.3), one can get

$$\begin{aligned} Z_{1n}(t) &\leq \left|\beta-\hat{\beta}_c\right|\cdot\max_{1\le j\le n}\left|\xi_j\right|\cdot\sum_{j=1}^{n}W_{nj}(t)=o(n^{-\frac{1}{4}})\text{ a.s.}\\ Z_{3n}(t) &\leq \left|\beta-\hat{\beta}_c\right|\cdot\left|\sum_{j=1}^{n}W_{nj}(t)\mu_j\right|=o(n^{-\frac{1}{2}})\text{ a.s.}\\ Z_{5n}(t) &\leq \max_{1\le j\le n}\left|\hat{g}_n^c(t_i)-g(t_i)\right|\cdot\sum_{j=1}^{n}W_{nj}(t)=o(n^{-\frac{1}{4}})\text{ a.s.}\end{aligned}$$

The proof of $Z_{kn}=o(n^{-1/4})$ a.s. for $k=2,4,6$ is analogous. Thus, the proof of Theorem 3.3*(b)* is completed.■

Proof of Lemma 5.3(a). We prove only $\max_{1\le i\le n}|\tilde{A}_i^c|=o(n^{-1/4})$ a.s. The proof of $\max_{1\le i\le n}|\tilde{A}_i|=o(n^{-1/4})$ is similar. For any $a>0$, by (A0)-(A3), one can deduce that:

$$\begin{aligned}\max_{1\le i\le n}|\tilde{A}_i^c| &\leq \max_{1\le i\le n}\Big|\sum_{j=1}^{n}\delta_jW_{nj}^c(t_i)\Big[A(t_i)-A(t_j)\Big]I(|t_i-t_j|>a\cdot n^{-\frac{1}{4}})\Big|\\ &\quad+\max_{1\le i\le n}\Big|\sum_{j=1}^{n}\delta_jW_{nj}^c(t_i)\Big[A(t_i)-A(t_j)\Big]I(|t_i-t_j|\le a\cdot n^{-1/4})\Big|.\end{aligned}$$

Therefore, it is easy to see that $\max_{1\le i\le n}|\tilde{A}_i^c|=o(n^{-1/4})$ a.s.

(b). In the sequel, we use the Abel Inequality (Härdle et al.[9], page 183). Let $A_1,A_2,\cdots,A_n$; $B_1,B_2,\cdots,B_n$ ($B_1\ge B_2\ge\cdots\ge B_n\ge 0$) be two sequences of numbers, and $S_k=\sum_{i=1}^{k}A_i$, $M_1=\min_{1\le k\le n}S_k$, $M_2=\max_{1\le k\le n}S_k$. Then, $B_1M_1\le\sum_{k=1}^{n}A_kB_k\le B_1M_2$. Let $E_i,F_i(1\le i\le n)$ be arbitrary numbers and $(j_1,j_2,\cdots,j_n)$ be a permutation of $(1,\cdots,n)$ such that $F_{j_1}\ge F_{j_2}\ge\cdots\ge F_{j_n}$. Then from the above equation, we have

$$\begin{aligned}\Big|\sum_{i=1}^{n}E_iF_i\Big| &= \Big|\sum_{i=1}^{n}E_{j_i}F_{j_i}\Big|\le\Big|\sum_{i=1}^{n}E_{j_i}(F_{j_i}-F_{j_n})\Big|+\Big|\sum_{i=1}^{n}E_{j_i}F_{j_n}\Big|\\ &\le C\max_{1\le i\le n}|F_i|\max_{1\le m\le n}\Big|\sum_{i=1}^{m}E_{j_i}\Big|.\end{aligned}$$

We prove only $n^{-1}\sum_{i=1}^{n}\delta_i(\tilde{\xi}_i^c)^2\to\Sigma_1$ a.s. The proof of the other three formulas is similar. From $\tilde{\xi}_i^c=\tilde{h}_i^c+v_i-\sum_{j=1}^{n}\delta_jW_{nj}^c(t_i)v_j$ and $\tilde{h}_i^c=h(t_i)-\sum_{j=1}^{n}\delta_iW_{nj}^c(t_i)h(t_j)$, we can write

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\delta_i(\tilde{\xi}_i^c)^2 &= \frac{1}{n}\sum_{i=1}^{n}\delta_iv_i^2+\frac{1}{n}\sum_{i=1}^{n}\delta_i\big(\tilde{h}_i^c\big)^2+\frac{1}{n}\sum_{i=1}^{n}\delta_i\Big[\sum_{j=1}^{n}\delta_jW_{nj}^c(t_i)v_j\Big]^2+\frac{2}{n}\sum_{i=1}^{n}\delta_iv_i\tilde{h}_i^c\\ &\quad-\frac{2}{n}\sum_{i=1}^{n}\delta_iv_i\Big[\sum_{j=1}^{n}\delta_jW_{nj}^c(t_i)v_j\Big]-\frac{2}{n}\sum_{i=1}^{n}\delta_i\tilde{h}_i^c\Big[\sum_{j=1}^{n}\delta_jW_{nj}^c(t_i)v_j\Big]\\ &:= \Omega_{1n}+\Omega_{2n}+\Omega_{3n}+2\Omega_{4n}-2\Omega_{5n}-2\Omega_{6n}\end{aligned}$$

Lemma 5.3*(a)* shows that $\max_{1\le i\le n}|\tilde{h}^c(t_i)|=o(n^{-1/4})$ a.s., which implies that $\Omega_{2n}=o(n^{-1/2})$ a.s.
Hence, it follows from (A1)-(A4) that

$$\begin{aligned}|\Omega_{3n}| &= \Big|\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\delta_i\delta_jW_{nj}^c(t_i)\delta_kW_{nk}^c(t_i)v_jv_k\Big|=\Big|\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\Big[\sum_{i=1}^{n}\delta_i\delta_jW_{nj}^c(t_i)\delta_kW_{nk}^c(t_i)\Big]v_jv_k\Big|\\ &\le \frac{C}{n}\cdot\max_{1\le m\le n}\Big|\sum_{i=1}^{m}v_{j_i}\Big|\cdot\max_{1\le m\le n}\Big|\sum_{i=1}^{m}v_{k_i}\Big|\cdot\max_{1\le i,j\le n}\delta_i\delta_jW_{nj}^c(t_i)\cdot\max_{1\le k\le n}\sum_{i=1}^{n}\delta_kW_{nk}^c(t_i)\end{aligned}$$

$$
\begin{aligned}
&= o(n^{-\frac{1}{2}}\log n) \text{ a.s.}\\
|\Omega_{4n}| &\leq \frac{2}{n}\sum_{i=1}^{n}|\delta_i|\cdot\max_{1\leq i\leq n}|v_i|\cdot\max_{1\leq i\leq n}|\tilde{h}_i^c| = o(n^{-\frac{1}{4}}) \text{ a.s.}\\
|\Omega_{5n}| &\leq \frac{C}{n}\cdot\max_{1\leq j\leq n}\Big|\sum_{i=1}^{n}\delta_i\delta_j W_{nj}^c(t_i)v_i\Big|\cdot\max_{1\leq m\leq n}\Big|\sum_{i=1}^{m}v_{j_i}\Big|\\
&\leq \frac{C}{n}\cdot\max_{1\leq m\leq n}\Big|\sum_{k=1}^{m}v_{i_k}\Big|\cdot\max_{1\leq i,j\leq n}\delta_i\delta_j W_{nj}^c(t_i)\cdot\max_{1\leq m\leq n}\Big|\sum_{j=1}^{m}v_{j_i}\Big| = o(n^{-\frac{1}{2}}\log n) \text{ a.s.}\\
|\Omega_{6n}| &\leq \frac{2}{n}\sum_{i=1}^{n}|\delta_i|\cdot\max_{1\leq i\leq n}|\tilde{h}_i^c|\cdot\max_{1\leq i\leq n}\Big|\sum_{j=1}^{n}\delta_j W_{nj}^c(t_i)\Big|\cdot\max_{1\leq j\leq n}|v_j| = o(n^{-\frac{1}{4}}) \text{ a.s.}
\end{aligned}
$$

Then, we obtain that $n^{-1}\sum_{i=1}^{n}\delta_i(\tilde{\xi}_i^c)^2 \to \Sigma_1$ a.s. from (A1)(i).

(c). Using the assumptions (A1), (A2) and (A4), it is easy to obtain that:

$$
\max_{1\leq i\leq n}|\tilde{\xi}_i^c| \leq \max_{1\leq i\leq n}|\tilde{h}_i^c| + \max_{1\leq i\leq n}|v_i| + \max_{1\leq i\leq n}\sum_{j=1}^{n}\delta_j W_{nj}^c(t_i)\cdot\max_{1\leq j\leq n}|v_j| = O(1) \text{ a.s.}
$$

The proof of $\max_{1\leq i\leq n}|\tilde{\xi}_i| = O(1)$ is similar. Then, the proof of Lemma 5.3 is completed. ■

References

1. Baek, J. I. and Liang, H. Y., Asymptotic of estimators in semi-parametric model under NA samples. Journal of Statist. Plann. Infer. 136 (2006) 3362–3382.
2. Bianco, A., Boente, G., Gonzlez-Manteiga, W. and Prez-Gonzlez, A., Estimation of the marginal location under a partially linear model with missing responses. Comput. Statist. Data Anal. 54 (2010) 546–564.
3. Chen, H., Convergence rates for parametric components in a partly linear model. Ann. Statist. 16 (1988) 136–146.
4. Cheng, P.E., Nonparametric estimation of mean functionals with data missing at random. J. Amer. Statist. Assoc. 89 (1994) 81–87.
5. Cui, H. J. and Li, R. C., On parameter estimation for semi-linear errors-in-variables models. Journal of Multivariate Analysis 64(1) (1998) 1–24.
6. Engle, R. F., Granger, C. W. J., Rice, J. and Weiss, A., Semiparametric estimation of the relation between weather and electricity sales. Journal of the American Statistical Association 81 (1986) 310–320.
7. Fan, G.L., Liang, H.Y., Wang, J.F. and Xu, H.X., Asymptotic properties for LS estimators in EV regression model with dependent errors. Adv. Stat. Anal. 94 (2010) 89–103.
8. Gao, J. T., Chen, X. R. and Zhao, L. C., Asymptotic normality of a class of estimators in partial linear models. Acta. Math. Sinica 37(2) (1994) 256–268.
9. Härdle, W., Liang, H. and Gao, J. T., Partial Linear Models, Physica-Verlag, Heidelberg, 2000.
10. Healy, M.J.R. and Westmacott, M., Missing values in experiments analysis on automatic computers. Appl. Statist. 5 (1956) 203–206.
11. Liang, H., Härdle, W. and Carrol, R. J., Estimation in a semiparametric partially linear errosr-in-variables model. The Annals of Statistics 27(5) (1999) 1519–1935.
12. Liu, J.X. and Chen, X.R., Consistency of LS estimator in simple linear EV regression models. Acta Math. Sci. Ser. B 25 (2005) 50–58.
13. Miao, Y. and Liu, W., Moderate deviations for LS estimator in simple linear EV regression model. J. Stat. Plann. Inference 139(9) (2009) 3122–3131.
14. Miao, Y., Yang, G. and Shen, L., The central limit theorem for LS estimator in simple linear EV regression models. Comm. Stat. Theory Methods 36 (2007) 2263–2272.
15. Parzen, E., On the estimation of a probability density function and mode. Ann. Math. Statist 33 (1962) 1065–1076.
16. Wang, Q., Linton, O. and Härdle, W., Semiparametric regression analysis with missing response at random. J. Amer. Statist. Assoc. 99(466)(2004) 334–345.
17. Wang, Q. and Sun, Z., Estimation in partially linear models with missing responses at random. J. Multivariate Anal. 98 (2007) 1470–1493.
18. Zhou, H.B., You, J.H. and Zhou, B., Statistical inference for fixed-effects partially linear regression models with errors in variables. Stat. Papers 51 (2010) 629–650.