



## Spectral Analysis of Discrete Dirac Equation with Generalized Eigenparameter in Boundary Condition

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**Abstract.** Let  $L$  denote the discrete Dirac operator generated in  $\ell_2(\mathbb{N}, \mathbb{C}^2)$  by the non-selfadjoint difference operators of first order

$$\begin{cases} a_{n+1}y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ a_{n-1}y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)} \end{cases}, \quad n \in \mathbb{N}, \quad (0.1)$$

with boundary condition

$$\sum_{k=0}^p (y_1^{(2)} \gamma_k + y_0^{(1)} \beta_k) \lambda^k = 0, \quad (0.2)$$

where  $(a_n)$ ,  $(b_n)$ ,  $(p_n)$  and  $(q_n)$ ,  $n \in \mathbb{N}$  are complex sequences,  $\gamma_i, \beta_i \in \mathbb{C}$ ,  $i = 0, 1, 2, \dots, p$  and  $\lambda$  is a eigenparameter. We discuss the spectral properties of  $L$  and we investigate the properties of the spectrum and the principal vectors corresponding to the spectral singularities of  $L$ , if

$$\sum_{n=1}^{\infty} |n| (|1 - a_n| + |1 + b_n| + |p_n| + |q_n|) < \infty$$

holds.

### 1. Introduction

Spectral theory of difference equations is one of the main branches of modern functional analysis and applications. Therefore, spectral properties of discrete boundary value problems has been intensively studied in the last decade and the spectral analysis of the difference equations have been treated by various authors in connection with the classical moment problem ([1-5] and the references therein). Moreover the modeling of certain linear and nonlinear problems from economics, optimal control theory, engineering,

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medicine and other areas of study have led to the rapid development of the theory of difference equations. Also, the spectral theory of the difference equations has been applied to the solution of classes of nonlinear discrete Kortevég-de Vriez equations and Toda lattices ([6,7]).

Consider the discrete boundary value problem (BVP)

$$\begin{cases} y_{n+1}^{(2)} - y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ -y_n^{(1)} + y_{n-1}^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)} \end{cases} \tag{1.1}$$

$$y_0^{(1)} = 0,$$

where  $(p_n)$  and  $(q_n)$  are complex sequences for  $n = 1, 2, \dots$  and  $\lambda$  is a spectral parameter. The spectral analysis of the BVP (1.1) with principal functions has been studied in [8]. In this article, the authors have determined that the spectrum of the BVP (1.1), the eigenvalues and the spectral singularities. They have proved that it has a finite number of eigenvalues and spectral singularities with finite multiplicities. Also the integral representation for the Weyl function and the spectral expansion of (1.1) was found in terms of the principal functions. Some problems related to the spectral analysis of difference equations with spectral singularities have been discussed in [9-16]. The spectral analysis of discrete Dirac equation which is eigenparameter dependent was studied in [17-19].

In this paper, we consider a general form of the difference equation (1.1) with a boundary condition which depends on the eigenparameters in a polynomial form.

Before we precisely state our problem, we describe a few notations for convenience of expressing the results about the properties of spectrum, resolvent operator and principal functions. Such problems are of interest in Engineering, Optimal control and Biology.

Let  $L$  denote the discrete Dirac operator generated in  $\ell_2(\mathbb{N}, \mathbb{C}^2)$  by

$$(\ell y)_n := \begin{pmatrix} \widetilde{\ell y}_n \\ \widehat{\ell y}_n \end{pmatrix} = \begin{pmatrix} a_{n+1} y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} \\ a_{n-1} y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)} \end{pmatrix}$$

and consider the non-selfadjoint BVP for the system of difference equations of first order

$$(\ell y)_n = \lambda y_n \tag{1.2}$$

$$\sum_{k=0}^p (y_1^{(2)} \gamma_k + y_0^{(1)} \beta_k) \lambda^k = 0, \tag{1.3}$$

where  $y_n = \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}$ ,  $n \in \mathbb{N}$  are vector sequences,  $a_n \neq 0, b_n \neq 0$  for all  $n$ . Moreover  $\sum_{k=0}^p (|\gamma_k| + |\beta_k|) \neq 0$ ,  $\gamma_p \neq \frac{\beta_{p-1}}{a_0}$  and  $\gamma_0 \beta_1 - \gamma_1 \beta_0 \neq 0$  where  $\gamma_k, \beta_k \in \mathbb{C}, i = 0, 1, 2, \dots, p$ . In addition, if  $a_n \equiv 1$  and  $b_n \equiv -1$  for all  $n \in \mathbb{N}$ , then the system (1.2) reduces to

$$\begin{pmatrix} \Delta y_n^{(2)} + p_n y_n^{(1)} \\ -\Delta y_{n-1}^{(1)} + q_n y_n^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda y_n^{(1)} \\ \lambda y_n^{(2)} \end{pmatrix}, n \in \mathbb{N} \tag{1.4}$$

where  $\Delta$  is a forward difference operator. The system (1.4) is the discrete analogue of the well-known Dirac system

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

([20], Chap. 2). Therefore the system (1.4) (also (1.2)) is called the discrete Dirac system.

In this study, we analyze the properties of the spectrum and the principal vectors corresponding to the spectral singularities of  $L$  under the condition

$$\sum_{n=1}^{\infty} |n| (|1 - a_n| + |1 + b_n| + |p_n| + |q_n|) < \infty.$$

### 2. Jost solution and Jost function of $L$

Let the complex sequences  $(a_n)$  and  $(b_n)$  satisfy

$$\sum_{n=1}^{\infty} |n| (|1 - a_n| + |1 + b_n| + |p_n| + |q_n|) < \infty. \tag{2.1}$$

Under the condition (2.1), it is well-known from [15] and eq. (1.2), that the bounded solutions are

$$e_n(z) = \begin{pmatrix} e_n^{(1)}(z) \\ e_n^{(2)}(z) \end{pmatrix} = \alpha_n \left( I_2 + \sum_{m=1}^{\infty} A_{nm} e^{imz} \right) \begin{pmatrix} e^{i\frac{z}{2}} \\ -i \end{pmatrix} e^{imz}, \quad n \in \mathbb{N}, \tag{2.2}$$

$$e_0^{(1)}(z) = \alpha_0^{11} \left\{ e^{i\frac{z}{2}} \left[ 1 + \sum_{m=1}^{\infty} A_{0m}^{11} e^{imz} \right] - i \sum_{m=1}^{\infty} A_{0m}^{12} e^{imz} \right\} \tag{2.3}$$

for  $\lambda = 2 \sin \frac{z}{2}$  and  $z \in \overline{\mathbb{C}}_+ := \{z : z \in \mathbb{C}, \text{Im } z \geq 0\}$ , where

$$\alpha_n = \begin{pmatrix} \alpha_n^{11} & \alpha_n^{12} \\ \alpha_n^{21} & \alpha_n^{22} \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{nm} = \begin{pmatrix} A_{nm}^{11} & A_{nm}^{12} \\ A_{nm}^{21} & A_{nm}^{22} \end{pmatrix}.$$

Here,  $\alpha_n^{ij}$  and  $A_{nm}^{ij}$  ( $i, j = 1, 2$ ) are expressed in terms of  $(a_n)$ ,  $(b_n)$ ,  $(p_n)$  and  $(q_n)$ ,  $n \in \mathbb{N}$  as

$$\alpha_n^{11} = \frac{1}{\prod_{k=n+1}^{\infty} (-1)^{n-k} b_k a_{k-1}}; \quad \alpha_n^{12} = 0; \quad \alpha_n^{22} = \frac{1}{b_n \prod_{k=n+1}^{\infty} (-1)^{n-k+1} b_k a_{k-1}}; \quad \alpha_n^{21} = \alpha_n^{22} \left[ p_n + \sum_{k=n+1}^{\infty} (p_k + q_k) \right],$$

$$A_{n1}^{12} = - \sum_{k=n+1}^{\infty} (p_k + q_k),$$

$$A_{n1}^{11} = \sum_{k=n+1}^{\infty} [a_{k+1} a_k + b_k^2 - p_k q_k + (p_k + q_k) A_{k1}^{12} - 2],$$

$$A_{n1}^{22} = -1 + a_{n+1} a_n + (A_{n1}^{12})^2 + A_{n1}^{11},$$

$$A_{n1}^{21} = - \sum_{k=n}^{\infty} \left\{ (q_{k+1} + A_{k1}^{12}) [a_{k+1} a_k + q_{k+1} (p_{k+1} + q_{k+1}) + q_{k+1} A_{k1}^{12} + b_{k+1}^2 + A_{k+1,1}^{11} - 1] - A_{k1}^{12} (1 + A_{k1}^{11}) \right\} + \sum_{k=n+1}^{\infty} (q_k A_{k1}^{22} - b_k^2 p_k),$$

$$A_{n2}^{12} = -a_{n+1} a_n (q_{n+1} + A_{n1}^{12}) + A_{n1}^{12} A_{n1}^{11} + A_{n1}^{12} - A_{n1}^{21},$$

$$A_{n2}^{11} = \sum_{k=n+1}^{\infty} \left\{ (b_k^2 - 1) A_{k1}^{11} - a_{k+1} a_k [(q_{k+1} + A_{k1}^{12}) A_{k+1,1}^{12} - A_{k+1,1}^{22}] - (p_k - A_{k1}^{12}) [q_k A_{k1}^{11} + A_{k1}^{12} - A_{k2}^{12}] - q_k A_{k1}^{21} + A_{k1}^{12} A_{k2}^{12} - A_{k1}^{22} \right\},$$

$$A_{n2}^{22} = -a_{n+1} a_n (q_{n+1} + A_{n1}^{12}) A_{n+1,1}^{12} + a_{n+1} a_n A_{n+1,1}^{22} + A_{n1}^{12} A_{n2}^{12} - A_{n1}^{11} + A_{n2}^{11},$$

$$A_{n2}^{21} = \sum_{k=n}^{\infty} \left\{ A_{k1}^{12} A_{k2}^{11} + A_{k2}^{21} - a_{k+1} a_k [(q_{k+1} + A_{k1}^{12}) A_{k+1,1}^{11} - A_{k+1,1}^{21}] \right\} - \sum_{k=n+1}^{\infty} \left[ (q_k + A_{k-1,1}^{12}) (q_k A_{k2}^{12} - A_{k1}^{11} + A_{k2}^{11}) + b_k^2 A_{k2}^{21} - p_k A_{k2}^{22} + A_{k1}^{21} \right],$$

and for  $m \geq 3$

$$\begin{aligned}
 A_{nm}^{12} &= -a_{n+1}a_n \left[ (q_{n+1} + A_{n1}^{12})A_{n+1,m-2}^{11} + A_{n+1,m-2}^{21} \right] + A_{n1}^{12}A_{n,m-1}^{11} + A_{n,m-1}^{12} - A_{n,m-1}^{21}, \\
 A_{nm}^{11} &= - \sum_{k=n+1}^{\infty} a_{k+1}a_k \left[ (q_{k+1} + A_{k1}^{12})A_{k+1,m-1}^{12} - A_{k+1,m-1}^{22} \right] - \sum_{k=n+1}^{\infty} (p_k - A_{k1}^{12}) \left( q_k A_{k,m-1}^{11} + A_{k,m-1}^{12} - A_{km}^{12} \right) \\
 &\quad + \sum_{k=n+1}^{\infty} (b_k^2 - 1)A_{k,m-1}^{11} - \sum_{k=n+1}^{\infty} q_k A_{k,m-1}^{21} + \sum_{k=n+1}^{\infty} A_{k1}^{12}A_{km}^{12} - \sum_{k=n+1}^{\infty} A_{k,m-1}^{22}, \\
 A_{nm}^{22} &= -a_{n+1}a_n \left[ (q_{n+1} + A_{n1}^{12})A_{n+1,m-1}^{11} - A_{n+1,m-1}^{22} \right] + A_{n1}^{12}A_{nm}^{12} + A_{nm}^{11} - A_{n,m-1}^{11}, \\
 A_{nm}^{21} &= - \sum_{k=n}^{\infty} a_{k+1}a_k \left[ (q_{k+1} + A_{k1}^{12})A_{k+1,m-1}^{11} - A_{k+1,m-1}^{21} \right] - \sum_{k=n+1}^{\infty} (q_k - A_{k-1,1}^{12}) \left( q_k A_{km}^{21} + A_{k,m-1}^{11} - A_{km}^{22} \right) \\
 &\quad - \sum_{k=n+1}^{\infty} (b_k^2 - 1)A_{km}^{12} + \sum_{k=n}^{\infty} A_{k1}^{12}A_{km}^{22} + \sum_{k=n+1}^{\infty} q_k A_{km}^{22} + \sum_{k=n}^{\infty} A_{km}^{12} - \sum_{k=n+1}^{\infty} A_{k,m-1}^{21}.
 \end{aligned}$$

Moreover

$$\left| A_{nm}^{ij} \right| \leq C \sum_{k=n+1}^{\infty} \left( |1 - a_k| + |1 + b_k| + |p_k| + |q_k| \right) \tag{2.4}$$

holds, where  $C > 0$  is a constant and  $\left[ \frac{m}{2} \right]$  is the integer part of  $\frac{m}{2}$ . Therefore  $e_n$  is vector-valued analytic function in  $\mathbb{C}_+ := \{z : z \in \mathbb{C}, \text{Im } z > 0\}$  and continuous in  $\overline{\mathbb{C}}_+$  with respect to  $z$  ([15]). The solution  $e_n(z) = \begin{pmatrix} e_n^{(1)}(z) \\ e_n^{(2)}(z) \end{pmatrix}$  is called Jost solution of (1.2). Let  $\widehat{\varphi}_n(\lambda) = \begin{pmatrix} \widehat{\varphi}_n^{(1)}(\lambda) \\ \widehat{\varphi}_n^{(2)}(\lambda) \end{pmatrix}$ ,  $n \in \mathbb{N} \cup \{0\}$  be the another solution of (1.2) subject to the initial conditions

$$\widehat{\varphi}_0^{(1)}(\lambda) = - \sum_{k=0}^p \gamma_k \lambda^k, \quad \widehat{\varphi}_1^{(2)}(\lambda) = \sum_{k=0}^p \beta_k \lambda^k.$$

If we characterize

$$\varphi_n(z) = \left\{ \widehat{\varphi}_n \left( 2 \sin \frac{z}{2} \right) \right\}, \quad n \in \mathbb{N} \cup \{0\},$$

then  $\varphi_n$  is an entire function and is  $4\pi$  periodic.

Let us take the semi-strips  $S_0 := \{z : z \in \mathbb{C}, z = x + iy, -\pi \leq x \leq 3\pi, y > 0\}$  and  $S := S_0 \cup [-\pi, 3\pi]$ . Then the wronskian of the solutions  $e_n(z)$  and  $\varphi_n(z)$  does not depend on  $n$  is given by

$$\begin{aligned}
 W[e_n(z), \varphi_n(z)] &= a_n \left[ \widehat{\varphi}_{n+1}^{(2)} \left( 2 \sin \frac{z}{2} \right) e_n^{(1)}(z) - e_{n+1}^{(2)}(z) \widehat{\varphi}_n^{(1)} \left( 2 \sin \frac{z}{2} \right) \right] \\
 &= a_0 \left[ \widehat{\varphi}_1^{(2)} \left( 2 \sin \frac{z}{2} \right) e_0^{(1)}(z) - e_1^{(2)}(z) \widehat{\varphi}_0^{(1)} \left( 2 \sin \frac{z}{2} \right) \right].
 \end{aligned}$$

If we define

$$f(z) = \widehat{\varphi}_1^{(2)} \left( 2 \sin \frac{z}{2} \right) e_0^{(1)}(z) - e_1^{(2)}(z) \widehat{\varphi}_0^{(1)} \left( 2 \sin \frac{z}{2} \right),$$

then  $f$  is analytic in  $\mathbb{C}_+$ , continuous in  $\overline{\mathbb{C}}_+$  and  $f(z) = f(z + 4\pi)$ . When  $f(z) \neq 0$  for all  $z \in S$ ,  $e_n(z)$  and  $\varphi_n(z)$  are linearly independent. Here

$$\widehat{f}(z) = W[e_n(z), \varphi_n(z)] = a_0 f(z) \tag{2.5}$$

is called Jost function of  $L$ . Moreover, if we define  $g_n = (g_n^{(1)}, g_n^{(2)})$  then,

$$R_\lambda(L)g_n := -\frac{1}{\widehat{f}(z)} \left\{ \sum_{k=1}^n (g_{k-1}^{(1)}, g_k^{(2)}) \begin{pmatrix} \frac{a_{k-1}}{a_k} \varphi_{k-1}^{(1)} \\ \varphi_k^{(2)} \end{pmatrix} \begin{pmatrix} e_n^{(1)} \\ e_n^{(2)} \end{pmatrix} + \sum_{k=n+1}^{\infty} (g_{k-1}^{(1)}, g_k^{(2)}) \begin{pmatrix} \frac{a_{k-1}}{a_k} e_{k-1}^{(1)} \\ e_k^{(2)} \end{pmatrix} \begin{pmatrix} \varphi_n^{(1)} \\ \varphi_n^{(2)} \end{pmatrix} \right\}$$

is the resolvent of the operator  $L$ .

### 3. Eigenvalues and spectral singularities of $L$

From (2.5), we clearly obtain that the function

$$\widehat{f}(z) = a_0 \left[ e_1^{(2)}(z) \sum_{k=0}^p \gamma_k \lambda^k + e_0^{(1)}(z) \sum_{k=0}^p \beta_k \lambda^k \right] \tag{3.1}$$

is analytic in  $\mathbb{C}_+$ , continuous up to the real axis and is  $4\pi$  periodic. Also if we denote the set of all eigenvalues and spectral singularities of  $L$  by  $\sigma_d(L)$  and  $\sigma_{ss}(L)$  respectively, then it is clear that

$$\sigma_d(L) = \left\{ \lambda : \lambda = 2 \sin \frac{z}{2}, z \in S_0, \widehat{f}(z) = 0 \right\}, \tag{3.2}$$

$$\sigma_{ss}(L) = \left\{ \lambda : \lambda = 2 \sin \frac{z}{2}, z \in [-\pi, 3\pi], \widehat{f}(z) = 0 \right\}. \tag{3.3}$$

From (2.2), (2.3) and (3.1) we obtain

$$\begin{aligned} \widehat{f}(z) = & a_0 \left\{ i^p \beta_p \alpha_0^{11} e^{-i(\frac{p-1}{2})z} + i^{p-1} (\beta_{p-1} \alpha_0^{11} + \gamma_p \alpha_1^{22}) e^{-i(\frac{p-2}{2})z} \right. \\ & + \sum_{k=0}^{p-2} i^k \beta_k \alpha_0^{11} e^{i(\frac{1}{2}-\frac{k}{2})z} - \sum_{k=0}^{p-1} i^{k+1} \gamma_k \alpha_1^{22} e^{i(1-\frac{k}{2})z} + \sum_{k=0}^p i^k \gamma_k \alpha_1^{21} e^{i(\frac{3}{2}-\frac{k}{2})z} \\ & - \sum_{m=1}^{\infty} \sum_{k=0}^p i^{k+1} \beta_k \alpha_0^{11} A_{0m}^{12} e^{i(m-\frac{k}{2})z} + \sum_{m=1}^{\infty} \sum_{k=0}^p i^k \beta_k \alpha_0^{11} A_{0m}^{11} e^{i(m+\frac{1}{2}-\frac{k}{2})z} \\ & - \sum_{m=1}^{\infty} \sum_{k=0}^p i^{k+1} \gamma_k (\alpha_1^{21} A_{1m}^{12} + \alpha_1^{22} A_{1m}^{22}) e^{i(m+1-\frac{k}{2})z} + \sum_{m=1}^{\infty} \sum_{k=0}^p i^k \gamma_k (\alpha_1^{21} A_{1m}^{11} + \alpha_1^{22} A_{1m}^{21}) e^{i(m+\frac{3}{2}-\frac{k}{2})z} \\ & + \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^k \binom{k}{s} \beta_k \alpha_0^{11} e^{i(s+\frac{1}{2}-\frac{k}{2})z} - \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^{k+1} \binom{k}{s} \gamma_k \alpha_1^{22} e^{i(s+1-\frac{k}{2})z} \\ & + \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^k \binom{k}{s} \gamma_k \alpha_1^{21} e^{i(s+\frac{3}{2}-\frac{k}{2})z} - \sum_{m=1}^{\infty} \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^{k+1} \binom{k}{s} \beta_k \alpha_0^{11} A_{0m}^{12} e^{i(m+s-\frac{k}{2})z} \\ & + \sum_{m=1}^{\infty} \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^k \binom{k}{s} \beta_k \alpha_0^{11} A_{0m}^{11} e^{i(m+s+\frac{1}{2}-\frac{k}{2})z} \\ & - \sum_{m=1}^{\infty} \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^{k+1} \binom{k}{s} \gamma_k (\alpha_1^{21} A_{1m}^{12} + \alpha_1^{22} A_{1m}^{22}) e^{i(m+s+1-\frac{k}{2})z} \\ & \left. + \sum_{m=1}^{\infty} \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^k \binom{k}{s} \gamma_k (\alpha_1^{21} A_{1m}^{11} + \alpha_1^{22} A_{1m}^{21}) e^{i(m+s+\frac{3}{2}-\frac{k}{2})z} \right\}. \tag{3.4} \end{aligned}$$

However  $\widehat{f}(z)$  is unbounded in  $\mathbb{C}_+$  when  $p$  is big enough. So, in order to overcome this difficulty, we take

$$F(z) := \widehat{f}(z) e^{i(\frac{p-1}{2})z}, \tag{3.5}$$

then, the function  $F$  is analytic in  $\mathbb{C}_+$ , continuous in  $\overline{\mathbb{C}_+}$ . Moreover zeros of  $\widehat{f}$  and  $F$  in  $S$  are same. Clearly we can write,

$$\begin{aligned}
 F(z) = & a_0 \left\{ i^p \beta_p \alpha_0^{11} + i^{p-1} (\beta_{p-1} \alpha_0^{11} + \gamma_p \alpha_1^{22}) e^{i\frac{z}{2}} \right. \\
 & + \sum_{k=0}^{p-2} i^k \beta_k \alpha_0^{11} e^{i(\frac{p-k}{2})z} - \sum_{k=0}^{p-1} i^{k+1} \gamma_k \alpha_1^{22} e^{i(\frac{p-k+1}{2})z} + \sum_{k=0}^p i^k \gamma_k \alpha_1^{21} e^{i(\frac{p-k+2}{2})z} \\
 & - \sum_{m=1}^{\infty} \sum_{k=0}^p i^{k+1} \beta_k \alpha_0^{11} A_{0m}^{12} e^{i(m+\frac{p-k-1}{2})z} + \sum_{m=1}^{\infty} \sum_{k=0}^p i^k \beta_k \alpha_0^{11} A_{0m}^{11} e^{i(m+\frac{p-k}{2})z} \\
 & - \sum_{m=1}^{\infty} \sum_{k=0}^p i^{k+1} \gamma_k (\alpha_1^{21} A_{1m}^{12} + \alpha_1^{22} A_{1m}^{22}) e^{i(m+\frac{p-k+1}{2})z} + \sum_{m=1}^{\infty} \sum_{k=0}^p i^k \gamma_k (\alpha_1^{21} A_{1m}^{11} + \alpha_1^{22} A_{1m}^{21}) e^{i(m+\frac{p-k+2}{2})z} \\
 & + \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^k \binom{k}{s} \beta_k \alpha_0^{11} e^{i(s+\frac{p-k}{2})z} - \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^{k+1} \binom{k}{s} \gamma_k \alpha_1^{22} e^{i(s+\frac{p-k+1}{2})z} \\
 & + \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^k \binom{k}{s} \gamma_k \alpha_1^{21} e^{i(s+\frac{p-k+2}{2})z} - \sum_{m=1}^{\infty} \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^{k+1} \binom{k}{s} \beta_k \alpha_0^{11} A_{0m}^{12} e^{i(m+s+\frac{p-k-1}{2})z} \\
 & + \sum_{m=1}^{\infty} \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^k \binom{k}{s} \beta_k \alpha_0^{11} A_{0m}^{11} e^{i(m+s+\frac{p-k}{2})z} \\
 & - \sum_{m=1}^{\infty} \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^{k+1} \binom{k}{s} \gamma_k (\alpha_1^{21} A_{1m}^{12} + \alpha_1^{22} A_{1m}^{22}) e^{i(m+s+\frac{p-k+1}{2})z} \\
 & \left. + \sum_{m=1}^{\infty} \sum_{k=1}^p \sum_{s=1}^k (-1)^s i^k \binom{k}{s} \gamma_k (\alpha_1^{21} A_{1m}^{11} + \alpha_1^{22} A_{1m}^{21}) e^{i(m+s+\frac{p-k+2}{2})z} \right\} \tag{3.6}
 \end{aligned}$$

and

$$F(z + 4\pi) = F(z).$$

Also, from using (3.2)-(3.5),

$$\sigma_d(L) = \left\{ \lambda : \lambda = 2 \sin \frac{z}{2}, z \in S_0, F(z) = 0 \right\}, \tag{3.7}$$

$$\sigma_{ss}(L) = \left\{ \lambda : \lambda = 2 \sin \frac{z}{2}, z \in [-\pi, 3\pi], F(z) = 0 \right\}. \tag{3.8}$$

**Definition 3.1.** The multiplicity of a zero of  $F$  in  $S$  is called the multiplicity of the corresponding eigenvalue or spectral singularity of  $L$ .

From (3.7) and (3.8), in order to investigate the quantitative properties of the eigenvalues and the spectral singularities of  $L$ , we need to discuss the quantitative properties of the zeros of  $F$  in  $S$ .

Let us define

$$\begin{aligned}
 P_1 & := \{z : z \in S_0, F(z) = 0\}, \\
 P_2 & := \{z : z \in [-\pi, 3\pi], F(z) = 0\}, \\
 P_3 & := \{\text{All limit points of } M_1\}, \\
 P_4 & := \{\text{All zeros of } F \text{ with infinite multiplicity}\}.
 \end{aligned} \tag{3.9}$$

So from (3.2), (3.3) and (3.9) we find that

$$\begin{aligned} \sigma_d(L) &= \left\{ \lambda : \lambda = 2 \sin \frac{z}{2}, z \in P_1 \right\}, \\ \sigma_{ss}(L) &= \left\{ \lambda : \lambda = 2 \sin \frac{z}{2}, z \in P_2 \right\}. \end{aligned} \tag{3.10}$$

**Theorem 3.2.** Under the condition (2.1),

- (i) The set  $P_1$  is bounded and countable.
- (ii)  $P_1 \cap P_3 = \emptyset, P_1 \cap P_4 = \emptyset$ .
- (iii) The set  $P_2$  is compact and  $\mu(P_2) = 0$ , where  $\mu$  denotes the Lebesgue measure in the real axis.
- (iv)  $P_3 \subset P_4 \subset P_2; \mu(P_3) = \mu(P_4) = 0$ .

*Proof.* From (2.4) and (3.6), we have

$$F(z) = \begin{cases} i^{p+2} \beta_p \alpha_1^{22} + o(e^{-y}) & , \beta_p \neq 0, \\ i^{p+1} \alpha_1^{22} \left[ \beta_{p-1} + \gamma_p \alpha_1^{22} (\alpha_0^{11})^{-1} \right] e^{i \frac{z}{2}} + o(e^{-y}), & \beta_p = 0, \end{cases} \tag{3.11}$$

when  $z \in S$  and  $y \rightarrow \infty$ . Eq. (3.11) shows that  $P_1$  is bounded. By using the uniqueness theorems of analytic functions ([21]) and the continuity of all derivatives of  $F$  on  $[-\pi, 3\pi]$  we complete the proof.  $\square$

Theorem 3.2 and the sets (3.10) lead to the following:

**Theorem 3.3.** Under the condition (2.1), the following hold:

- (i) the set of eigenvalues of  $L$  is bounded and countable and its limit points can lie only in  $[-2, 2]$ .
- (ii)  $\sigma_{ss}(L) \subset [-2, 2], \sigma_{ss}(L) = \overline{\sigma_{ss}(L)}$  and  $\mu[\sigma_{ss}(L)] = 0$ .

**Theorem 3.4.** If the condition

$$\sum_{n=1}^{\infty} \exp(\varepsilon n) (|1 - a_n| + |1 + b_n| + |p_n| + |q_n|) < \infty \tag{3.12}$$

holds then  $L$  has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

*Proof.* From (2.4) we find that

$$|A_{nm}^{ij}| \leq C \exp\left[-\frac{\varepsilon}{p+2}(n+m)\right]; i, j = 1, 2; n, m \in \mathbb{N}, \tag{3.13}$$

where  $C > 0$  is a constant. Additionally, using (3.6) and (3.13), we observe that the function  $F$  has an analytic continuation to the half-plane  $\text{Im } z > -\frac{\varepsilon}{p+2}$ . Since  $F$  is a  $4\pi$  periodic function, the limit points of its zeros in  $S$  cannot lie in  $[-\pi, 3\pi]$ . Also from Theorem 3.2 we find that the bounded sets  $P_1$  and  $P_2$  have a finite number of elements. Using analyticity of  $F$  in  $\text{Im } z > -\frac{\varepsilon}{p+2}$ , we get that all the zeros of  $F$  in  $S$  are of finite multiplicity. Therefore, we obtain the finiteness of eigenvalues and spectral singularities of  $L$  from (3.10).  $\square$

We can see that the condition (3.12) guarantees the analytic continuation of  $F$  from the real axis to lower half-plane. So the finiteness of the eigenvalues and spectral singularities of  $L$  are obtained as a result of this analytic continuation.

On the other hand, let us assume that

$$\sum_{n=1}^{\infty} \exp(\varepsilon n^\delta) (|1 - a_n| + |1 + b_n| + |p_n| + |q_n|) < \infty, \varepsilon > 0, \delta \in \left[\frac{1}{2}, 1\right) \tag{3.14}$$

which is weaker than (3.12). Under this condition, the function  $F$  is analytic in  $\mathbb{C}_+$  and infinitely differentiable on the real axis. But it does not have an analytic continuation from the real axis to lower half-plane. Therefore, the finiteness of eigenvalues and spectral singularities of  $L$  could be shown in another way by using Theorem 3.4 under the condition (3.14). For that we will use the following:

**Theorem 3.5.** ([8]) Let the  $4\pi$  periodic function  $g$  be analytic in  $\mathbb{C}_+$ , all its derivatives be continuous in  $\overline{\mathbb{C}}_+$  and

$$\sup_{z \in S} |g^{(k)}(z)| \leq B_k, \quad k \in \mathbb{N} \cup \{0\}.$$

Moreover, if the set  $G \subset [-\pi, 3\pi]$  with Lebesgue measure zero is the set of all zeros of the function  $g$  with infinite multiplicity in  $S$  and if

$$\int_0^\omega \ln M(s) d\mu(G_s) = -\infty, \tag{3.15}$$

where  $M(s) = \inf_k \frac{B_k s^k}{k!}$  and  $\mu(G_s)$  is the Lebesgue measure of  $s$ -neighborhood of  $G$  and  $\omega \in (-\pi, 3\pi)$  is an arbitrary constant, then  $g \equiv 0$  in  $\overline{\mathbb{C}}_+$ .

Under the condition (3.14), we find from (2.4) and (3.6) that

$$|F^{(k)}(z)| \leq B_k, \quad k \in \mathbb{N} \cup \{0\},$$

where

$$B_k = (p+2)^k C \sum_{m=1}^\infty m^k \exp\left(-\frac{\varepsilon}{p+2} m^\delta\right),$$

and  $C > 0$  is a constant. Also we can get the following estimate,

$$B_k \leq (p+2)^k C \int_0^\infty x^k \exp\left(-\frac{\varepsilon}{p+2} x^\delta\right) dx \leq D d^k k! k^{\frac{1-\delta}{\delta}}, \tag{3.16}$$

where  $D$  and  $d$  are constants depending  $C$ ,  $\varepsilon$  and  $\delta$ .

**Theorem 3.6.** Under the condition (3.14),  $P_4 = \emptyset$ .

*Proof.* From using Theorem 3.5 we obtain that the function  $F$  satisfies the condition

$$\int_0^\omega \ln M(s) d\mu(P_{4,s}) > -\infty \tag{3.17}$$

instead of (3.15), where  $M(s) = \inf_k \frac{B_k s^k}{k!}$ ,  $k \in \mathbb{N} \cup \{0\}$ ,  $\mu(P_{4,s})$  is the Lebesgue measure of  $s$ -neighborhood of  $P_4$  and  $B_k$  is defined by (3.16). Then we get

$$M(s) = D \exp\left\{-\frac{1-\delta}{\delta} e^{-1} d^{-\frac{\delta}{1-\delta}} s^{-\frac{\delta}{1-\delta}}\right\}. \tag{3.18}$$

by (3.16) and it follows from (3.17) and (3.18) that

$$\int_0^\omega s^{-\frac{\delta}{1-\delta}} d\mu(P_{4,s}) < \infty. \tag{3.19}$$

However, since  $\frac{\delta}{1-\delta} \geq 1$ , we see that (3.19) holds for arbitrary  $s$  if and only if  $\mu(P_{4,s}) = 0$ , i.e.  $P_4 = \emptyset$ .  $\square$

**Theorem 3.7.**  $L$  has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity under the condition (3.14).

*Proof.* By using Theorem 3.2 and Theorem 3.6 we find that  $P_3 = \emptyset$ . So the bounded sets  $P_1$  and  $P_2$  have no limit points, it means that the function  $F$  has only a finite number of zeros in  $S$ . Moreover because of  $P_4 = \emptyset$ , these zeros are of finite multiplicity.  $\square$



#### 4. Principal Functions of L

Let  $\lambda_1, \lambda_2, \dots, \lambda_k$  and  $\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_\nu$  denote respectively the zeros of  $F$  in  $S_0$  and  $[-\pi, 3\pi]$  with multiplicities  $m_1, m_2, \dots, m_k$  and  $m_{k+1}, m_{k+2}, \dots, m_\nu$ .

**Definition 4.1.** Let  $\lambda = \lambda_0$  be an eigenvalue of  $L$ . If the vectors  $y_n, \frac{d}{d\lambda} y_n, \frac{d^2}{d\lambda^2} y_n, \dots, \frac{d^\nu}{d\lambda^\nu} y_n$ ;

$$\frac{d^j}{d\lambda^j} y := \left\{ \frac{d^j}{d\lambda^j} y_n \right\}_{n \in \mathbb{N}}, \quad j = 0, 1, \dots, \nu; \quad n \in \mathbb{N}$$

satisfy the conditions

$$\begin{aligned} (\ell y)_n - \lambda_0 y_n &= 0, \\ \left( \ell \left( \frac{d^j}{d\lambda^j} y \right) \right)_n - \lambda_0 \frac{d^j}{d\lambda^j} y_n - \frac{d^{j-1}}{d\lambda^{j-1}} y_n &= 0, \quad j = 1, 2, \dots, \nu; \quad n \in \mathbb{N} \end{aligned}$$

then the vector  $y_n$  is called the eigenvector corresponding to the eigenvalue  $\lambda = \lambda_0$  of  $L$ . The vectors  $\frac{d}{d\lambda} y_n, \frac{d^2}{d\lambda^2} y_n, \dots, \frac{d^\nu}{d\lambda^\nu} y_n$  are called the associated vectors corresponding to  $\lambda = \lambda_0$ . The eigenvector and the associated vectors corresponding to  $\lambda = \lambda_0$  are called the principal vectors of the eigenvalue  $\lambda = \lambda_0$ . The principal vectors of the spectral singularities of  $L$  are defined similarly.

We define the vectors for  $\lambda = 2 \sin \frac{\lambda}{2}$

$$\frac{d^j}{d\lambda^j} V_n(\lambda_i) = \left( \begin{array}{c} \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda=\lambda_i} \\ \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda=\lambda_i} \end{array} \right), \quad n \in \mathbb{N} \tag{4.1}$$

and

$$\begin{aligned} E_n(\lambda) &= \left( \begin{array}{c} E_n^{(1)}(\lambda) \\ E_n^{(2)}(\lambda) \end{array} \right) := e_n \left( 2 \arcsin \frac{\lambda}{2} \right) \\ &= \left( \begin{array}{c} e_n^{(1)} \left( 2 \arcsin \frac{\lambda}{2} \right) \\ e_n^{(2)} \left( 2 \arcsin \frac{\lambda}{2} \right) \end{array} \right), \end{aligned} \tag{4.2}$$

where  $j = 0, 1, \dots, m_i - 1; i = 1, 2, \dots, k, k + 1, \dots, \nu$ . Moreover, if  $y(\lambda) = \{y_n(\lambda)\} := \left( \begin{array}{c} y_n^{(1)}(\lambda) \\ y_n^{(2)}(\lambda) \end{array} \right)_{n \in \mathbb{N}}$  is a solution of (1.2), then

$$\frac{d^j}{d\lambda^j} y(\lambda) = \left\{ \left( \frac{d^j}{d\lambda^j} y_n(\lambda) \right) \right\}_{n \in \mathbb{N}} := \left\{ \left( \begin{array}{c} \left( \frac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) \right) \\ \left( \frac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) \right) \end{array} \right) \right\}$$

satisfies the following system of equation

$$\left( \begin{array}{c} a_{n+1} \frac{d^j}{d\lambda^j} y_{n+1}^{(2)}(\lambda) + b_n \frac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) + p_n \frac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) \\ a_{n-1} \frac{d^j}{d\lambda^j} y_{n-1}^{(1)}(\lambda) + b_n \frac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) + q_n \frac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) \end{array} \right) = \left( \begin{array}{c} \lambda \frac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) + j \frac{d^{j-1}}{d\lambda^{j-1}} y_n^{(1)}(\lambda) \\ \lambda \frac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) + j \frac{d^{j-1}}{d\lambda^{j-1}} y_n^{(2)}(\lambda) \end{array} \right). \tag{4.3}$$

Using (4.1)-(4.3) we obtain that

$$\begin{aligned} (\ell V(\lambda_i))_n - \lambda_0 V_n(\lambda_i) &= 0, \\ \left( \ell \left( \frac{d^j}{d\lambda^j} V(\lambda_i) \right) \right)_n - \lambda_0 \frac{d^j}{d\lambda^j} V_n(\lambda_i) - \frac{d^{j-1}}{d\lambda^{j-1}} V_n(\lambda_i) &= 0, \quad n \in \mathbb{N} \\ j &= 1, 2, \dots, m_i - 1; \quad i = 1, 2, \dots, \nu. \end{aligned}$$

In here, the vectors  $\frac{d^j}{d\lambda^j} V_n(\lambda_i)$  for  $i = 1, 2, \dots, k$  and  $\frac{d^j}{d\lambda^j} V_n(\lambda_i)$  for  $i = k + 1, k + 2, \dots, \nu$  are respectively the principal vectors of eigenvalues and spectral singularities of  $L$  where  $j = 0, 1, 2, \dots, m_i - 1$ .

**Theorem 4.2.** For  $j = 0, 1, 2, \dots, m_i - 1$ ,

$$\frac{d^j}{d\lambda^j} V_n(\lambda_i) \in \ell_2(\mathbb{N}, \mathbb{C}^2), \quad i = 1, 2, \dots, k$$

and

$$\frac{d^j}{d\lambda^j} V_n(\lambda_i) \notin \ell_2(\mathbb{N}, \mathbb{C}^2), \quad i = k + 1, k + 2, \dots, v.$$

*Proof.* From (4.2) we obtain that

$$\left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda=\lambda_i} = \sum_{t=0}^j C_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(1)}(z) \right\}_{z=z_i}, \quad n \in \mathbb{N}$$

and

$$\left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda=\lambda_i} = \sum_{t=0}^j D_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(2)}(z) \right\}_{z=z_i}, \quad n \in \mathbb{N},$$

where  $\lambda_i = 2 \sin \frac{z_i}{2}$ ,  $z_i \in S = S_0 \cup [-\pi, 3\pi]$  for  $i = 1, 2, \dots, k$  and  $C_t, D_t$  are constants depending on  $\lambda$ . Using (2.2) we get that

$$\begin{aligned} \left\{ \frac{d^t}{d\lambda^t} e_n^{(1)}(z) \right\}_{z=z_i} &= \alpha_n^{11} t^t \left( n + \frac{1}{2} \right)^t e^{iz_i(n+\frac{1}{2})} \\ &+ \sum_{m=1}^{\infty} \alpha_n^{11} \left\{ A_{nm}^{11} t^t \left( m + n + \frac{1}{2} \right)^t e^{i(m+n+\frac{1}{2})z_i} - A_{nm}^{12} t^{t+1} (m+n)^t e^{i(m+n)z_i} \right\} \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} \left\{ \frac{d^t}{d\lambda^t} e_n^{(2)}(z) \right\}_{z=z_i} &= \alpha_n^{21} t^t \left( n + \frac{1}{2} \right)^t e^{iz_i(n+\frac{1}{2})} - i(in)^t \alpha_n^{22} e^{inz_i} \\ &+ \sum_{m=1}^{\infty} \alpha_n^{21} \left\{ A_{nm}^{11} t^t \left( m + n + \frac{1}{2} \right)^t e^{i(m+n+\frac{1}{2})z_i} - A_{nm}^{12} t^{t+1} (m+n)^t e^{i(m+n)z_i} \right\} \\ &+ \sum_{m=1}^{\infty} \alpha_n^{22} \left\{ A_{nm}^{21} t^t \left( m + n + \frac{1}{2} \right)^t e^{i(m+n+\frac{1}{2})z_i} - A_{nm}^{22} t^{t+1} (m+n)^t e^{i(m+n)z_i} \right\}. \end{aligned} \quad (4.5)$$

For the principal vectors  $\frac{d^j}{d\lambda^j} V_n(\lambda_i) = \left\{ \frac{d^j}{d\lambda^j} V_n(\lambda_i) \right\}_{n \in \mathbb{N}}$  for  $j = 0, 1, \dots, m_i - 1$ ;  $i = 1, 2, \dots, k$  corresponding to the eigenvalues of  $L$ , we have

$$\frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda=\lambda_i} = \frac{1}{j!} \sum_{t=0}^j C_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(1)}(z_i) \right\} \quad (4.6)$$

$$\frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda=\lambda_i} = \frac{1}{j!} \sum_{t=0}^j D_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(2)}(z_i) \right\}. \quad (4.7)$$

Since  $\text{Im } \lambda_i > 0$  for  $i = 1, 2, \dots, k$  from (4.6) and (4.7) we find that

$$\begin{aligned} \left\| \frac{d^j}{d\lambda^j} V_n \right\|^2 &= \left( \frac{1}{j!} \right)^2 \sum_{n=1}^{\infty} \left( \left| \sum_{t=0}^j C_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(1)}(z_i) \right\} \right|^2 + \left| \sum_{t=0}^j D_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(2)}(z_i) \right\} \right|^2 \right) \\ &\leq \left( \frac{1}{j!} \right)^2 \left[ \sum_{n=1}^{\infty} \sum_{t=0}^j \max\{|C_t|, |D_t|\} \times \left( \left| \left\{ \frac{d^t}{d\lambda^t} e_n^{(1)}(z_i) \right\} \right| + \left| \left\{ \frac{d^t}{d\lambda^t} e_n^{(2)}(z_i) \right\} \right| \right) \right]^2 \end{aligned}$$

or

$$\begin{aligned} \left\| \frac{d^j}{d\lambda^j} V_n \right\|^2 &\leq \left( \frac{1}{j!} \right)^2 \left\{ \sum_{n=1}^{\infty} \left[ \sum_{t=0}^j \max \{ |C_t|, |D_t| \} \left[ (|\alpha_n^{11}| + |\alpha_n^{21}|) \left( n + \frac{1}{2} \right)^t e^{-(n+1/2) \operatorname{Im} z_i} \right) + |\alpha_n^{22}| |n|^t e^{-n \operatorname{Im} z_i} \right] \right. \\ &\quad + \sum_{t=0}^j \max \{ |C_t|, |D_t| \} \left\{ \sum_{m=1}^{\infty} (|\alpha_n^{11}| + |\alpha_n^{21}|) \left( |A_{nm}^{11}| \left| m + n + \frac{1}{2} \right|^t e^{-(m+n+\frac{1}{2}) \operatorname{Im} z_i} \right) \right. \\ &\quad \left. \left. + |A_{nm}^{12}| |m + n|^t e^{-(m+n) \operatorname{Im} z_i} \right\} \right. \\ &\quad \left. \left. + \sum_{t=0}^j |D_t| \left[ \sum_{m=1}^{\infty} |\alpha_n^{22}| \left( |A_{nm}^{21}| \left| m + n + \frac{1}{2} \right|^t e^{-(m+n+\frac{1}{2}) \operatorname{Im} z_i} + |A_{nm}^{22}| |m + n|^t e^{-(m+n) \operatorname{Im} z_i} \right) \right] \right\} \right\}^2. \end{aligned} \tag{4.8}$$

Using (4.8), if we say

$$\Omega = \left( \frac{1}{j!} \right)^2 \sum_{n=1}^{\infty} \sum_{t=0}^j \max \{ |C_t|, |D_t| \} \left\{ (|\alpha_n^{11}| + |\alpha_n^{21}|) \left( n + \frac{1}{2} \right)^t e^{-(n+\frac{1}{2}) \operatorname{Im} z_i} + |\alpha_n^{22}| |n|^t e^{-n \operatorname{Im} z_i} \right\}$$

then

$$\begin{aligned} \Omega &\leq \frac{\xi (j+1)}{(j!)^2} \sum_{n=1}^{\infty} \left[ \left( n + \frac{1}{2} \right)^j e^{-(n+\frac{1}{2}) \operatorname{Im} z_i} + n^j e^{-n \operatorname{Im} z_i} \right] \\ &< \infty \end{aligned} \tag{4.9}$$

holds, where

$$\xi = \max \{ |C_t|, |D_t| \} \max \{ (|\alpha_n^{11}| + |\alpha_n^{21}|), |\alpha_n^{22}| \}.$$

Now we define the function

$$\begin{aligned} \Psi_n(z) &= \sum_{t=0}^j \max \{ |C_t|, |D_t| \} \\ &\quad \times \left\{ \sum_{m=1}^{\infty} (|\alpha_n^{11}| + |\alpha_n^{21}|) \left( |A_{nm}^{11}| \left| m + n + \frac{1}{2} \right|^t e^{-(m+n+\frac{1}{2}) \operatorname{Im} z_i} + |A_{nm}^{12}| |m + n|^t e^{-(m+n) \operatorname{Im} z_i} \right) \right\} \\ &\quad + \sum_{t=0}^j |D_t| \left\{ \sum_{m=1}^{\infty} |\alpha_n^{22}| \left( |A_{nm}^{21}| \left| m + n + \frac{1}{2} \right|^t e^{-(m+n+\frac{1}{2}) \operatorname{Im} z_i} + |A_{nm}^{22}| |m + n|^t e^{-(m+n) \operatorname{Im} z_i} \right) \right\}. \end{aligned}$$

So we get

$$\begin{aligned} &\left( \frac{1}{j!} \right)^2 \sum_{n=1}^{\infty} \left[ \sum_{t=0}^j \max \{ |C_t|, |D_t| \} \right. \\ &\quad \times \left\{ \sum_{m=1}^{\infty} (|\alpha_n^{11}| + |\alpha_n^{21}|) \left( |A_{nm}^{11}| \left| m + n + \frac{1}{2} \right|^t e^{-(m+n+\frac{1}{2}) \operatorname{Im} z_i} + |A_{nm}^{12}| |m + n|^t e^{-(m+n) \operatorname{Im} z_i} \right) \right\} \\ &\quad \left. + \sum_{t=0}^j |D_t| \left\{ \sum_{m=1}^{\infty} |\alpha_n^{22}| \left( |A_{nm}^{21}| \left| m + n + \frac{1}{2} \right|^t e^{-(m+n+\frac{1}{2}) \operatorname{Im} z_i} + |A_{nm}^{22}| |m + n|^t e^{-(m+n) \operatorname{Im} z_i} \right) \right\} \right]^2 \\ &= \left( \frac{1}{j!} \right)^2 \left( \sum_{n=1}^{\infty} \Psi_n(z) \right)^2. \end{aligned}$$

Also, using the boundedness of  $A_{nm}^{ij}$  and  $\alpha_n^{ij}$  for  $i, j = 1, 2$ , we obtain that

$$\Psi_n(z) \leq \max\{|C_t|, |D_t|\} M \sum_{t=0}^j \sum_{m=1}^{\infty} \left\{ \left| m + n + \frac{1}{2} \right|^t e^{-(m+n+\frac{1}{2})\text{Im}z_i} + |m+n|^t e^{-(m+n)\text{Im}z_i} \right\},$$

where

$$M = \max\left\{ (|\alpha_n^{11}| + |\alpha_n^{21}|) |A_{nm}^{11}|, |\alpha_n^{22}| |A_{nm}^{21}|, (|\alpha_n^{11}| + |\alpha_n^{21}|) |A_{nm}^{12}|, |\alpha_n^{22}| |A_{nm}^{22}| \right\}.$$

If we take  $\max\{|C_t|, |D_t|\} M = N$ , we can write

$$\begin{aligned} \Psi_n(z) &\leq N \sum_{t=0}^j e^{-n\text{Im}z_i} \sum_{m=1}^{\infty} \left\{ \left( m + n + \frac{1}{2} \right)^t e^{-m\text{Im}z_i} + (m+n)^t e^{-m\text{Im}z_i} \right\} \\ &= N e^{-n\text{Im}z_i} \left\{ \sum_{m=1}^{\infty} 2 e^{-m\text{Im}z_i} + \sum_{m=1}^{\infty} e^{-m\text{Im}z_i} \left( \left( m + n + \frac{1}{2} \right) + (m+n) \right) \right. \\ &\quad \left. + \dots + \sum_{m=1}^{\infty} e^{-m\text{Im}z_i} \left( \left( m + n + \frac{1}{2} \right)^j + (m+n)^j \right) \right\} \\ &\leq N e^{-n\text{Im}z_i} \sum_{m=1}^{\infty} \sum_{t=0}^j e^{-m\text{Im}z_i} \left( \left( m + n + \frac{1}{2} \right)^t + (m+n)^t \right) \\ &\leq \kappa e^{-n\text{Im}z_i}, \end{aligned}$$

where

$$\kappa = N \sum_{t=0}^j e^{-m\text{Im}z_i} \left( \left( m + n + \frac{1}{2} \right)^t + (m+n)^t \right).$$

Hence we have

$$\left( \frac{1}{j!} \sum_{n=1}^{\infty} \Psi_n(z) \right)^2 \leq \left( \frac{1}{j!} \sum_{n=1}^{\infty} \kappa e^{-n\text{Im}z_i} \right)^2 < \infty.$$

Using (4.9) and (4.10),  $\frac{d^j}{d\lambda^j} V_n(\lambda_i) \in \ell_2(\mathbb{N}, \mathbb{C}^2)$  for  $j = 0, 1, \dots, m_i - 1; i = 1, 2, \dots, k$ .

On the other hand; since  $\text{Im}z_i = 0$  for  $j = 0, 1, \dots, m_i - 1; i = k + 1, k + 2, \dots, \nu$  from (4.4) we find that

$$\sum_{n=1}^{\infty} \left| \alpha_n^{11} t^t \left( n + \frac{1}{2} \right)^t e^{iz_i(n+\frac{1}{2})} \right|^2 = \infty.$$

However the other terms in (4.4) belongs  $\ell_2(\mathbb{N}, \mathbb{C}^2)$ , so  $\frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \notin \ell_2(\mathbb{N}, \mathbb{C}^2)$ . Likewise using (4.5) we get

$\frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \notin \ell_2(\mathbb{N}, \mathbb{C}^2)$ , and so we obtain that

$\frac{d^j}{d\lambda^j} V_n(\lambda_i) \notin \ell_2(\mathbb{N}, \mathbb{C}^2)$  for  $j = 0, 1, \dots, m_i - 1; i = k + 1, k + 2, \dots, \nu$ .  $\square$

Let us introduce the Hilbert space

$$H_{-j}(\mathbb{N}) = \left\{ y = \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix} : \sum_{n \in \mathbb{N}} (1 + |n|)^{-2j} \left( |y_n^{(1)}|^2 + |y_n^{(2)}|^2 \right) < \infty \right\}$$

for  $j = 0, 1, 2, \dots$ , where

$$\|y\|_{-j}^2 = \sum_{n \in \mathbb{N}} (1 + |n|)^{-2j} \left( |y_n^{(1)}|^2 + |y_n^{(2)}|^2 \right).$$

Then, we can obtain the following result:

**Theorem 4.3.**  $\frac{d^j}{d\lambda^j} V_n(\lambda_i) \in H_{-(j+1)}(\mathbb{N})$  for  $j = 0, 1, 2, \dots, m_i - 1$ ;  $i = k + 1, k + 2, \dots, v$ .

*Proof.* From (4.1), (4.6) and (4.7), we have

$$\begin{aligned} & \sum_{n \in \mathbb{N}} (1 + |n|)^{-2(j+1)} \left( \left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda=\lambda_i} \right|^2 + \left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda=\lambda_i} \right|^2 \right) \\ &= \sum_{n \in \mathbb{N}} \frac{(1 + |n|)^{-2(j+1)}}{(j!)^2} \left\{ \left| \sum_{t=0}^j C_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(1)}(z_i) \right\} \right|^2 + \left| \sum_{t=0}^j D_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(2)}(z_i) \right\} \right|^2 \right\} \\ &\leq \frac{1}{(j!)^2} \sum_{n=1}^{\infty} (1 + |n|)^{-2(j+1)} \left\{ \left( \sum_{t=0}^j \left| C_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(1)}(z_i) \right\} \right| \right)^2 + \left( \sum_{t=0}^j \left| D_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(2)}(z_i) \right\} \right| \right)^2 \right\} \end{aligned} \tag{4.11}$$

for  $j = 0, 1, 2, \dots, m_i - 1$ ;  $i = k + 1, k + 2, \dots, v$ . Since  $\text{Im } z_i = 0$ , using (4.11), we obtain

$$\begin{aligned} & \sum_{n \in \mathbb{N}} (1 + |n|)^{-2(j+1)} \frac{1}{(j!)^2} \left( \sum_{t=0}^j \left| C_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(1)}(z_i) \right\} \right| \right)^2 \\ &\leq \frac{1}{(j!)^2} \sum_{n=1}^{\infty} \left\{ \sum_{t=0}^j (1 + |n|)^{-(j+1)} \left( n + \frac{1}{2} \right)^t |\alpha_n^{11}| |C_t| + \sum_{t=0}^j |C_t| |\alpha_n^{11}| (1 + |n|)^{-(j+1)} \right. \\ &\quad \left. \times \left[ \sum_{m=1}^{\infty} |A_{nm}^{11}| \left( m + n + \frac{1}{2} \right)^t + |A_{nm}^{12}| (m + n)^t \right] \right\}^2 \\ &= \frac{1}{(j!)^2} \sum_{n=1}^{\infty} \left\{ \left( \sum_{t=0}^j (1 + |n|)^{-(j+1)} \left( n + \frac{1}{2} \right)^t |\alpha_n^{11}| |C_t| \right)^2 + 2 (1 + |n|)^{-2(j+1)} |\alpha_n^{11}|^2 \left[ \sum_{t=0}^j \left( n + \frac{1}{2} \right)^t |C_t| \right] \right. \\ &\quad \left. \times \left[ \sum_{t=0}^j |C_t| \sum_{m=1}^{\infty} |A_{nm}^{11}| \left( m + n + \frac{1}{2} \right)^t + |A_{nm}^{12}| (m + n)^t \right] \right. \\ &\quad \left. + \left( \sum_{t=0}^j |C_t| (1 + |n|)^{-(j+1)} |\alpha_n^{11}| \sum_{m=1}^{\infty} |A_{nm}^{11}| \left( m + n + \frac{1}{2} \right)^t + |A_{nm}^{12}| (m + n)^t \right)^2 \right\}. \end{aligned} \tag{4.12}$$

From (4.12), (2.1) and (2.4), we first get

$$\begin{aligned}
 & \left( \sum_{t=0}^j |C_t| |\alpha_n^{11}| (1 + |n|)^{-(j+1)} \sum_{m=1}^{\infty} \left( |A_{nm}^{11}| \left(m + n + \frac{1}{2}\right)^t + |A_{nm}^{12}| (m + n)^t \right) \right)^2 \\
 & \leq 4 \left( \sum_{t=0}^j |C_t| |\alpha_n^{11}| \sum_{m=1}^{\infty} (1 + |n|)^{-(j+1)} \left(m + n + \frac{1}{2}\right)^t \right. \\
 & \quad \left. \times C \sum_{j=n+\lceil \frac{m}{2} \rceil}^{\infty} (|1 - a_j| + |1 + b_j| + |p_j| + |q_j|) e^{-\varepsilon j^\delta} e^{\varepsilon j^\delta} \right)^2 \\
 & \leq 4 \left\{ \sum_{t=0}^j |\alpha_n^{11}| \sum_{m=1}^{\infty} (1 + |n|)^{-(j+1)} \left(m + n + \frac{1}{2}\right)^t C \exp(-\varepsilon ((n + m) / 4)^\delta) \right. \\
 & \quad \left. \times \sum_{j=n+\lceil \frac{m}{2} \rceil}^{\infty} e^{\varepsilon j^\delta} (|1 - a_j| + |1 + b_j| + |p_j| + |q_j|) \right\}^2 \\
 & \leq C_1 \left( \sum_{t=0}^j (1 + |n|)^{-(j+1)} \sum_{m=1}^{\infty} \left(m + n + \frac{1}{2}\right)^t \exp(-\varepsilon ((n + m) / 4)^\delta) \right)^2 \\
 & \leq C_1 \left( \sum_{t=0}^j (1 + |n|)^{-(j+1)} \sum_{m=1}^{\infty} \left(m + n + \frac{1}{2}\right)^t \exp(-\varepsilon ((n + m) / 4)^{1/2}) \right)^2 \\
 & \leq C_1 \left( \sum_{t=0}^j (1 + |n|)^{-(j+1)} \sum_{m=1}^{\infty} \left(m + n + \frac{1}{2}\right)^t \exp(-\varepsilon \sqrt{2} (n^{1/2} + m^{1/2}) / 4) \right)^2 \\
 & = C_1 (1 + |n|)^{-2(j+1)} \exp(-\varepsilon \sqrt{2} n^{1/2} / 2) \left( \sum_{t=0}^j \sum_{m=1}^{\infty} \left(m + n + \frac{1}{2}\right)^t \exp(-\varepsilon \sqrt{2} m^{1/2} / 4) \right)^2 \\
 & = Y \exp(-\varepsilon \sqrt{2} n^{1/2} / 2) (1 + |n|)^{-2(j+1)}, \tag{4.13}
 \end{aligned}$$

where

$$\begin{aligned}
 C_1 &= \left( 2C |\alpha_n^{11}| \sum_{j=n+\lceil \frac{m}{2} \rceil}^{\infty} e^{\varepsilon j^\delta} (|1 - a_j| + |1 + b_j| + |p_j| + |q_j|) \right)^2 \\
 Y &= C_1 \left( \sum_{t=0}^j \sum_{m=1}^{\infty} (m + n + 1/2)^t \exp(-\varepsilon \sqrt{2} m^{1/2} / 4) \right)^2.
 \end{aligned}$$

So we get from (4.13)

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \left( \sum_{t=0}^j |C_t| (1 + |n|)^{-(j+1)} |\alpha_n^{11}| \sum_{m=1}^{\infty} |A_{nm}^{11}| \left(m + n + \frac{1}{2}\right)^t + |A_{nm}^{12}| (m + n)^t \right)^2 \\
 & \leq Y \sum_{n=1}^{\infty} \exp(-\varepsilon \sqrt{2} n^{1/2} / 2) (1 + |n|)^{-2(j+1)} \\
 & < \infty. \tag{4.14}
 \end{aligned}$$

Finally, using (4.12) and (4.13) we find

$$\begin{aligned} & \sum_{n=1}^{\infty} 2 \left\{ \left[ \sum_{t=0}^j |\alpha_n^{11}| |C_t| (1 + |n|)^{-(j+1)} \left( n + \frac{1}{2} \right)^t \right] \right. \\ & \quad \times \left[ \sum_{t=0}^j |C_t| |\alpha_n^{11}| \sum_{m=1}^{\infty} (1 + |n|)^{-(j+1)} \left( \left( m + n + \frac{1}{2} \right)^t |A_{nm}^{11}| \right) + (m + n)^t |A_{nm}^{12}| \right] \left. \right\} \\ & \leq T \sum_{n=1}^{\infty} \left[ \sum_{t=0}^j (1 + |n|)^{-2(j+1)} \left( n + \frac{1}{2} \right)^t \exp(-\varepsilon \sqrt{2}n^{1/2}/4) \right] \\ & < \infty, \end{aligned} \tag{4.15}$$

where

$$T = |\alpha_n^{11}| G^{1/2} \max |C_t|$$

and also expression of the left side of (4.13) is obviously convergent. Therefore, we get from (4.14) and (4.15)

$$\sum_{n \in \mathbb{N}} (1 + |n|)^{-2(j+1)} \frac{1}{(j!)^2} \left( \sum_{t=0}^j \left| C_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(1)}(z_i) \right\} \right| \right)^2 < \infty$$

and similarly

$$\sum_{n \in \mathbb{N}} (1 + |n|)^{-2(j+1)} \frac{1}{(j!)^2} \left( \sum_{t=0}^j \left| D_t \left\{ \frac{d^t}{d\lambda^t} e_n^{(2)}(z_i) \right\} \right| \right)^2 < \infty.$$

As a result  $\frac{d^j}{d\lambda^j} V_n(\lambda_i) \in H_{-(j+1)}(\mathbb{N})$  for  $j = 0, 1, 2, \dots, m_i - 1$ ;  $i = k + 1, k + 2, \dots, \nu$ .  $\square$

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