Filomat 33:18 (2019), 6015–6024
https://doi.org/10.2298/FIL1918015Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Geometry of Trans-Para-Sasakian Manifolds

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Abstract. In this paper, we introduce the trans-para-Sasakian manifolds and we study their geometry. These manifolds are an analogue of the trans-Sasakian manifolds in the Riemannian geometry. We shall investigate many curvature properties of these manifolds and we shall give many conditions under which the manifolds are either $\eta-$Einstein or Einstein manifolds.

1. Introduction

In Grey-Hervella classification of almost Hermitian manifolds (see [3]), there appears a class, $\mathcal{W}_4$, of Hermitian manifolds which are closely related to locally conformal *Kähler* manifolds. An almost contact structure on a manifold M is called a *trans-Sasakian structure* (see [8]) if the product manifold $M\times\mathbb{R}$ belongs to the class $\mathcal{W}_4$. The class $C_6\bigoplus C_5$ (see [6], [7]) coincides with the class of trans-Sasakian structures of type (α,β). In fact, in (see [7]), local nature of the two subclasses, namely the C_5 and the C_6 structures, of trans-Sasakian structures are characterized completely. We note the that trans-Sasakian structures of type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are cosympletic (see [1]), $\beta-$Kenmotsu (see [4]) and $\alpha-$Sasakian (see [4]), respectively. We consider the trans-para-Sasakian manifolds as an analogue of the trans-Sasakian manifolds. A trans-para-Sasakian manifold is a trans-para-Sasakian structure of type (α,β), where α and β are smooth functions. The trans-para-Sasakian manifolds of types (α,β), and are respecively the para-cosympletic, para-Sasakian (in case $\alpha=1$, these are just the para-Sasakian manifolds; in case $\alpha=-1$, these are the quasi-para-Sasakian manifolds, see [11]) and para-Kenmotsu (for the case $\beta=1$ see [12]). In the second section, we give the formal definition of trans-para-Sasakian manifolds of type (α,β) and we prove some basic properties. We give an example for a 3-dimensional trans-para-Sasakian manifold. In the last section, we investigate the curvature properties of the trans-para-Sasakian manifolds. Further, we find many conditions under which the manifolds are either $\eta-$Einstein or Einstein manifolds.

2010 *Mathematics Subject Classification*. 53D15
Keywords. trans-para-Sasakian manifolds, 3-dimensional trans-para-Sasakian manifolds, $\xi-$sectional curvature
Received: 04 April 2019; Accepted: 24 August 2019
Communicated by Ljubica Velimirović
Research supported by Contract DN 12/3/12.12.2017 and Contract 80-10-31/10.04.2019 with the Sofia University "St.Kl.Ohridski".
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2. Preliminaries

A (2n+1)-dimensional smooth manifold $M^{(2n+1)}$ has an *almost paracontact structure* (φ, ξ, η) if it admits a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following compatibility conditions

$$\begin{aligned} &(i) \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \\ &(ii) \quad \eta(\xi) = 1 \quad \varphi^2 = id - \eta \otimes \xi, \\ &(iii) \quad \text{distribution } \mathbb{D} : p \in M \longrightarrow \mathbb{D}_p \subset T_pM : \\ &\qquad \mathbb{D}_p = Ker\eta = \{X \in T_pM : \eta(X) = 0\} \text{ is called } \textit{paracontact} \\ &\qquad \textit{distribution} \text{ generated by } \eta. \end{aligned} \tag{1}$$

The tensor field φ induces an almost paracomplex structure [5] on each fibre on $\mathbb{D}$ and $(\mathbb{D}, \varphi, g_{|\mathbb{D}})$ is a $2n$-dimensional almost paracomplex distribution. Since g is non-degenerate metric on M and ξ is non-isotropic, the paracontact distribution $\mathbb{D}$ is non-degenerate.

An immediate consequence of the definition of the almost paracontact structure is that the endomorphism φ has rank $2n$, $\varphi\xi = 0$ and $\eta \circ \varphi = 0$, (see [1, 2] for the almost contact case).

If a manifold $M^{(2n+1)}$ with (φ, ξ, η)-structure admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2}$$

then we say that $M^{(2n+1)}$ has an almost paracontact metric structure and g is called *compatible*. Any compatible metric g with a given almost paracontact structure is necessarily of signature $(n+1, n)$.

Note that setting $Y = \xi$, we have $\eta(X) = g(X, \xi)$.

Further, any almost paracontact structure admits a compatible metric.

Definition 2.1. *If* $g(X, \varphi Y) = d\eta(X, Y)$ *(where* $d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y])$ *then* η *is a paracontact form and the almost paracontact metric manifold* $(M, \varphi, \eta, \xi, g)$ *is said to be a* paracontact metric manifold.

A paracontact metric manifold for which ξ is Killing is called a $K-$ *paracontact manifold*. A paracontact structure on $M^{(2n+1)}$ naturally gives rise to an almost paracomplex structure on the product $M^{(2n+1)} \times \mathfrak{R}$. If this almost paracomplex structure is integrable, then the given paracontact metric manifold is said to be a *para-Sasakian*. Equivalently, (see [10]) a paracontact metric manifold is a para-Sasakian if and only if

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{3}$$

for all vector fields X and Y (where ∇ is the Livi-Civita connection of g).

Definition 2.2. *If* $(\nabla_X \varphi)Y = \alpha(-g(X, Y)\xi + \eta(Y)X) + \beta(g(X, \varphi Y)\xi + \eta(Y)\varphi X)$*, then the manifold* $(M^{(2n+1)}, \varphi, \eta, \xi, g)$ *is said to be a* trans-para-Sasakian manifold.

From *Definition* 2.2 we have

$$\nabla_X \xi = -\alpha\varphi X - \beta(X - \eta(X)\xi). \tag{4}$$

Definition 2.3. *A* $(2n+1)$*-dimensional almost paracontact metric manifold is called normal if* $N(X, Y) - 2d\eta(X, Y)\xi = 0$*, where* $N(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$ *is the Nijenhuis torsion tensor of* φ *(see [10]).*

Denoting by £ the Lie differentiation of g, we see

Proposition 2.4. *Let* $(M^{(2n+1)}, \varphi, \eta, \xi, g)$ *be a trans-para-Sasakian manifold. Then we have*

$$(\nabla_X \eta)Y = \alpha g(X, \varphi Y) - \beta(g(X, Y) - \eta(X)\eta(Y)), \tag{5}$$

$$d\eta(X, Y) = \alpha g(X, \varphi Y), \tag{6}$$

$$(\pounds_\xi g)(X,Y) = -2\beta(g(X,Y) - \eta(X)\eta(Y)), \tag{7}$$

$$\pounds_\xi \varphi = 0, \tag{8}$$

$$\pounds_\xi \eta = 0, \tag{9}$$

where $X, Y \in T_pM$.

Since the proof of *Proposition* 2.4 follows by routine calculation, we shall omit it.

From *Proposition* 2.4 we see that $(M^{(2n+1)}, \varphi, \eta, \xi, g)$ is normal.

Example 2.5. *Let us consider the 3-dimensional manifold $M^3 = \{(x,y,z) : (x,y,z) \in \Re^3_1\}, z \neq 0$, where (x,y,z) are the standard coordinates in $\Re^3_1$.We choose the vector fields*

$$E_1 = e^z(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}). \quad E_2 = e^z\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of M. We define an almost paracontact structure (φ, ξ, η) and a pseudo-Riemannian metric g in the following way:

$$\varphi E_1 = E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0$$
$$\xi = E_3, \quad \eta(E_3) = 1, \quad \eta(E_1) = \eta(E_2) = 0,$$
$$g(E_1, E_1) = g(E_3, E_3) = -g(E_2, E_2) = 1,$$
$$g(E_i, E_j) = 0, \quad i \neq j \in \{1, 2, 3\}.$$

By the definition of Lie bracket, we have

$$[E_1, E_2] = ye^zE_2 - e^{2z}E_3, \quad [E_2, E_3] = -E_2, \quad [E_1, E_3] = -E_3.$$

Then $(M, \varphi, \xi, \eta, g)$ is a 3-dimensional almost paracontact manifold. The Koszul equality becomes

$$\nabla_{E_1}E_1 = E_3, \quad \nabla_{E_1}E_2 = -\tfrac{1}{2}e^{2z}E_3, \quad \nabla_{E_1}E_3 = -E_1 - \tfrac{1}{2}e^{2z}E_2,$$
$$\nabla_{E_2}E_1 = -ye^zE_2 + \tfrac{1}{2}e^{2z}E_3, \quad \nabla_{E_2}E_2 = -ye^zE_1 - E_3, \quad \nabla_{E_2}E_3 = -\tfrac{1}{2}e^{2z}E_1 - E_2,$$
$$\nabla_{E_3}E_1 = -\tfrac{1}{2}e^{2z}E_2, \quad \nabla_{E_3}E_2 = -\tfrac{1}{2}e^{2z}E_1, \quad \nabla_{E_3}E_3 = 0.$$

We have $\nabla_{E_1}\xi = -\alpha\varphi E_1 - \beta E_2$, $\nabla_{E_2}\xi = -\alpha\varphi E_2 - \beta E_2$, $\nabla_\xi\xi = 0$ for $E_3 = \xi$, where $\alpha = \frac{1}{2}e^{2z}$ and $\beta = 1$.

Again, by virtue of (5) *and $(\nabla_X\eta)Y = X(\eta(Y)) - \eta(\nabla_XY)$ we obtain*

$$(\nabla_{E_1}\eta)E_1 = -\beta = -1, \quad (\nabla_{E_2}\eta)E_1 = -\alpha = -\frac{1}{2}e^{2z}, \quad (\nabla_{E_3}\eta)E_1 = 0.$$

Thus from above the calculation the condition (4) *and* (5) *are satisfied and the structure (φ, ξ, η, g) is a trans-para-Sasakian structure of type (α, β), where $\alpha = \frac{1}{2}e^{2z}$ and $\beta = 1$. Consequently $(M^3, \varphi, \xi, \eta, g)$ is a trans-para-Sasakian manifold.*

Finally, the sectional curvature $K(\xi, X) = \epsilon_X R(X, \xi, \xi, X)$, where $|X| = \epsilon_X = \pm 1$, of a plane section spanned by ξ and the vector X orthogonal to ξ is called *ξ-sectional curvature*, where denoting by R the curvature tensor of ∇.

3. Some curvature properties of trans-para-Sasakian manifolds

We begin with the following Lemma.

Lemma 3.1. *Let $(M^{(2n+1)}, \varphi, \eta, \xi, g)$ be a trans-para-Sasakian manifold. Then we have*

$$R(X,Y)\xi = -(\alpha^2 + \beta^2)(\eta(Y)X - \eta(X)Y) - 2\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y) - \\ -X(\alpha)\varphi Y + Y(\alpha)\varphi X + Y(\beta)\varphi^2X - X(\beta)\varphi^2Y. \tag{10}$$

Proof. Using $Definition$ 2.2, we obtain

$$\nabla_X\nabla_Y\xi = \nabla_X(-\alpha\varphi Y - \beta(Y - \eta(Y)\xi) =$$

$$= -X(\alpha)\varphi Y - \alpha\nabla_X\varphi Y - X(\beta)\varphi^2 Y - \beta\nabla_X Y - \beta(X\eta(Y))\xi -$$

$$-\alpha\beta\eta(Y)\varphi X - \beta^2\eta(Y)X + \beta^2\eta(X)\eta(Y)\xi,$$

From here and (4), we get

$$R(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi =$$

$$= -X(\alpha)\varphi Y + Y(\alpha)\varphi X - \alpha((\nabla_X\varphi)Y - (\nabla_Y\varphi)X) -$$

$$-X(\beta)\varphi^2 Y + Y(\beta)\varphi^2 X + \beta((\nabla_X\eta)Y - (\nabla_Y\eta)X)\xi -$$

$$-\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y) - \beta^2(\eta(Y)X - \eta(X)Y),$$

which in view of $Definition$ 2.2 and (5) gives (10). □

Lemma 3.1 yields the following

Proposition 3.2. *If* $(M^{(2n+1)},\varphi,\eta,\xi,g)$ *is a trans-para-Sasakian manifold, then it is of* $\xi-$*sectional curvature* $K(\xi,X) = -\epsilon_X(\alpha^2+\beta^2-\xi(\beta))$.

In a trans-para-Sasakian manifolds the functions α and β can not be arbitrary. This fact is shown in the following

Theorem 3.3. *In trans-para-Sasakian manifold, we have*

$$R(\xi,X)\xi = (\alpha^2+\beta^2-\xi(\beta))(X-\eta(X)\xi), \quad (11)$$

$$2\alpha\beta - \xi(\alpha) = 0. \quad (12)$$

Proof. Using (10) in $R(\xi,Z,X,Y) = R(X,Y,\xi,Z)$, we get

$$R(\xi,Z)X = -(\alpha^2+\beta^2)(g(X,Z)-\eta(X)Z) - 2\alpha\beta(g(\varphi X,Z)\xi + \eta(X)\varphi Z) + \quad (13)$$

$$+X(\alpha)\varphi Z + g(\varphi X,Z)grad\alpha - X(\beta)(Z-\eta(Z)\xi) - g(\varphi X,\varphi Z)grad\beta.$$

From (10), we get

$$R(\xi,X)\xi = (\alpha^2+\beta^2-\xi(\beta))(X-\eta(X)\xi) + (2\alpha\beta-\xi(\alpha))\varphi Y,$$

while gives us (10)

$$R(\xi,X)\xi = (\alpha^2+\beta^2-\xi(\beta))(X-\eta(X)\xi) - (2\alpha\beta-\xi(\alpha))\varphi Y.$$

The above two equations provide (11) and (12). □

From *Lemma* 3.1, we have the following

Proposition 3.4. *In a* $(2n+1)-$*dimensional tras-para-Sasakian manifold, we have*

$$Ric(X,\xi) = -(2n(\alpha^2+\beta^2)-\xi(\beta))\eta(X) + (2n-1)X(\beta) - \varphi X(\alpha), \quad (14)$$

$$Q\xi = -(2n(\alpha^2+\beta^2)-\xi(\beta))\xi + (2n-1)grad\beta + \varphi(grad\alpha), \quad (15)$$

where Ric is the Ricci tensor and Q is the Ricci operator given by

$$Ric(X,Y) = g(QX,Y). \quad (16)$$

Corollary 3.5. *If in a* $(2n+1)-$*dimensional trans-para-Sasakian manifold we have* $\varphi(grad\alpha) = -(2n-1)grad\beta$, *then*

$$\xi(\beta) = g(\xi, grad\beta) = -\frac{1}{2n-1}g(\xi, \varphi(grad\alpha)) = 0,$$

and hence

$$Ric(X, \xi) = -2n(\alpha^2 + \beta^2)\eta(X), \tag{17}$$

$$Q\xi = -2n(\alpha^2 + \beta^2)\xi. \tag{18}$$

From here on, we shall assume that $\varphi(grad\alpha) = -(2n-1)grad\beta$.
The Weyl-projective curvature tensor P is defined as

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}(Ric(Y, Z)X - Ric(X, Z)Y). \tag{19}$$

Hence we can state the following

Theorem 3.6. *A Weyl projectively flat trans-para-Sasakian manifold is an Einstein manifold.*

Proof. Suppose that $P = 0$. Then from equation (19), we have

$$R(X, Y)Z = \frac{1}{2n}(Ric(Y, Z)X - Ric(X, Z)Y). \tag{20}$$

From (20), we obtain

$$R(X, Y, Z, W) = \frac{1}{2n}(Ric(Y, Z)g(X, W) - Ric(X, Z)g(Y, W)). \tag{21}$$

Putting $W = \xi$ in (21), we get

$$\eta(R(X, Y)Z) = \frac{1}{2n}(Ric(Y, Z)\eta(X) - Ric(X, Z)\eta(Y)). \tag{22}$$

Again taking $X = \xi$, and using (10) and (17), we get

$$Ric(X, Y) = -2n(\alpha^2 + \beta^2)g(X, Y). \tag{23}$$

□

Theorem 3.7. *A trans-para-Sasakian manifold satisfying* $R(X, Y)P = 0$ *is an Einstein manifold and also it is a manifold of scalar curvature* $scal = -2n(2n + 1)(\alpha^2 + \beta^2)$.

Proof. Using (10) and (17) in (19), we get

$$\eta(P(X, Y)\xi) = 0 \tag{24}$$

and

$$\eta(P(\xi, Y)Z) = -(\alpha^2 + \beta^2)g(Y, Z) - \frac{1}{2n}Ric(Y, Z) \tag{25}$$

Now,

$$(R(X, Y)P(U, V)Z = R(X, Y)P(U, V)Z - P(R(X, Y)U, V)Z - P(U, R(X, Y)V)Z - P(U, V)R(X, Y)Z.$$

By assumption $R(X, Y)P = 0$, so we have

$$R(X, Y)P(U, V)Z - P(R(X, Y)U, V)Z - P(U, R(X, Y)V)Z - P(U, V)R(X, Y)Z = 0. \tag{26}$$

Therefore

$$g(R(\xi,Y)P(U,V)Z,\xi)-g(P(R(\xi,Y)U,V)Z,\xi)-g(P(U,R(\xi,Y)V)Z,\xi)-g(P(U,V)R(\xi,Y)Z,\xi)=0.$$

From this, it follows that,

$$-P(U,V,Z,Y)+\eta(Y)\eta(P(U,V)Z)-\eta(U)\eta(P(Y,V)Z)+ \tag{27}$$
$$+g(Y,U)\eta(P(\xi,V)Z)-\eta(V)\eta(P(U,Y)Z)+g(Y,V)\eta(P(U,\xi)Z)-\eta(Z)\eta(P(U,V)Y)=0.$$

Let $\{e_i\}$, $i=1,...,2n+1$ be an orthonormal basis. Then summing up for $1\leq i\leq 2n+1$ of the relation (27) for $Y=U=e_i$ yields

$$2n\eta(P(\xi,V)Z)+\eta(Z)P(V,e_i,e_i,\xi)=0. \tag{28}$$

From (25), we have

$$Ric(V,Z)=-2n(\alpha^2+\beta^2)g(Y,Z)-((2n+1)(\alpha^2+\beta^2)+\frac{scal}{2n}). \tag{29}$$

Taking $Z=\xi$ in (29) and using (17) we obtain

$$scal=-2n(2n+1)(\alpha^2+\beta^2) \quad and \quad Ric(V,Z)=-2n(\alpha^2+\beta^2)g(Y,Z) \tag{30}$$

□

The Weyl-conformal tensor C is defined by

$$C(X,Y)Z=R(X,Y)Z-\frac{1}{2n-1}(g(Y,Z)QX-g(X,Z)QY+Ric(Y,Z)X- \tag{31}$$
$$-Ric(X,Z)Y)+\frac{scal}{2n(2n-1)}(g(Y,Z)X-g(X,Z)Y).$$

We have the following

Theorem 3.8. *A conformally flat trans-para-Sasakian manifold is an $\eta-$Einstein manifold.*

Proof. Suppose that $C=0$. Then from (31), we get

$$R(X,Y)Z=\frac{1}{2n-1}(g(Y,Z)QX-g(X,Z)QY+Ric(Y,Z)X- \tag{32}$$
$$-Ric(X,Z)Y)-\frac{scal}{2n(2n-1)}(g(Y,Z)X-g(X,Z)Y).$$

From the identity (32), we have

$$\eta(R(X,Y)Z)=\frac{1}{2n-1}(g(Y,Z)Ric(X,\xi)-g(X,Z)Ric(Y,\xi)+\eta(X)Ric(Y,Z)- \tag{33}$$
$$-\eta(Y)Ric(X,Z))-\frac{scal}{2n(2n-1)}(g(Y,Z)\eta(X)-g(X,Z)\eta(Y)).$$

Again taking $X=\xi$ in (33), and using (10) and (17) we get

$$Ric(X,Y)=((\alpha^2+\beta^2)+\frac{scal}{2n})g(Y,Z)-((2n+1)(\alpha^2+\beta^2)+\frac{scal}{2n})\eta(X)\eta(Y). \tag{34}$$

□

Theorem 3.9. *A trans-para-Sasakian manifold satisfying $R(X,Y)C = 0$ is an $\eta-$Einstein manifold.*

Proof. From identity (31), we have $\eta(C(X,Y)\xi) = 0$ and

$$\eta(C(\xi,Y)Z) = \frac{1}{2n-1}((\alpha^2+\beta^2)+\frac{scal}{2n})(g(Y,Z)-\eta(Y)\eta(Z)) - \frac{1}{2n-1}(Ric(Y,Z)+2n(\alpha^2+\beta^2)\eta(Y)\eta(Z)). \quad (35)$$

Now,

$$(R(X,Y)C(U,V)Z = R(X,Y)C(U,V)Z - C(R(X,Y)U,V)Z - C(U,R(X,Y)V)Z - C(U,V)R(X,Y)Z.$$

By assumption $R(X,Y)C = 0$, so we have

$$R(X,Y)C(U,V)Z - C(R(X,Y)U,V)Z - C(U,R(X,Y)V)Z - C(U,V)R(X,Y)Z = 0. \quad (36)$$

Therefore

$$g(R(\xi,Y)C(U,V)Z,\xi) - g(C(R(\xi,Y)U,V)Z,\xi) - g(C(U,R(\xi,Y)V)Z,\xi) - g(C(U,V)R(\xi,Y)Z,\xi) = 0.$$

From this, it follows that,

$$-C(U,V,Z,Y) + \eta(Y)\eta(C(U,V)Z) - \eta(U)\eta(C(Y,V)Z) + \quad (37)$$
$$+g(Y,U)\eta(C(\xi,V)Z) - \eta(V)\eta(C(U,Y)Z) + g(Y,V)\eta(C(U,\xi)Z) - \eta(Z)\eta(C(U,V)Y) = 0.$$

Let $\{e_i\}$, $i = 1,...,2n+1$ be an orthonormal basis. Then summing up for $1 \le i \le 2n+1$ of the relation (37) for $Y = U = e_i$ yields

$$\eta(C(\xi,V)Z) = 0. \quad (38)$$

From (35), we have

$$Ric(Y,Z) = (\frac{scal}{2n} + (\alpha^2+\beta^2))g(Y,Z) - ((2n+1)(\alpha^2+\beta^2) + \frac{scal}{2n})\eta(Y)\eta(Z). \quad (39)$$

□

The concicular curvature tensor $\overline{C}$ is defined by

$$\overline{C}(X,Y)Z = R(X,Y)Z - \frac{scal}{2n(2n+1)}(g(Y,Z)X - g(X,Z)Y). \quad (40)$$

We have the following

Theorem 3.10. *A trans-para-Sasakian manifold satisfying $R(X,Y)\overline{C} = 0$ is an Einstein manifold and a manifold of scalar curvature $scal = -2n(2n-1)(\alpha^2+\beta^2)$.*

Proof. From equality (40), we have $\eta(\overline{C}(X,Y)\xi) = 0$ and

$$\eta(\overline{C}(\xi,Y)Z) = (-\frac{scal}{2n(2n+1)} + (\alpha^2+\beta^2))(g(Y,Z) - \eta(Y)\eta(Z)). \quad (41)$$

Now,

$$(R(X,Y)\overline{C}(U,V)Z = R(X,Y)\overline{C}(U,V)Z - \overline{C}(R(X,Y)U,V)Z - \overline{C}(U,R(X,Y)V)Z - \overline{C}(U,V)R(X,Y)Z.$$

By assumption $R(X,Y)\overline{C} = 0$, so we have

$$R(X,Y)\overline{C}(U,V)Z - \overline{C}(R(X,Y)U,V)Z - \overline{C}(U,R(X,Y)V)Z - \overline{C}(U,V)R(X,Y)Z = 0. \quad (42)$$

Therefore

$$g(R(\xi,Y)\overline{C}(U,V)Z,\xi)-g(\overline{C}(R(\xi,Y)U,V)Z,\xi)-g(\overline{C}(U,R(\xi,Y)V)Z,\xi)-g(\overline{C}(U,V)R(\xi,Y)Z,\xi)=0.$$

From this, it follows that,

$$-\overline{C}(U,V,Z,Y)+\eta(Y)\eta(\overline{C}(U,V)Z)-\eta(U)\eta(\overline{C}(Y,V)Z)+ \tag{43}$$

$$+g(Y,U)\eta(\overline{C}(\xi,V)Z)-\eta(V)\eta(\overline{C}(U,Y)Z)+g(Y,V)\eta(\overline{C}(U,\xi)Z)-\eta(Z)\eta(\overline{C}(U,V)Y)=0.$$

Let $\{e_i\}$, $i=1,...,2n+1$ be an orthonormal basis. Then summing up for $1\leq i\leq 2n+1$ of the relation (43) for $Y=U=e_i$ yields

$$-Ric(V,Z)+\frac{scal}{2n+1}g(V,Z)-2n(\alpha^2+\beta^2)(g(V,Z)-\eta(V)\eta(Z))- \tag{44}$$

$$-\frac{scal}{2n+1}(g(V,Z)-\eta(V)\eta(Z))+\eta(Z)Ric(V,\xi)-\frac{scal}{2n+1}\eta(V)\eta(Z).$$

Using (17) in (44), we have

$$Ric(Y,Z)=-2n(\alpha^2+\beta^2)g(Y,Z) \tag{45}$$

and $scal=-2n(2n-1)(\alpha^2+\beta^2)$. □

The projective Ricci tensor is defined by

$$\widetilde{P}(X,Y)=\frac{(2n+1)}{2n}Ric(X,Y)-\frac{scal}{2n}g(X,Y). \tag{46}$$

We have the following

Theorem 3.11. *A trans-para-Sasakian manifold satisfying $R(X,Y)\widetilde{P}=0$ is an Einstein manifold and a manifold of scalar curvature $scal=-2n(2n+1)(\alpha^2+\beta^2)$.*

Proof. From the identity $R(X,Y)\widetilde{P}=0$, we get

$$\widetilde{P}(R(X,Y)U,V)+\widetilde{P}(U,R(X,Y)V)=0. \tag{47}$$

Putting $X=U=\xi$ and using (10) and (47) we have

$$-(\alpha^2+\beta^2)(\eta(Y)\widetilde{P}(\xi,V)+g(Y,V)\widetilde{P}(\xi,\xi)-\widetilde{P}(Y,V)-\eta(V)\widetilde{P}(\xi,Y))=0. \tag{48}$$

Using (47) in (48), we obtain that $Ric(X,Y)=-2n(\alpha^2+\beta^2)g(X,Y)$ and $scal=2n(2n-1)(\alpha^2+\beta^2)$. □

The pseudo-projective curvature tensor is defined by

$$\overline{P}(X,Y)Z=aR(X,Y)Z+b(Ric(Y,Z)X-Ric(X,Z)Y)-\frac{(a+2nb)scal}{2n(2n+1)}(g(Y,Z)X-g(X,Z)Y), \tag{49}$$

where a,b are constants such that $a,b\neq 0$.

We have the following

Theorem 3.12. *If a trans-para-Sasakian manifold is pseudo-projectively flat, then it is an Einstein manifold and a manifold of scalar curvature $scal=-2n(2n+1)(\alpha^2+\beta^2)$.*

Proof. Suppose that $\overline{P}(X,Y)Z = 0$, then from (49), we get

$$aR(X,Y)Z + b(Ric(Y,Z)X - Ric(X,Z)Y) - \frac{(a+2nb)scal}{2n(2n+1)}(g(Y,Z)X - g(X,Z)Y) = 0. \tag{50}$$

Taking the inner product on both sides of (50) by ξ, we get

$$a\eta(R(X,Y)Z) + b(Ric(Y,Z)\eta(X) - Ric(X,Z)\eta(Y)) - \tag{51}$$

$$-\frac{(a+2nb)scal}{2n(2n+1)}(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) = 0.$$

Putting $X = \xi$ and using (10) and (17) in (51), we get

$$-a(\alpha^2+\beta^2)(g(Y,Z) - \eta(Y)\eta(Z)) + b(Ric(Y,Z) + 2n(\alpha^2+\beta^2)\eta(Y)\eta(Z)) + \tag{52}$$

$$+(a+2nb)(\alpha^2+\beta^2)(g(Y,Z) - \eta(Y)\eta(Z)) = 0.$$

From the identity (52), we obtain that $Ric(X,Y) = -2n(\alpha^2+\beta^2)g(Y,Z)$ and $scal = -2n(2n+1)(\alpha^2+\beta^2)$. □

Theorem 3.13. *A trans-para-Sasakian manifold is satisfying the relation $R(X,Y)\overline{P} = 0$ is an Einstein manifold and a manifold of scalar curvature $scal = -2n(2n+1)(\alpha^2+\beta^2)$.*

Proof. From equality (49), we have $\eta(\overline{P}(X,Y)\xi) = 0$. Now,

$$(R(X,Y)\overline{P}(U,V)Z = R(X,Y)\overline{P}(U,V)Z - \overline{P}(R(X,Y)U,V)Z - \overline{P}(U,R(X,Y)V)Z - \overline{P}(U,V)R(X,Y)Z.$$

By assumption $R(X,Y)\overline{P} = 0$, so we have

$$R(X,Y)\overline{P}(U,V)Z - \overline{P}(R(X,Y)U,V)Z - \overline{P}(U,R(X,Y)V)Z - \overline{P}(U,V)R(X,Y)Z = 0. \tag{53}$$

Therefore

$$g(R(\xi,Y)\overline{P}(U,V)Z,\xi) - g(\overline{P}(R(\xi,Y)U,V)Z,\xi) - g(\overline{P}(U,R(\xi,Y)V)Z,\xi) - g(\overline{P}(U,V)R(\xi,Y)Z,\xi) = 0.$$

From this, it follows that,

$$-\overline{P}(U,V,Z,Y) + \eta(Y)\eta(\overline{P}(U,V)Z) - \eta(U)\eta(\overline{P}(Y,V)Z) + \tag{54}$$

$$+g(Y,U)\eta(\overline{P}(\xi,V)Z) - \eta(V)\eta(\overline{P}(U,Y)Z) + g(Y,V)\eta(\overline{P}(U,\xi)Z) - \eta(Z)\eta(\overline{P}(U,V)Y) = 0.$$

Let $\{e_i\}$, $i = 1,...,2n+1$ be an orthonormal basis. Then summing up for $1 \leq i \leq 2n+1$ of the relation (54) for $Y = U = e_i$ yields

$$\overline{P}(e_i,V,Z,e_i) - 2n\eta(\overline{P}(\xi,V)Z) + \eta(Z)\eta(\overline{P}(e_i,V)e_i) = 0. \tag{55}$$

Taking the trace of the identity, we obtain

$$-\overline{P}(e_i,V,Z,e_i) + 2n\overline{P}(\xi,V,Z,\xi) + \eta(Z)\overline{P}(\xi,e_i,e_i,\xi) = 0. \tag{56}$$

From identity (56), we get

$$aRic(V,Z) = -2n.a(\alpha^2+\beta^2)g(V,Z) + (b.scal + 2n(2n+1)b(\alpha^2+\beta^2))\eta(V)\eta(Z). \tag{57}$$

Taking $Z = \xi$ in (57) and using (17) we obtain

$$scal = -2n(2n+1)(\alpha^2+\beta^2) \quad and \quad Ric(V,Z) = -2n(\alpha^2+\beta^2)g(V,Z). \tag{58}$$

□

The PC-Bochner curvature tensor on M is defined by [9]

$$\mathbf{B}(X,Y,Z,W) = R(X,Y,Z,W) + \frac{1}{2n+4}(Ric(X,Z)g(Y,W) - Ric(Y,Z)g(X,W) + $$
$$+Ric(Y,W)g(X,Z) - Ric(X,W)g(Y,Z) + Ric(\varphi X,Z)g(Y,\varphi W) -$$
$$-Ric(\varphi Y,Z)g(X,\varphi W) + Ric(\varphi Y,W)g(X,\varphi Z) - Ric(\varphi X,W)g(Y,\varphi Z) +$$
$$+2Ric(\varphi X,Y)g(Z,\varphi W) + 2Ric(\varphi Z,W)g(X,\varphi Y) - Ric(X,Z)\eta(Y)\eta(W) +$$
$$+Ric(Y,Z)\eta(X)\eta(W) - Ric(Y,W)\eta(X)\eta(Z) + Ric(X,W)\eta(Y)\eta(Z)) +$$
$$+\frac{k-4}{2n+4}(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)) - \frac{k+2n}{2n+4}(g(Y,\varphi W)g(X,\varphi Z) -$$
$$-g(X,\varphi W)g(Y,\varphi Z) + 2g(X,\varphi Y)g(Z,\varphi W)) - \frac{k}{2n+4}(g(X,Z)\eta(Y)\eta(W) -$$
$$-g(Y,Z)\eta(X)\eta(W) + g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)),$$

where $k = -\frac{scal-2n}{2n+2}$.

Using the PC-Bochner curvature tensor we have

Theorem 3.14. *If a trans-para-Sasakian manifold is paracontact conformally flat, then* $\alpha^2 + \beta^2 = 1$.

Proof. Suppose that the manifold is paracontact conformally flat. Then the condition $\mathbf{B}(X,Y)Z = 0$ holds. Putting $X = Z = \xi$ and using (11), we obtain

$$(\alpha^2 + \beta^2 - 1)(Y - \eta(Y)\xi) = 0. \tag{59}$$

Since $Y - \eta(Y)\xi = \varphi^2 Y \neq 0$, we have $\alpha^2 + \beta^2 - 1 = 0$. □

Theorem 3.15. *If a trans-para-Sasakian manifold satisfies the condition* $\mathbf{B}(\xi,Y)Ric = 0$*, then it is either an Einstein manifold with scalar curvature* $scal = -2n(2n+1)(\alpha^2+\beta^2)$ *or* $\alpha^2 + \beta^2 = 1$.

Proof. Suppose that the condition $\mathbf{B}(\xi,Y)Ric(Z,V) = 0$ holds.This condition implies that

$$Ric(\mathbf{B}(\xi,Y)Z,V) + Ric(Z,\mathbf{B}(\xi,Y)V) = 0. \tag{60}$$

Putting $V = \xi$ and using (11), we obtain

$$(\alpha^2 + \beta^2 - 1)(Ric(Y,Z) + 2n(\alpha^2+\beta^2)g(Y,Z)) = 0. \tag{61}$$

□

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