



Affine Spheres with Prescribed Blaschke Metric

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Abstract. It is proved that the equality $\Delta \ln |\kappa - \lambda| = 6\kappa$, where κ is the Gaussian curvature of a metric tensor g on a 2-dimensional manifold is a sufficient and necessary condition for local realizability of the metric as the Blaschke metric of some affine sphere. Consequently, the set of all improper local affine spheres with nowhere-vanishing Pick invariant can be parametrized by harmonic functions.

1. Introduction

Affine spheres are still a mysterious class, even in the 2-dimensional case. It is a big contrast with the Riemannian or the pseudo-Riemannian case. It is known that the class of affine spheres is huge and only very few subclasses have been classified. Therefore the results, which allow at least to estimate the amount of affine spheres within some geometrically described subclasses are desirable. In this respect we prove that the set of all improper local affine spheres with nowhere-vanishing Pick invariant can be parametrized by harmonic functions, see Corollary 1.3 and Remark 1.4.

The following fact has been known since Blaschke's times, [1]. For an affine sphere in \mathbf{R}^3 whose Pick invariant nowhere vanishes the following equality is satisfied

$$\Delta \ln |J| = 6\kappa, \tag{1}$$

where κ is the Gaussian curvature of the Blaschke metric and J is the Pick invariant. The Laplacian is taken relative to the Blaschke metric. The affine theorem egregium says that $J = \kappa - \rho$, where ρ is the affine scalar curvature. For an affine sphere with the shape operator $S = \lambda \text{id}$ the scalar curvature is equal to λ . Therefore, the equality (1) can be written as $\Delta \ln |\kappa - \lambda| = 6\kappa$. It has not been noticed, however, that this equality is also a sufficient condition for local realizability of a metric as the Blaschke metric on an affine sphere in \mathbf{R}^3 . Realizability of prescribed objects on submanifolds belongs to fundamental problems in all types of geometries of submanifolds. Each fundamental theorem have the existence and the uniqueness part. The existence part says, roughly speaking, that having some objects on an abstract manifold, one can realize them as the induced objects on some types of submanifolds in a fixed ambient space (usually a homogeneous one) if the given objects satisfy some conditions – differential equations, called the fundamental equations. The set of the given objects should be optimal, that is, the objects should give one realization up to a specified group of transformations of the ambient space and they should be independent (except for the

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relations given by the fundamental equations). The basic version of the fundamental theorem in the theory of affine hypersurfaces says about three prescribed objects (a connection, a symmetric bilinear form and a $(1, 1)$ -tensor field) and four fundamental equations. The so called Radon's type theorems need two objects, but with additional assumptions about the rank of the bilinear form, see e.g. [3]. The main work on fundamental theorems in affine differential geometry was done in the last two decades of the last century, although the founders of affine differential geometry, e.g. J. Radon [7], proved their versions of fundamental theorems much earlier. The new attempt to affine differential geometry proposed by K. Nomizu allowed to formulate old theorems in a better way and to improve them significantly. By a realization problem we mean a question whether, or under which conditions, a given object can be a component of some geometric structure, for instance, it can be realized as an induced object on some specified type of submanifolds. Usually the realization is not unique. We also want that the conditions imposed on the given object which are sufficient for the realizability are as elegant as possible. In fact, it is quite unusual that such a theorem can be formulated, especially in a simple way. In [6] the realization problem was studied for connections in affine differential geometry. Another example of the realization problem is a recently studied question whether a given tensor field can be realized as the Ricci tensor of some types of connections and how many realizations exist, see e.g. [2], [5].

In this paper we consider the question when a metric tensor field can be realized as the Blaschke metric on an affine sphere. It turns out that only one linear differential equation is needed for formulating the sufficient and necessary condition for the realizability, if we restrict to surfaces with nowhere-vanishing Pick invariant.

Note that the 2-dimensional affine sphere with vanishing Pick invariant are classified. Namely, by the affine theorem egregium we know that if an affine sphere has vanishing Pick invariant then the Gaussian curvature of the Blaschke metric is constant. A complete classification of such affine spheres in \mathbf{R}^3 is given, for instance, in Section 5 of Chapter III in [3].

The main aim of this paper is to prove the following result

Theorem 1.1. *Let g be a metric tensor field on a 2-dimensional manifold M . It can be locally realized as the Blaschke metric on an affine sphere with nowhere-vanishing Pick invariant if and only if*

$$\Delta \ln |\kappa - \lambda| = 6\kappa \tag{2}$$

for some constant λ such that $\kappa - \lambda \neq 0$ everywhere.

This is a local result. In the compact case there is no realization problem. Indeed, any compact affine sphere is an ellipsoid with the Blaschke metric being the standard metric of constant curvature on a sphere. Also in the case where a metric is positive definite and complete the realization problem on elliptic or parabolic spheres is trivial, because complete elliptic affine spheres are ellipsoids and complete parabolic affine spheres are elliptic paraboloids.

As corollaries of Theorem 1.1 we shall prove

Corollary 1.2. *An improper locally strongly convex affine sphere in \mathbf{R}^3 with nowhere-vanishing Pick invariant is analytic.*

Not all improper affine spheres are analytic. For instance, the surface given by the equation $z = xy + \Phi(x)$, where Φ a smooth but non-analytic function is an affine sphere. This sphere has vanishing Pick invariant and its Blaschke metric is indefinite, see [3].

In the following corollary g_0 stands for the canonical metric tensor field in \mathbf{R}^2 and the harmonicity is relative to g_0 .

Corollary 1.3. *Let $\bar{g} = e^h g_0$ be a metric tensor of constant Gaussian curvature ± 2 defined in a neighborhood U of $p \in \mathbf{R}^2$. Let h be a harmonic function defined on U . Then $g = e^{\frac{h-1}{2}} g_0$ can be locally realized as the Blaschke metric in a neighborhood of p on some improper affine sphere.*

Remark 1.4. Since metric structures of the same constant Gaussian curvature are locally isometric, the above theorem says, roughly speaking, that local improper affine spheres with nowhere-vanishing Pick invariant can be parametrized by harmonic functions.

2. Affine spheres

Let $f : M \rightarrow \mathbf{R}^{n+1}$ be a hypersurface in \mathbf{R}^{n+1} . At the beginning, for simplicity, we assume that M is connected and oriented. On \mathbf{R}^{n+1} we have the volume form given by the standard determinant. The standard flat connection on \mathbf{R}^{n+1} will be denoted by $\tilde{\nabla}$. Let ξ be a transversal vector field for f (consistent with the orientation on M). The induced volume form on M is given by

$$v_\xi(X_1, \dots, X_n) = \det(f_*X_1, \dots, f_*X_n, \xi). \tag{3}$$

We have the following Gauss formula

$$\tilde{\nabla}_X(f_*Y) = f_*(\nabla_X Y) + g(X, Y)\xi, \tag{4}$$

for vector fields X, Y on M . It is known that ∇ is a torsion-free connection and g is a symmetric bilinear form on M . The connection ∇ is called the induced connection and g - the second fundamental form of f . The conformal class of g is independent of the choice of a transversal vector field ξ . A hypersurface is called non-degenerate if g is non-degenerate at each point of M .

From now on we shall consider only non-degenerate hypersurfaces. Hence the second fundamental form g on M is a metric tensor field (maybe indefinite). The induced volume form on M is, in general, different than the volume form v_g determined by g . If $v_\xi = v_g$ we say that the apolarity condition is satisfied. A transversal vector field ξ is called equiaffine if $\nabla v_\xi = 0$ on M . If a transversal vector field is equiaffine then ∇g as a cubic form (that is, $\nabla g(X, Y, Z) = (\nabla_Z g)(Y, Z)$) is totally symmetric.

The following theorem is central in the classical affine differential geometry, see [4],[3].

Theorem 2.1. *Let $f : M \rightarrow \mathbf{R}^{n+1}$ be a non-degenerate hypersurface. There exists a unique equiaffine transversal vector field ξ such that $v_\xi = v_g$.*

The unique transversal vector field is called the affine normal vector field. If the affine lines determined by the affine normal vector field meet at one point or are parallel then the hypersurface is called an affine sphere - proper in the first case and improper in the second one.

Affine spheres are also described by the affine shape operator. Namely, let ξ be the affine normal vector field for a hypersurface immersion f . By differentiating ξ relative to $\tilde{\nabla}$ we get

$$\tilde{\nabla}_X \xi = -f_*(SX) \tag{5}$$

for some $(1, 1)$ -tensor field S on M . The tensor field S is called the affine shape operator. The fact that f is an affine sphere is equivalent to the condition $S = \lambda \text{id}$, where λ is a real number, non-zero for a proper sphere and zero for an improper sphere.

We have the following fundamental theorem for equiaffine hypersurfaces, see [4], [3].

Theorem 2.2. *Let ∇ be a torsion-free connection on a simply connected manifold M , g be a symmetric bilinear non-degenerate form and S is a $(1, 1)$ -tensor field on M such that the following fundamental equations are satisfied:*

$$R(X, Y)Z = g(Y, Z)SX - g(X, Z)SY \quad - \text{ Gauss}, \tag{6}$$

$$g(X, SY) = g(SX, Y) \quad - \text{ Ricci}, \tag{7}$$

$$\nabla g(X, Y, Z) = \nabla g(Y, X, Z) \quad - \text{ I Codazzi}, \tag{8}$$

$$\nabla S(X, Y) = \nabla S(Y, X) \quad - \text{II Codazzi} \tag{9}$$

for every $X, Y, Z \in T_xM$, $x \in M$, where R is the curvature tensor for ∇ . There is an immersion $f : M \rightarrow \mathbf{R}^{n+1}$ and its equiaffine transversal vector field ξ such that ∇, g, S are the induced connection, the second fundamental form and the shape operator for the immersion f equipped with the transversal vector field ξ . The immersion is unique up to an equiaffine transformation of \mathbf{R}^{n+1} . If moreover $\nabla v_g = 0$ then ξ is the affine normal (up to a constant) for f .

For an affine sphere with $S = \lambda \text{ id}$ the fundamental equations reduce to the two equations

$$\begin{aligned} R(X, Y)Z &= \lambda\{g(Y, Z)X - g(X, Z)Y\}, \\ (\nabla g)(X, Y, Z) &= (\nabla g)(Y, X, Z). \end{aligned} \tag{10}$$

As a consequence of the fundamental theorem we have

Corollary 2.3. *Let M be a manifold equipped with a metric tensor field g , a torsion-free connection ∇ such that the equations (10) are satisfied for some constant real number λ and $\nabla v_g = 0$. For each point x of M there is a neighborhood U of x and an immersion $f : U \rightarrow \mathbf{R}^{n+1}$ which is an affine sphere whose shape operator is equal to $\lambda \text{ id}$.*

Proof. It is sufficient to define $S = \lambda \text{ id}$. \square

From now on we shall deal with the 2-dimensional case. For a connection on a 2-dimensional manifold the curvature tensor is determined by its Ricci tensor. Namely we have

$$R(X, Y)Z = \text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y \tag{11}$$

for any vectors $X, Y, Z \in T_xM$, $x \in M$. Therefore the Gauss equation for a 2-dimensional sphere is equivalent to the condition $\text{Ric} = \lambda g$. Hence we have

Corollary 2.4. *Let a metric tensor g and a torsion-free connection ∇ be given on a two-dimensional manifold M . They can be locally realized on an affine sphere if and only if the cubic form ∇g is totally symmetric, $\text{Ric} = \lambda g$ for some constant real number λ and $\nabla v_g = 0$.*

3. Affine connections, volume forms and the Ricci tensor

All connections considered in this paper are torsion-free. For any connection ∇ and a metric tensor field g we denote by K the difference tensor, that is, $K_X Y = \nabla_X Y - \hat{\nabla}_X Y$, where $\hat{\nabla}$ the Levi-Civita connection of g . Set $K(X, Y) = K_X Y$. Since both connections ∇ and $\hat{\nabla}$ are without torsion, K is symmetric for X, Y . The cubic form ∇g is symmetric if and only if K is symmetric relative to g , i.e. $g(K(X, Y), Z) = g(K(X, Z), Y)$ for every X, Y, Z . Indeed, we have

$$(\nabla_X g)(Y, Z) = (K_X g)(Y, Z) = -g(K(X, Y), Z) - g(Y, K(X, Z)).$$

Assume that the cubic form ∇g is symmetric. Since

$$2\nabla_X v_g = \text{tr}_g(\nabla_X g)(\cdot, \cdot)v_g, \tag{12}$$

the condition $\nabla v_g = 0$ is equivalent to the condition

$$\text{tr}_g(\nabla_X g)(\cdot, \cdot) = 0 \tag{13}$$

for every $X \in TM$. Since

$$\nabla g(X, Y, Z) = -2g(K(X, Y), Z), \tag{14}$$

we have $\nabla v_g = 0$ if and only if and only if $\text{tr } K_X = 0$ for every X .

Recall that the divergence of $X \in \mathcal{X}(M)$ relative to a connection ∇ is defined as $\operatorname{div}^\nabla X = \operatorname{tr} \{Y \rightarrow \nabla_Y X\}$. Recall also that for a torsion-free connection ∇ there is (locally) a volume form ν such that $\nabla \nu = 0$ if and only if the Ricci tensor of ∇ is symmetric. A pair (∇, ν) is then called an equiaffine structure. For a fixed coordinate system we have

$$\begin{aligned} (\nabla_{\partial_i} \nu)(\partial_1, \dots, \partial_n) &= \partial_i(\nu(\partial_1, \dots, \partial_n)) - \nu(\nabla_{\partial_i} \partial_1, \dots, \partial_n) - \dots - \nu(\partial_1, \dots, \nabla_{\partial_i} \partial_n) \\ &= \partial_i(\nu(\partial_1, \dots, \partial_n)) - \nu(\nabla_{\partial_1} \partial_i, \dots, \partial_n) - \dots - \nu(\partial_1, \dots, \nabla_{\partial_n} \partial_i) \\ &= \partial_i(\nu(\partial_1, \dots, \partial_n)) - (\operatorname{div}^\nabla \partial_i) \nu(\partial_1, \dots, \partial_n). \end{aligned}$$

It follows that $\nabla \nu = 0$ if and only if $\partial_i(\ln(\nu(\partial_1, \dots, \partial_n))) = \operatorname{div}^\nabla \partial_i$. Assume now that ν_g is a volume form determined by a metric tensor g (not necessarily positive definite). Then

$$\nu_g(\partial_1, \dots, \partial_n)^2 = G,$$

where $G = |\det [g_{kl}]|$ and $g_{kl} = g(\partial_k, \partial_l)$ for $1 \leq k, l \leq n$. Hence $\nabla \nu_g = 0$ if and only if $\operatorname{div}^\nabla \partial_i = (\ln G)_i / 2$ for $i = 1, \dots, n$.

Let Γ_{ij} be the Christoffel symbols of a connection ∇ . In general, we have the following formula for the Ricci tensor Ric of ∇

$$\operatorname{Ric}(\partial_i, \partial_j) = \sum_{k=1}^n (\Gamma_{ij}^k)_k - (D_j)_i + \Lambda_{ij} \tag{15}$$

where $D_i = \operatorname{div}^\nabla \partial_i$ and

$$\Lambda_{ij} = \sum_{k,l=1}^n [\Gamma_{kj}^l \Gamma_{il}^k - \Gamma_{ij}^l \Gamma_{kl}^l]. \tag{16}$$

Since the connection is torsion-free, Λ_{ij} is symmetric for i, j . From now on we assume that M is 2-dimensional and Ric is symmetric. Let r_{ij} be the components of the Ricci tensor in a coordinate system. Then we have

$$\begin{aligned} \Lambda_{11} &= \Gamma_{11}^2 \Gamma_{12}^1 + (\Gamma_{12}^2)^2 - \Gamma_{11}^1 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2, \\ \Lambda_{12} &= \Gamma_{22}^1 \Gamma_{11}^2 - \Gamma_{12}^1 \Gamma_{12}^2, \\ \Lambda_{22} &= (\Gamma_{21}^1)^2 + \Gamma_{12}^2 \Gamma_{22}^1 - \Gamma_{22}^1 \Gamma_{11}^1 - \Gamma_{22}^2 \Gamma_{12}^1, \end{aligned} \tag{17}$$

$$\begin{aligned} (\Gamma_{12}^2)_1 &= (\Gamma_{11}^2)_2 - \Lambda_{11} - r_{11}, \\ (\Gamma_{12}^1)_1 &= -(\Gamma_{12}^2)_2 + \Lambda_{12} + (D_1)_2 + r_{12}, \\ (\Gamma_{22}^1)_1 &= -(\Gamma_{22}^2)_2 + \Lambda_{22} + (D_2)_2 + r_{22}, \end{aligned} \tag{18}$$

$$\begin{aligned} D_1 &= \Gamma_{11}^1 + \Gamma_{21}^2, \\ D_2 &= \Gamma_{12}^1 + \Gamma_{22}^2. \end{aligned} \tag{19}$$

Let g be a metric tensor field on M and $\hat{\nabla}$ its Levi-Civita connection. Let (x^1, x^2) be an isothermal coordinate system for g , that is, $g_{22} = \varepsilon g_{11}$, $g_{11} = e^\varphi$ and $g_{12} = 0$ for some function φ and $\varepsilon = \pm$ depending on the signature of g . For an isothermal coordinate system we have

$$\begin{aligned} \hat{\nabla}_{\partial_1} \partial_1 &= \frac{\varphi_1}{2} \partial_1 - \varepsilon \frac{\varphi_2}{2} \partial_2, \\ \hat{\nabla}_{\partial_1} \partial_2 &= \frac{\varphi_2}{2} \partial_1 + \frac{\varphi_1}{2} \partial_2, \\ \hat{\nabla}_{\partial_2} \partial_2 &= -\varepsilon \frac{\varphi_1}{2} \partial_1 + \frac{\varphi_2}{2} \partial_2. \end{aligned} \tag{20}$$

The curvature κ of g is given by

$$\kappa = -\frac{\Delta \varphi}{2}. \tag{21}$$

Recall that for any function α we have

$$\Delta\alpha = \frac{\alpha_{11} + \varepsilon\alpha_{22}}{e^\varphi}. \tag{22}$$

Assume ∇ is a connection on M such that the cubic form ∇g is totally symmetric. The difference tensor $K = \nabla - \hat{\nabla}$ is symmetric relative to g . For the isothermal coordinate system the symmetry is equivalent to the conditions

$$K_{21}^2 = \varepsilon K_{22}^1, \quad K_{12}^1 = \varepsilon K_{11}^2. \tag{23}$$

Assume moreover that $\nabla_{\nu}g = 0$. Then $\operatorname{div}^{\nabla}\partial_1 = \varphi_1$ and $\operatorname{div}^{\nabla}\partial_2 = \varphi_2$. Since

$$\begin{aligned} \varphi_1 &= \operatorname{div}^{\nabla}\partial_1 = \hat{\Gamma}_{11}^1 + K_{11}^1 + \hat{\Gamma}_{21}^2 + K_{21}^2, \\ \varphi_2 &= \operatorname{div}^{\nabla}\partial_2 = \hat{\Gamma}_{12}^1 + K_{12}^1 + \hat{\Gamma}_{22}^2 + K_{22}^2, \end{aligned}$$

we get

$$K_{11}^1 = -K_{21}^2, \quad K_{12}^2 = -K_{12}^1. \tag{24}$$

We see that among the functions K_{jk}^i , $i, j, k = 1, 2$, only two functions are independent. We choose the functions K_{12}^1 and K_{21}^2 . Using (17), (24) and (23), by a straightforward computation, one gets

$$\begin{aligned} \Lambda_{11} &= \varphi_1 K_{21}^2 - \varepsilon\varphi_2 K_{12}^1 + 2[(K_{21}^2)^2 + \varepsilon(K_{12}^1)^2], \\ \Lambda_{12} &= -\varphi_1 K_{12}^1 - \varphi_2 K_{21}^2, \\ \Lambda_{22} &= \varphi_2 K_{12}^1 - \varepsilon\varphi_1 K_{21}^2 + 2[(K_{12}^1)^2 + \varepsilon(K_{21}^2)^2]. \end{aligned} \tag{25}$$

We have

$$g(K, K) = \frac{4}{e^\varphi} [\varepsilon(K_{12}^1)^2 + (K_{21}^2)^2]. \tag{26}$$

In the theory of affine surfaces, the function $\frac{1}{2}g(K, K)$ is called the Pick invariant and it is usually denoted by J .

Formulas (18) for the Ricci tensor $\operatorname{Ric} = r_{ij}$ of the connection ∇ , after using (25), receive the following form:

$$\begin{aligned} (K_{21}^2)_1 &= -\frac{\varphi_{11} + \varepsilon\varphi_{22}}{2} + \varepsilon(K_{12}^1)_2 \\ &\quad - \varphi_1 K_{21}^2 + \varepsilon\varphi_2 K_{12}^1 - 2[\varepsilon(K_{12}^1)^2 + (K_{21}^2)^2] - r_{11}, \\ (K_{12}^1)_1 &= -(K_{21}^2)_2 - \varphi_1 K_{12}^1 - \varphi_2 K_{21}^2 + r_{12}, \\ (K_{21}^2)_1 &= \frac{\varphi_{11} + \varepsilon\varphi_{22}}{2} + \varepsilon(K_{12}^1)_2 \\ &\quad + \varepsilon\varphi_2 K_{12}^1 - \varphi_1 K_{21}^2 + 2\varepsilon[\varepsilon(K_{12}^1)^2 + (K_{21}^2)^2] + \varepsilon r_{22}. \end{aligned} \tag{27}$$

When adding and subtracting the first and the last equalities we get the following system of equalities equivalent to (27)

$$\begin{aligned} r_{11} + \varepsilon r_{22} &= -(\varphi_{11} + \varepsilon\varphi_{22}) - 4[\varepsilon(K_{12}^1)^2 + (K_{21}^2)^2], \\ (K_{21}^2)_1 &= \varepsilon(K_{12}^1)_2 + \varepsilon\varphi_2 K_{12}^1 - \varphi_1 K_{21}^2 + \frac{-r_{11} + \varepsilon r_{22}}{2}, \\ (K_{12}^1)_1 &= -(K_{21}^2)_2 - \varphi_1 K_{12}^1 - \varphi_2 K_{21}^2 + r_{12}. \end{aligned} \tag{28}$$

The first equality is the affine theorem egregium, that is, the equality

$$\lambda = \kappa - J, \tag{29}$$

where $\lambda = (\operatorname{tr}_g \operatorname{Ric})/2$.

4. Proof of Theorem 1.1

Direct proofs of necessity of the condition (2) can be found, for instance, in [3] or [4]. The proof below also gives the necessity, but we focus on the sufficiency of the condition. Assume that for a given g and some constant λ the equality (2) is satisfied. By the fundamental theorem we are looking for a connection ∇ such that the cubic form ∇g is symmetric, the Ricci tensor of ∇ is equal to λg and $\nabla \nu_g = 0$. Let $\nabla = \hat{\nabla} + K$. As in the previous section we will carry considerations for a fixed isothermal coordinate system for g . Instead of looking for a connection ∇ we will look for a tensor field K satisfying appropriate symmetry conditions and the system of differential equations (28), where $\lambda = \frac{r_{11} + \varepsilon r_{22}}{2e^\varphi}$. It will turn out that (2) is the integrability condition for the system.

Since $g(K, K)$ should be non-zero, the tensor K should be non-zero. Suppose that $K_{21}^2 \neq 0$. Set

$$L = \varepsilon(K_{12}^1)^2 + (K_{21}^2)^2, \quad l = \frac{K_{12}^1}{K_{21}^2}. \tag{30}$$

The two functions L and l determine the difference tensor K up to sign. This is sufficient for our consideration because affine spheres go in pairs. More precisely, if $g, \nabla = \hat{\nabla} + K, S = \lambda \text{id}$ constitute the induced structure on an affine sphere then $\bar{g}, \bar{\nabla} = \hat{\nabla} - K, S = \lambda \text{id}$ form the induced structure on another affine sphere. It follows from the fact that for an affine sphere $R = \bar{R}$, where R and \bar{R} are the curvature tensors for ∇ and $\bar{\nabla}$.

We now have

$$(K_{21}^2)^2 = \frac{L}{\varepsilon l^2 + 1}. \tag{31}$$

Set $r = \lambda g$. The system (28) now becomes

$$\begin{aligned} 2 \frac{L}{e^\varphi} &= \kappa - \lambda, \\ (K_{21}^2)_1 &= \varepsilon(K_{12}^1)_2 + \varepsilon\varphi_2 K_{12}^1 - \varphi_1 K_{21}^2, \\ (K_{12}^1)_1 &= -(K_{21}^2)_2 - \varphi_1 K_{12}^1 - \varphi_2 K_{21}^2. \end{aligned} \tag{32}$$

Note that the denominator in (31) is different than 0 if and only if $L \neq 0$, that is, by the first equality from (32), if and only if $\kappa - \lambda = 2L/e^\varphi \neq 0$. Of course, we should define $L = \frac{(\kappa - \lambda)e^\varphi}{2}$. The function L is now given.

We want to prove that the system of the last two equations from (32) relative to unknown functions K_{12}^1, K_{21}^2 has a solution. Using (30) and (32) we obtain

$$\begin{aligned} L_1 &= 2\{\varepsilon(K_{12}^1)_1 K_{12}^1 + (K_{21}^2)_1 K_{21}^2\} \\ &= 2\{\varepsilon[-(K_{21}^2)_2 - \varphi_1 K_{12}^1 - \varphi_2 K_{21}^2] K_{12}^1 \\ &\quad + [\varepsilon(K_{12}^1)_2 + \varepsilon\varphi_2 K_{12}^1 - \varphi_1 K_{21}^2] K_{21}^2\} \\ &= 2\{\varepsilon l_2 (K_{21}^2)^2 - \varphi_1 L\} \end{aligned} \tag{33}$$

and consequently

$$\frac{L_1}{2L} = \frac{l_2}{l^2 + \varepsilon} - \varphi_1. \tag{34}$$

Similarly, using (30) and (32), we get

$$\begin{aligned} l_1 &= \frac{(K_{12}^1)_1 K_{21}^2 - (K_{21}^2)_1 K_{12}^1}{(K_{21}^2)^2} \\ &= \frac{[-(K_{21}^2)_2 - \varphi_1 K_{12}^1 - \varphi_2 K_{21}^2] K_{21}^2 - [\varepsilon(K_{12}^1)_2 + \varepsilon\varphi_2 K_{12}^1 - \varphi_1 K_{21}^2] K_{12}^1}{(K_{21}^2)^2} \\ &= \frac{-\frac{l_2}{2} - \varphi_2 L}{(K_{21}^2)^2} \end{aligned} \tag{35}$$

and consequently (by (31))

$$\frac{l_1}{l^2 + \varepsilon} = -\varepsilon \frac{L_2}{2L} - \varepsilon \varphi_2. \tag{36}$$

We have got the following system of differential equations relative to l

$$\begin{aligned} \frac{l_1}{l^2 + \varepsilon} &= -\varepsilon \left(\frac{1}{2} \ln |L| + \varphi \right)_2, \\ \frac{l_2}{l^2 + \varepsilon} &= \left(\frac{1}{2} \ln |L| + \varphi \right)_1. \end{aligned} \tag{37}$$

We have $\frac{l_1}{l^2 + \varepsilon} = F_1$ and $\frac{l_2}{l^2 + \varepsilon} = F_2$, where $F = \arctg l$ or $F = \ln \left| \frac{l-1}{l+1} \right|$ depending on whether $\varepsilon = 1$ or -1 . The integrability condition for (37) is

$$-\varepsilon \left(\frac{1}{2} \ln |L| + \varphi \right)_{22} = \left(\frac{1}{2} \ln |L| + \varphi \right)_{11},$$

which is equivalent to

$$\frac{(\ln |L|)_{11} + \varepsilon (\ln |L|)_{22}}{4e^\varphi} = -\frac{\varphi_{11} + \varepsilon \varphi_{22}}{2e^\varphi},$$

that is,

$$\Delta \ln |L| = 4\kappa.$$

The last condition is equivalent to

$$\Delta \ln \frac{2|L|}{e^\varphi} = 6\kappa, \tag{38}$$

which is the desired condition (2).

Thus the system (37) has a solution l around any fixed point $p \in M$. As the initial condition we take $l(p) = \beta$, where β can be any real number if $\varepsilon = 1$ and $\beta^2 \neq 1$ if $\varepsilon = -1$ (which again is the condition $J \neq 0$). From (31) and (30) we get K_{12}^1 and K_{21}^2 . The other components of the tensor K are defined by using formulas (23) and (24).

We shall now check that the obtained functions K_{12}^1, K_{21}^2 satisfy the last two equations of (32) if $L = \varepsilon(K_{12}^1)^2 + (K_{21}^2)^2$. Using also the definition of l one gets

$$\begin{aligned} L_1 &= 2\{\varepsilon(K_{12}^1)_1 K_{12}^1 + (K_{21}^2)_1 K_{21}^2\} \\ (K_{21}^2)^2 l_1 &= (K_{12}^1)_1 K_{21}^2 - (K_{21}^2)_1 K_{12}^1. \end{aligned} \tag{39}$$

We have already checked that if we substitute the quantities $(K_{12}^1)_1, (K_{21}^2)_1$ by the right hand sides of the last two formulas of (32), then we get the equalities. We shall now regard (39) as a system of algebraic linear equations with unknowns $(K_{12}^1)_1, (K_{21}^2)_1$. The main determinant of (39) is equal to $-2L \neq 0$, hence the system of equations has only one solution. It means that the functions K_{12}^1, K_{21}^2 satisfy the last two equations of (32). It follows (by (28)) that the Ricci tensor of $\nabla = \hat{\nabla} + K$ satisfies the conditions $\text{Ric}(\partial_1, \partial_1) = \varepsilon \text{Ric}(\partial_2, \partial_2)$ and $\text{Ric}(\partial_1, \partial_2) = 0$. Since the scalar curvature, say ρ , of ∇ satisfies the equality $J + \rho = \kappa$ and we have $J + \lambda = \kappa$, we see that $\text{Ric} = \lambda g$. \square

Remark 4.1. It is clear from the above proof that an affine sphere whose Blaschke metric is prescribed (and satisfies the condition (2)) is not unique. First of all the function l can be prescribed at a point p . Different functions l give different (non-equivalent modulo the affine special group acting on \mathbf{R}^3) affine spheres. Of course, the difference tensors K and $-K$ also give two different spheres if $K \neq 0$.

5. Proofs of Corollaries 1.2 and 1.3

Assume first that g satisfies (1) and (x^1, x^2) is an isothermal coordinate system for g as in the previous sections. Denote by Δ_0 the Laplacian for the coordinate system, that is, $\Delta_0\alpha = \alpha_{11} + \varepsilon\alpha_{22}$ for a function α . This is the Laplacian for the flat metric tensor field g_0 adapted to the coordinate system, that is, $g_0(\partial_1, \partial_1) = 1$, $g_0(\partial_1, \partial_2) = 0$, $g_0(\partial_2, \partial_2) = \varepsilon$, where $\varepsilon = \pm 1$. The equality (1) is equivalent to the equality

$$\Delta_0[\ln|\kappa - \lambda| + 3\varphi] = 0. \tag{40}$$

Hence

$$\ln|\kappa - \lambda| + 3\varphi = h \tag{41}$$

for some Δ_0 -harmonic function h . Thus $\kappa - \lambda = \pm e^{h-3\varphi}$. Since $\kappa = -\frac{\Delta_0\varphi}{2e^\varphi}$ (by (21) and (22)), the last equality is equivalent to the equality

$$-\frac{\Delta_0 c}{2e^c} = \mp 2 - 2\lambda e^{\frac{h-3c}{2}}, \tag{42}$$

where $c = h - 2\varphi$. If $\lambda = 0$ then the last equality becomes

$$-\frac{\Delta_0 c}{2e^c} = \pm 2. \tag{43}$$

It means, by (21) and (22), that c is such a function that the metric $\bar{g} = e^c g_0$ has constant Gaussian curvature ± 2 . Note that the metric \bar{g} is defined only locally, that is, on a domain of an isothermal coordinate map.

Proof of Corollary 1.2. Let $f : M \rightarrow \mathbf{R}^3$ be an improper locally strongly convex affine sphere with nowhere-vanishing Pick invariant, equivalently, with nowhere vanishing curvature κ of the Blaschke metric g . The atlas of isothermal coordinates is analytic. The Levi-Civita connection for \bar{g} is locally symmetric and therefore analytic. It follows that its curvature tensor is analytic and so is its Ricci tensor. The Ricci tensor is equal to $\pm 2\bar{g}$. Hence \bar{g} is analytic and c is analytic. If g is definite then a g_0 -harmonic function is analytic. It follows that $\varphi = \frac{h-c}{2}$ is analytic and consequently $g = e^{\frac{h-c}{2}} \bar{g} = e^{\frac{h-c}{2}} g_0$ is analytic.

We have proved that the Blaschke metric g is analytic. An improper affine sphere can be locally regarded as a graph of some function and its affine normal is a constant vector. More precisely, f is, up to an affine transformation of \mathbf{R}^3 , given locally by

$$\mathbf{R}^2 \supset U \ni (x^1, x^2) \rightarrow (x^1, x^2, \Psi(x^1, x^2)) \in \mathbf{R}^3,$$

where Ψ is a smooth function and its affine normal is equal to $(0, 0, 1)$. We have

$$g(\partial_i, \partial_j) = \Psi_{ij}.$$

Since g is analytic, so is Ψ . The proof of Corollary 1.2 is completed. \square

Proof of Corollary 1.3. Assume now that h is an arbitrary harmonic function on some open set $U \subset \mathbf{R}^2$ and c is such a function on U that $g = e^c g_0$ has constant Gaussian curvature 2 or -2 . Of course, such a function exists because there exist metrics of any constant Gaussian curvature. Set $\varphi = \frac{h-c}{2}$. By the consideration from the beginning of this section one sees that the equality (41) is satisfied for $\kappa = -\frac{\Delta_0\varphi}{2e^\varphi}$. \square

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