Filomat 33:18 (2019), 5949–5957
https://doi.org/10.2298/FIL1918949F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of Solutions for a System of Chandrasekhar's Equations in Banach Algebras Under Weak Topology

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Abstract. This paper is devoted to the study of a coupled system within fractional integral equations in suitable Banach algebra. In particular, we are concerned with a quadratic integral equations of Chandrasekhar type. The existence of solutions will be proved by applying fixed point theorem of a 2×2 block operator matrix defined on a nonempty, closed and convex subset of Banach algebra where the entries are weakly sequentially continuous operators.

1. Introduction

Quadratic integral equations have many useful application in numerous diverse fields of science and engineering. For example, the theory of radiative transfer, kinetic theory of gases, the theory of neutron transport and the traffic theory. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations. This study was performed via fixed point technique, see for exemple [1, 3–8, 17]. Above all, Chandrasekhar's integral equation which has been a subject of much research since its appearance for a longtime in the literature [10, 11]. In this work, we are concerned with the system of two quadratic integral equations of Chandrasekhar type

$$\begin{cases} x(t) = f_1(t, x(t)) + g(t, y(t)) \left(\int_0^t \frac{t}{t+s} p_1(s, y(s)) ds \right) \cdot u, \quad u \in X \backslash \{0\} \\ y(t) = f_2(t, y(t)) + \left(\int_0^t \frac{t}{t+s} p_2(s, x(s)) ds \right) \cdot v, \quad v \in X \backslash \{0\}, \end{cases} \tag{1}$$

for $t \in J = [0, b], b > 0$, where $f_i : J \times X \longrightarrow X$, for $i = 1, 2$ and $p_i : J \times X \longrightarrow \mathbb{R}$, for $i = 1, 2$ are weakly sequentially continuous. Here, X is a Banach algebra satisfying certain topological conditions of sequential nature. Our work is based on the fixed point theory and the measure of weak noncompactness of De Balsi

2010 *Mathematics Subject Classification*. Primary 32A65; Secondary 47H08; 47H30; 58C30
Keywords. Banach algebra; Chandrasekhar equation; Weakly sequentially continuous; Measure of weak noncompactness; Fixed point theorem.
Received: 02 March 2018; Accepted: 15 July 2018
Communicated by Snežana Živković-Zlatanović
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[14]. Note that the system (1) may be transformed into the following fixed point problem of the 2×2 block operator matrix

$$\begin{pmatrix} A & B \cdot B' \\ C & D \end{pmatrix}, \tag{2}$$

with nonlinear inputs defined on a product of Banach algebra. Our assumptions are as follows: A and C maps a unbounded closed convex nonempty subset S of a Banach algebra X into X, B, B' and D act from X into X.

In this direction, the authors A. Jeribi, N. Kaddachi and B. Krichen in [18] have established some fixed point for a 2×2 operator matrix (2), when X is a Banach algebra satisfying certain condition. An application to a system of nonlinear integral equation occurring in some physical and biological problem.

Recently, H. H. G. Hashem in [16] used some results of [19] to study the existence of solution for a system of quadratic integral equations of Chandrasekhar type by applying fixed point theorem for the block operator matrix (2) defined on a nonempty bounded closed convex subsets of Banach algebras where the entries are nonlinear operators.
Since the weak topology is the practice setting to investigate the problem of existence of solution nonlinear integral equations in Banach algebras, it turns out the results mentioned above cannot be easily applied. However, because of the lacks of stability of convergence for the product sequence under the weak topology, the authors in [9] have introduced a new class of Banach algebra satisfying the condition denoted $(\mathcal{P})$:

$$(\mathcal{P}) \begin{cases} \text{For any sequences } \{x_n\} \text{ and } \{y_n\} \text{ of } X \text{ such that } x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y \\ \text{then } x_n \cdot y_n \rightharpoonup x \cdot y; \text{ here } \rightharpoonup \text{ denotes weak convergence,} \end{cases}$$

and they have established some new variants point results on the notion of weak sequentially continuous.

The outline of the paper is as follows. In the next section, we give some preliminaries and results needed in the sequel. In addition, we give a reformulation of Theorem 3.4 in [18]. In Section 3, we apply Theorem 2.9 to discuss the existence of solutions to equations of system (1).

2. Preliminaries and mains results

In this section, we collect a few auxiliary results which will be applied further on. Assume that X is a Banach algebra with the norm $\|\cdot\|$ and the zero element θ. Denote by $\mathcal{B}(X)$ the collection of all nonempty bounded subsets of X and $\mathcal{W}(X)$ is the subfamily of $\mathcal{B}(X)$ consisting of all weakly compact subsets of X. Recall that the notion of the measure of weak noncompactness β on $\mathcal{B}(X)$ was introduced by De Blasi [14] in the following way:

$$\beta(S) = \inf\Big\{r > 0 : \text{ there exists } K \in \mathcal{W}(X) \text{ such that } S \subseteq K + B_r\Big\},$$

where B_r is the closed ball in X centered at 0 with a radius r. For convenience we recall some basic properties of $\beta(.)$ needed below [2, 14].

Lemma 2.1. *Let S_1, S_2 be two elements of $\mathcal{B}(X)$. Then the following conditions are satisfied:*

1. $\beta(S_1) = 0$, *if and only if* $\overline{S_1^w} \in \mathcal{W}(X)$, *where* $\overline{S_1^w}$ *is the weak closure of* S_1,
2. $S_1 \subset S_2$ *implies* $\beta(S_1) \leq \beta(S_2)$,
3. $\beta(\overline{S_1^w}) = \beta(S_1)$,
4. $\beta(S_1 \cup S_2) = max\{\beta(S_1), \beta(S_2)\}$,
5. $\beta(\lambda S_1) = |\lambda|\beta(S_1)$, *for all* $\lambda \in \mathbb{R}$,
6. $\beta(co(S_1)) = \beta(S_1)$, *where* $(co(S_1))$ *is the convex hull of* S_1,

7. $\beta(S_1 + S_2) \leq \beta(S_1) + \beta(S_2)$,
8. *If* $(S_n)_{n\in\mathbb{N}}$ *is a decreasing sequence of nonempty, bounded and weakly closed subsets of* X *with* $\lim_{n\to\infty} S_n = 0$, *then* $S_\infty := \cap_{n=0}^{\infty} S_n$ *is nonempty relatively weakly compact.* $\diamond$

Definition 2.2. *An operator* $T : X \longrightarrow X$ *is said to be weakly compact, if* $T(B)$ *is relatively weakly compact for every nonempty bounded subset* $B \subseteq X$. $\diamond$

Definition 2.3. *An operator* $T : X \longrightarrow X$ *is said to be weakly sequentially continuous on* X *if, for every sequence* $\{x_n\} \subset X$ *with* $x_n \rightharpoonup x$, *we have* $Tx_n \rightharpoonup Tx$. $\diamond$

Definition 2.4. *A mapping* $T : X \longrightarrow X$ *is called* $\mathcal{D}$*-Lipschitizan if there exists a continuous and nondecreasing function* $\phi_T : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *satisfying*

$$\|Tx - Ty\| \leq \phi_T(\|x - y\|),$$

for all $x, y \in X$, *where* $\phi_T(0) = 0$.
Moreover, if $\psi_T(r) < r$, $r > 0$, *then* T *is called a nonlinear contraction on* X. *In particular, if* $\psi_T(r) = kr$, *for some constant* $0 < k < 1$, *then* T *is a contraction.* $\diamond$

Let $S \subset X$ and $T : S \longrightarrow X$. If T is bounded and $\beta(T(S_1)) < \beta(S_1)$ for any $S_1 \in \mathcal{B}(S)$ with $\beta(S_1) > 0$, then T is called β-condensing.
In the sequel, we will use the following lemmas which were established in [9].

Lemma 2.5. *Let* S *be a nonempty, closed and convex subset of a Banach space* X. *Assume that* $F : S \longrightarrow S$ *is weakly sequentially continuous and* β*-condensing. In addition, if* $F(S)$ *is bounded, then* F *has, at least, one fixed point in* S.$\diamond$

Lemma 2.6. *The set* $K \cdot K' = \{x \cdot y \ ; \ x \in K \text{ and } x' \in K'\} \in \mathcal{W}(X)$, *for all* $K, K' \in \mathcal{W}(X)$. $\diamond$

Lemma 2.7. *If* $V \in \mathcal{B}(X)$ *and* $K \in \mathcal{W}(X)$, *then* $\beta(V \cdot K) \leq \|K\|\beta(V)$. $\diamond$

Lemma 2.8. *If* F *is Lipschitzian with constant* α *and is weakly sequentially continuous on* X, *then* $\beta(F(V)) \leq \alpha\beta(V)$, *for all* $V \in \mathcal{B}(X)$. $\diamond$

In [18], A. Jeribi, N. Kaddachi and B. Krichen give a proof of the next result in case of Lipschitzians mappings. Now, we give a proof for the case of $\mathcal{D}$-Lipschitizians maps.

Theorem 2.9. *Let* S *be a nonempty, closed and convex subset of a Banach algebras* X *satisfying the condition* $\mathcal{P}$. *Suppose that* $A, C : S \longrightarrow X$, *and* $B, B', D : X \longrightarrow X$ *are five weakly sequentially continuous operators satisfying the following conditions:*
(i) *A, B and C are* $\mathcal{D}$*-Lipschitzians with* $\mathcal{D}$*-functions* ϕ_A,ϕ_B *and* ϕ_C *respectively,*
(ii) $B'(S)$ *is weakly relatively compact and* $A(S)$ *and* $B(S)$ *are bounded,*
(iii) D *is a contraction with constant* k *and* $C(S) \subseteq (I - D)(S)$,
(iv) $Ax + B(I - D)^{-1}Cx \cdot B'(I - D)^{-1}Cx \in S$ *for all* $x \in S$.
Then the operator (2) has, at least, one fixed point as soon as $\phi_A(r) + M\phi_B \circ (\frac{1}{1-k}\phi_C)(r) < r$, *where* $M = \|B'(S)\|$. $\diamond$

Proof. Let us define a mapping $F : S \longrightarrow S$ by the formula

$$Fx = Ax + B(I - D)^{-1}Cx \cdot B'(I - D)^{-1}Cx.$$

Notice that $(I - D)^{-1}C$ as well as F are weakly sequentially continuous on S. In fact, let $\{x_n, \ n \in \mathbb{N}\}$ be a sequence in S which is weakly converge to a point x. Using both the equality

$$(I - D)^{-1}C = C + D(I - D)^{-1}C, \tag{3}$$

and keeping in mind the subadditivity of the De Blasi's measure of weak noncompactness, we infer that

$$\begin{aligned}\beta(\{(I-D)^{-1}C(x_n),\ n\in\mathbb{N}\}) &\leq \beta\left(\{C(x_n),\ n\in\mathbb{N}\}\right)+\beta\left(\{D(I-D)^{-1}C(x_n),\ n\in\mathbb{N}\}\right)\\ &\leq k\beta\left(\{(I-D)^{-1}C(x_n),\ n\in\mathbb{N}\}\right).\end{aligned}$$

This inequality means that $\{(I-D)^{-1}C(x_n),\ n\in\mathbb{N}\}$ is relatively weakly compact. Consequently, there is a subsequence (x_{n_k}) of $\{x_n,\ n\in\mathbb{N}\}$ such that

$$(I-D)^{-1}C(x_{n_k})\rightharpoonup y.$$

Taking into account the weak sequential continuity of the maps C and D and using (3), we obtain $y=(I-D)^{-1}C(x)$. Accordingly we have

$$(I-D)^{-1}C(x_{n_k})\rightharpoonup (I-D)^{-1}C(x).$$

Now, we show that $(I-D)^{-1}C(x_n)\rightharpoonup (I-D)^{-1}C(x)$. Suppose that this is not the case, then there is a weak neighborhood V^w of $(I-D)^{-1}C(x)$ and a subsequence (x_{n_j}) of $\{x_n,\ n\in\mathbb{N}\}$ such that $(I-D)^{-1}C(x_{n_j})\notin V^w$ for all $j\geq 1$. Since (x_{n_j}) converges weakly to x, and arguing as before, we find a subsequence $(x_{n_{j_k}})$ of (x_{n_j}) such that $(I-D)^{-1}C(x_{n_{j_k}})\rightharpoonup (I-D)^{-1}C(x)$. Which is absurd, since $(I-D)^{-1}C(x_{n_{j_k}})\notin V^w$. As a result, $(I-D)^{-1}C$ is weakly sequentially continuous. Moreover, taking into account that X is a Banach algebra satisfying the condition $(\mathcal{P})$, and using the assumption (iii), we deduce that F is weakly sequentially continuous on S. Next, we will prove that F is β-condensing. To see this, let S_1 be a bounded subset of S with $\beta(S_1)>0$. From assumption (ii), it follows that $F(S_1)$ is bounded. The use of Lemma 2.7 and also the subadditivity of the De Blasi's measure of weak noncompactness yields

$$\begin{aligned}\beta(F(S_1)) &= \beta(A(S_1))+\beta\left(B(I-D)^{-1}C(S_1)\cdot B'(I-D)^{-1}C(S_1)\right)\\ &\leq \beta(A(S_1))+\left\|B'(I-D)^{-1}C(S_1)\right\|\beta\left(B(I-D)^{-1}C(S_1)\right)\\ &\leq \phi_A(\beta(S_1))+\left(M\phi_B\circ(\frac{1}{1-k}\phi_C)\right)\beta(S_1)\\ &\leq \left(\phi_A+M\phi_B\circ(\frac{1}{1-k}\phi_C)\right)\beta(S_1).\end{aligned}$$

This shows that F is a β-condensing. Now, we may apply Lemma 2.5 to infer that F has, at least, one fixed point x in S. Consequently, the use of vector $y=(I-D)^{-1}Cx$ solves the problem. $\square$

3. Existence theorem

The main aim of this section is to apply Theorem 2.9 to prove the existence of solutions to the coupled system (1) in the space $C(J,X)$ of all continuous functions on $J=[0,b]$, $0<b<\infty$ endowed with the norm $\|\cdot\|_\infty$, where X is a Banach algebra satisfying the condition $(\mathcal{P})$. Clearly, $C(J,X)$ becomes a Banach algebra satisfying the condition $(\mathcal{P})$ (see [9]).
Let us now introduce the following assumptions:

$(\mathcal{H}_0)$ The function $f_i : J\times X\longrightarrow X$, $i=1,2$ is such that :
- (a) f_i is a k_i-contraction and weakly sequentially continuous with respect to the second variable, and
- (b) $M_i=\sup\limits_{t\in J}|f_i(t,x)|$.

$(\mathcal{H}_1)$ The function $p_i : J\times X\longrightarrow\mathbb{R}$, $i=1,2$ is such that :
- (a) The partial $t\longmapsto p_i(t,x)$ is measurable for each $x\in X$,
- (b) The partial $x\longmapsto p_i(t,x)$ is weakly sequentially continuous for almost all $t\in J$,

(c) The partial $x \longmapsto p_i(t,x)$ is contraction with constant l_i with respect to the second variable,
(d) there exist a function $m_i \in L^1(J)$ such that

$$\|p_i(t,x)\| \leq m_i(t) \text{ for all } (t,x) \in J \times X,$$

and $\lambda_i = \sup \int_0^b \frac{1}{t+s} m_i(s)ds.$

$(\mathcal{H}_2)$ The function $g : J \times X \longrightarrow X$ is such that :
(a) The partial $x \longmapsto g(t,x)$ is weakly sequentially continuous with respect to the second variable,
(b) There exists two constants $L, K > 0$ such that

$$0 < g(t,x(t)) - g(t,y(t)) \leq \frac{L(x-y)}{K+(x-y)}$$

for all $t \in J$ and $x, y \in X$ with $x \geq y$. Moreover, $L \leq K$ and $N = \|g\|$.

Theorem 3.1. *Let the assumptions $(\mathcal{H}_0) - (\mathcal{H}_2)$ be satisfied. Furthermore, if*

$$\begin{cases} L\|u\|\|m_1\| \leq K \\ \\ k_1 + \dfrac{1}{1-k_2} bl_2\|v\| \leq 1, \end{cases}$$

then the system of the quadratic integral equations (1) has, at least, one solution. ◊

Proof. Let us define the subset S of $C(J,X)$ by:

$$S = \Big\{ x \in C(J,X), \|x\| \leq \inf \big\{ \lambda_1 N\|v\| + M_1, \lambda_2\|v\| + M_2 \big\} \Big\}.$$

Consider the operators A, B, C, D and B' on S defined by:

$$\begin{cases} (Ax)(t) = f_1(t,x(t)) \\ (By)(t) = g(t,y(t)) \\ (Cx)(t) = \left(\displaystyle\int_0^t \frac{t}{t+s} p_2(s,x(s))ds \right) \cdot v, \quad t \in J \text{ and } v \in X \backslash \{0\} \\ (Dy)(t) = f_2(t,y(t)) \\ (B'y)(t) = \left(\displaystyle\int_0^t \frac{t}{t+s} p_1(s,y(s))ds \right) \cdot u, \quad t \in J \text{ and } u \in X \backslash \{0\}\,. \end{cases}$$

Then, the problem (1) is equivalent to the system:

$$\begin{cases} x(t) = (Ax)(t) + (By)(t) \cdot (B'y)(t) \\ \\ y(t) = (Cx)(t) + (Dy)(t). \end{cases}$$

In order to apply Theorem 2.9, we have to verify the following claims:
Claim 1. A, B and C are $\mathcal{D}$-Lipschitzian. To see this, for all $x, y \in S$ we have

$$\begin{aligned} \|Ax(t) - Ay(t)\| &= \|f_1(t,x(t)) - f_1(t,y(t))\| \\ &\leq k_1\|x-y\|. \end{aligned}$$

Taking the sepremum over t, we obtain

$$\|Ax - Ay\| \leq \psi_A(\|x-y\|).$$

This shows that A is $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function $\psi(r) = k_1 r$. Furthermore, by using assumption $(\mathcal{H}_2)$ we conclude that

$$\begin{aligned} \|Bx(t) - By(t)\| &= \|g(t, x(t)) - g(t, y(t))\| \\ &\le \frac{L\|x - y\|}{K + \|x - y\|}. \end{aligned}$$

Taking the supremum over t, we obtain

$$\|Bx - By\| \le \frac{L\|x - y\|}{K + \|x - y\|}.$$

This shows that B is $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function $\psi_B(r) = \dfrac{Lr}{K + r}$.
Again, the operator C is $\mathcal{D}$-lipschitzian with the $\mathcal{D}$-function $\psi_C(r) = bl_2\|v\|r$. Indeed, for all $x, y \in X$ we have

$$\begin{aligned} \|Cx(t) - Cy(t)\| &= \left\| \left(\int_0^t \frac{t}{t+s} p_2(s, x(s)) ds \right) \cdot v - \left(\int_0^t \frac{t}{t+s} p_2(s, y(s)) ds \right) \cdot v \right\| \\ &\le \|v\| \left\| \int_0^t \frac{t}{t+s} \Big(p_2(s, x(t)) - p_2(s, y(s)) \Big) ds \right\| \\ &\le \|v\| \int_0^t \frac{t}{t+s} \|p_2(s, x(t)) - p_2(s, y(s))\| ds \\ &\le \|v\| \int_0^t \frac{t}{t+s} l_2 \|x - y\| ds \\ &\le bl_2 \|v\| \|x - y\|. \end{aligned}$$

Taking the supremum over t, we obtain

$$\|Cx - Cy\| \le bl_2\|v\|\|x - y\|.$$

Claim 2. Let $\{x_n, \ n \in \mathbb{N}\}$ be any sequence in S, we have $(B'x_n)(t) = r_n(t) \cdot u$, where

$$r_n(t) = \int_0^t \frac{t}{t+s} p_1(s, y(s)) ds.$$

Since $\|r_n(t)\| \le \lambda_1$ in view of assumption $(\mathcal{H}_1)$, it follows that there is a renamed subsequence such that $r_n(t) \to r(t)$, which implies that

$$(B'x_n)(t) \rightharpoonup (B'x)(t) \ \text{ in } \ X.$$

As a result, $B'(S)(t)$ is sequentially relatively weakly compact. Next, we will show that $B'(S)$ is a weakly equi-continuous set. If we take $x^* \in X^*$ and $t_1, t_2 \in J$ (without loss of generality assume that $t_1 < t_2$), then we have

$$\begin{aligned} |x^*(B'x(t_2) - B'x(t_1))| &\le \left| \int_0^{t_2} \frac{t_2}{t_2 + s} p_1(s, x(s)) ds - \int_0^{t_1} \frac{t_1}{t_1 + s} p_1(s, x(s)) ds \right| \|x^*(u)\| \\ &\le \left| \int_0^{t_2} \frac{t_2}{t_2 + s} p_1(s, x(s)) ds - \int_0^{t_1} \frac{t_1}{t_1 + s} p_1(s, x(s)) ds \right. \\ &+ \left. \int_0^{t_2} \frac{t_1}{t_1 + s} p_1(s, x(s)) ds - \int_0^{t_2} \frac{t_1}{t_1 + s} p_1(s, x(s)) ds \right| \|x^*(u)\| \\ &\le \left(\int_0^{t_2} \frac{t_2 - t_1}{t_1 + s} \Big| p_1(s, x(s)) \Big| ds + \int_{t_1}^{t_2} \frac{t_1}{t_1 + s} \Big| p_1(s, x(s)) \Big| ds \right) \|x^*(u)\| \\ &\le \left(|t_2 - t_1| \lambda_1 + \int_{t_1}^{t_2} m_1(s) ds \right) \|x^*(u)\|. \end{aligned}$$

Therefore,

$$|x^*(B'x(t_2) - B'x(t_1))| \to 0, \text{ as } t_2 \to t_1.$$

Based on an application of the Arzela-Ascoli's theorem [22], we conclude that $B'(S)$ is sequentially relatively weakly compact in X. Again, an application of Eberlein-Smulian's theorem [13] shows that $B'(S)$ is relatively weakly compact.

Claim 3. We show that $C(S) \subseteq (I - D)(S)$. To see that, let $x \in S$ be fixed point. Define a mapping

$$\begin{cases} \phi_x : \mathcal{C}(J, X) \longrightarrow \mathcal{C}(J, X) \\ y \longmapsto Cx + Dy. \end{cases}$$

From assumption $(\mathcal{H}_0)$, it follows that the operator ϕ_x is a contraction with a constant k_2, then an application of Banach's fixed point theorem yields there is a unique point $y \in \mathcal{C}(J, X)$ such that $y = Cx + Dy$ and consequently $C(S) \subseteq (I - D)(\mathcal{C}(J, X))$.

Since $y \in \mathcal{C}(J, X)$, then there is $t^* \in J$ such that

$$\begin{aligned} \|y\|_\infty = \|y(t^*)\| &= |Cx(t^*) + Dy(t^*)| \\ &\leq \left| \int_0^{t^*} \frac{t^*}{t^* + s} p_2(s, x(s)) ds \right| \cdot \|v\| + |f_2(t^*, x(t^*))| \\ &\leq \lambda_2 \|v\| + M_2. \end{aligned}$$

This means, in particular, that is $C(S) \subseteq (I - D)(S)$.

Claim 4. By using the assumption $(\mathcal{H}_1)$, we have

$$\begin{aligned} M &= \|B'(I - D)^{-1}C(S)\| \\ &\leq \sup_{t \in J} \left\| \left(\int_0^t \frac{t}{t + s} p_1(s, x(s)) ds \right) \cdot u \right\| \\ &\leq \|m_1\| \|u\|, \end{aligned}$$

and therefore $M\psi_B \circ (\frac{1}{1-k_2}\psi_c)(r) + \psi_A(r) \leq r$.

Next, let us fix an arbitrary $y \in \mathcal{C}(J, X)$ and $x \in S$ such that

$$y = Ax + B(I - D)^{-1}Cx \cdot B'(I - D)^{-1}Cx,$$

or, equivalently for all $t \in [0, b]$

$$y(t) = Ax(t) + B(I - D)^{-1}Cx(t) \cdot B'(I - D)^{-1}Cx(t).$$

Then

$$\begin{aligned} \|y(t)\| &= \|Ax(t) + B(I - D)^{-1}Cx(t) \cdot B'(I - D)^{-1}Cx(t)\| \\ &\leq \|Ax(t)\| + \|B(I - D)^{-1}Cx(t)\| \|B'(I - D)^{-1}Cx(t)\| \\ &\leq M_1 + N\|v\|\lambda_1. \end{aligned}$$

This implies that

$$Ax + B(I - D)^{-1}Cx \cdot B'(I - D)^{-1}Cx \in S \text{ for all } x \in \mathcal{C}(J, X).$$

Since $y \in C(J, X)$, there is $t^* \in J$ such that $\|y\|_\infty = \|y(t^*)\|$ and consequently,

$$\|y\|_\infty \leq M_1 + N\|v\|\lambda_1.$$

As a result

$$Ax + B(I - D)^{-1}Cx \cdot B'(I - D)^{-1}Cx \in S \text{ for all } x \in S.$$

To end the proof, we apply Theorem 2.9, we get that the system (1) has, at least, one solution in $S \times S$. □

The quadratic integral equations (1) is new in the theory of integral functional equations and some special cases belonging to it have been extensively discussed in the literature. We have the following particular cases that constitute the versions of quadratic integral equations of Chandrasekhar type:

1. The special case when $x = y$, $f_1(t, x(t)) = 1$, $g(t, x(t)) = x(t)$, $p_1(t, x(t)) = \varphi(t)x(t)$, $f_2(t, y(t)) = y(t)$ and $p_2(t, x(t)) = 0$, and QIE (1) reduced to QIE
$$x(t) = 1 + x(t) \int_0^t \frac{t}{t+s} \varphi(s)x(s)ds. \tag{4}$$
The Chandrasekhar's integral equation (4) has been discussed in [11] for different aspects of the solutions under suitable conditions.

2. If we take $f_1(t, x(t)) = a(t) \int_0^a u(t, s, x(s))ds$, $f_2(t, y(t)) = y(t)$, $g(t, x(t)) = \int_0^t v(t, s, x(s))ds$ and $p_1(s, x(s)) = \frac{t+s}{t} u(t, s, x(s))$, we obtain the following integral equation
$$x(t) = a(t) \int_0^a u(t, s, x(s))ds + \left(\int_0^t v(t, s, x(s))ds \right) \cdot \left(\int_0^t u(t, s, x(s))ds \right). \tag{5}$$
The equation (5) was examined in the paper [15] and some special cases of this equation were considered in [20, 21].

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