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Remarks on an Equation of the Ginzburg-Landau Type

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Abstract. This paper is concerned with a priori estimate of the Ginzburg-Landau equation. We study the uniform bound of classical solutions on the whole space. In addition, we also obtain the Liouville-type result of finite energy solutions.

1. Introduction

In 1994, Brezis, Merle and Riviere [5] studied the quantization effects of the following equation

$$-\triangle u = (1 - |u|^2)u \quad in\ \mathbb{R}^2.$$

It is the Euler-Lagrange equation of the Ginzburg-Landau (GL) energy

$$E_{GL}(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{4}\|1 - |u|^2\|^2_{L^2(\mathbb{R}^2)}.$$

Here $u : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector value function. In particular, they proved

$$|u| \leq 1 \quad in \quad \mathbb{R}^2. \tag{1.1}$$

(The same result was also obtained in [8]). Based on this result, they also obtained a Liouville-type theorem for the finite energy solution by an argument due to Cazenave. Namely,

$$\nabla u \in L^2(\mathbb{R}^2) \Rightarrow u(x) \equiv C\ with\ |C| \in \{0, 1\}. \tag{1.2}$$

Those results can be generalized to the higher dimensions case (cf. [11]).

In 2010, Ma [13] gave a new proof of $|u| \leq 1$ on the whole space in the higher dimensions case, where only the maximum principle was employed. Afterwards, this priori estimate was also obtained for the Chern-Simons-Higgs type equation which has a more complicated right hand side (cf. [12]).

In 1995, Rubinstein (cf. [14]) introduced another GL functional with a new penalization

$$E_\beta(u, G) = \frac{1}{2}\|\nabla u\|^2_{L^2(G)} + \frac{1}{4\varepsilon^2}\|\beta^2 - |u|^2\|^2_{L^2(G)}.$$

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Here G is a bounded domain in $\mathbb{R}^2$, $\varepsilon > 0$, and $\beta \in L^\infty(G)$. This functional is helpful to study the pinning effects of GL vortices (cf. [2], [3], [6], [7] and [9]) and other analogous problems appear in the study of Bose-Einstein condensate (cf. [1] and the references therein).

In this paper, we are concerned with the following equation

$$-\triangle u = u(\beta^2 - |u|^2) \quad in\ \mathbb{R}^n, \tag{1.3}$$

where $u : \mathbb{R}^n \to \mathbb{R}^k$ is a vector-valued function, and $\beta \in L^\infty(\mathbb{R}^n)$ satisfies

$$0 < m \leq \beta(x) \leq M < \infty, \quad \forall x \in \mathbb{R}^n. \tag{1.4}$$

Here $n, k \geq 2, m, M$ are positive constants. We will use the idea in [13] to prove a result analogous to (1.1).

Theorem 1.1. *Let $u \in C^2(\mathbb{R}^n, \mathbb{R}^k)$ be a classical solution of (1.3) with (1.4). Then*

$$|u| \leq M, \quad \forall x \in \mathbb{R}^n. \tag{1.5}$$

Next, if u satisfies a stronger assumption $|u| \leq \beta$ than (1.5), we will prove a result analogous to a Liouville-type result (1.2) for the finite energy solutions.

Theorem 1.2. *Let $u \in C^2(\mathbb{R}^n, \mathbb{R}^k)$ be a finite energy solution (i.e. $\nabla u \in L^2(\mathbb{R}^n)$) of (1.3). Assume that β is a piecewise derivable function satisfying (1.4) and $x \cdot \nabla\beta \geq 0$. If*

$$|u(x)| \leq \beta(x), \quad \forall x \in \mathbb{R}^n, \tag{1.6}$$

then either $u \in L^2(\mathbb{R}^n, \mathbb{R}^k)$, or $|u| \equiv \beta$ a.e. in $\in \mathbb{R}^n$.

Remark 1.1.. An obvious fact is, if $u \in L^2(\mathbb{R}^n, \mathbb{R}^k) \cap C^2(\mathbb{R}^n, \mathbb{R}^k)$, then $|u| \to 0$ when $|x| \to \infty$. In addition, when $\beta = 0$ in B_ρ and $\beta = 1$ in $\mathbb{R}^n \setminus B_\rho$, Theorem 1.2 is consistent with Theorem 1.2 in [10].

Remark 1.2.. Let u be a finite energy solution of (1.3). By the regularity result (cf. Lemma A.1 in [4]), one has $\sup_{\mathbb{R}^n} |\nabla u| < \infty$, which together with $\nabla u \in L^2(\mathbb{R}^n)$, implies $\nabla u \in L^\gamma(\mathbb{R}^n)$ for all $\gamma \geq 2$.

2. Proof of Theorem 1.1

For any fixed unit vector $\vec{e} \in \mathbb{S}^{k-1}$, we define $v = \vec{e} \cdot u$. Then from (1.3) we have

$$\triangle v + v(\beta^2 - |u|^2) = 0, \quad in\ \mathbb{R}^n. \tag{2.1}$$

Let $V = v^2$. Then, using $|u|^2 \geq V$, from the result above we deduce that

$$\triangle V = 2(v\triangle v + |\nabla v|^2) \geq 2v\triangle v \geq 2V(V - \beta^2). \tag{2.2}$$

Given any small $R \in (0, 1)$ and large $\alpha > 1$, take

$$w(x) := (R^2 - |x - x_0|^2)^{-\alpha}.$$

By (1.4) and a direct computation, we can see that

$$\triangle w + 2w(\beta^2 - w) \leq 0, \ \ in\ B_R(x_0) \tag{2.3}$$

for sufficiently large α. Since $w(x) = \infty$ on $\partial B_R(x_0)$, we claim by the comparison lemma that

$$V(x) \leq w(x), \ \ in\ B_R(x_0).$$

In fact, if there exists a point $x_1 \in \mathbb{R}^n$ such that $V(x_1) > w(x_1)$, then the positive maximum of $V(x) - w(x)$ can be achieved at some point x_2. Therefore, at point x_2, one has

$$0 \le -\frac{1}{2}\triangle(V - w) \le V(\beta^2 - V) - w(\beta^2 - w) \tag{2.4}$$

by (2.2) and (2.3). On the other hand, if we write $g_1(t) = t(t - \beta^2(x))$, then $g_1'(t) > 0$ for $t > M^2/2$ and all $x \in \mathbb{R}^n$. This implies that

$$V(\beta^2 - V) - w(\beta^2 - w) < 0$$

at x_2 as long as we notice $V(x_2) > w(x_2) \ge R^{-2\alpha} > M^2/2$. This contradicts with (2.4) and the claim is verified. Then we have some constant $C(R) > 0$ such that $|v(x)| \le C(R)$ in $B_{\frac{R}{2}}(x_0)$. Since x_0 and $\vec{e}$ are arbitrary, we deduce that

$$|u(x)| \le C, \quad in\ \mathbb{R}^n. \tag{2.5}$$

Moreover, we prove $|u(x)| \le M$ on $\mathbb{R}^n$ by the contradiction argument.

Case 1. Suppose that there exists a point $x_0 \in \mathbb{R}^n$ such that $v(x_0) > M$. First, by (2.1) and $v \le |u|$ we see that

$$\triangle v + v(\beta^2 - v^2) \ge 0 \quad when\ v > 0. \tag{2.6}$$

For small $\epsilon > 0$, we take

$$W_1(x) := v(x) - v(x_0) + \epsilon - \epsilon|x - x_0|^2.$$

By (2.5), $W_1(x) \to -\infty$ as $|x| \to \infty$. In view of $W_1(x_0) > 0$, there exists a point $y \in \mathbb{R}^n$ such that

$$W_1(y) = \max_{\mathbb{R}^n} W_1(x) \ge W_1(x_0) = \epsilon,$$

which implies

$$v(y) \ge v(x_0) + \epsilon|y - x_0|^2 > M. \tag{2.7}$$

In view of $0 \ge \triangle W_1(y) = \triangle v(y) - 2n\epsilon$, by (2.6) we get

$$2n\epsilon \ge \triangle v(y) \ge v(y)(v^2(y) - \beta^2(y)). \tag{2.8}$$

If we write $g_2(t) = t(t^2 - \beta^2(x))$, then $g_2'(t) > 0$ for $t > M/\sqrt{3}$ and all $x \in \mathbb{R}^n$. This, together with (2.7), implies $g_2(v(y)) \ge g_2(v(x_0) + \epsilon|y - x_0|^2)$. Combining with (2.8) yields

$$2n\epsilon \ge g_2(v(x_0) + \epsilon|y - x_0|^2).$$

Letting $\epsilon \to 0$, we get

$$0 \ge v(x_0)(v^2(x_0) - \beta^2(x_0)),$$

which, together with $v(x_0) > M$, leads to a contradiction.

Case 2. Assume that we can find a point $x_0 \in \mathbb{R}^n$ such that $v(x_0) < -M$. If replacing W_1 by

$$W_2(x) := v(x) - v(x_0) - \epsilon + \epsilon|x - x_0|^2,$$

we can also see a contradiction by the same argument above. This shows that $v \ge -M$.

Combining Cases 1 and 2 together yields $|v(x)| \le M$ for arbitrary $x \in \mathbb{R}^n$ and arbitrary $\vec{e}$. We then conclude that $|u(x)| \le M$ in $\mathbb{R}^n$. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

By (1.3) we have

$$|u|^2(\beta^2-|u|^2)=|\nabla u|^2-\Delta(|u|^2/2). \tag{3.1}$$

In view of $\nabla u\in L^2(\mathbb{R}^n)$, the Sobolev inequality implies $u\in L^{2^*}(\mathbb{R}^n)$ when $n\geq 3$. Here $2^*=\frac{2n}{n-2}$. Thus, we can find $r_j\to\infty$ such that

$$r_j\int_{\partial B_{r_j}(0)}|\nabla u|^2ds=o(1),\quad when\ n\geq 2;$$

$$r_j\int_{\partial B_{r_j}(0)}|u|^{2^*}ds=o(1),\quad when\ n\geq 3.$$

Therefore, by the divergence theorem and the Hölder inequality, when $r_j\to\infty$, there hold

$$\begin{aligned}\left|\int_{B_{r_j}(0)}\Delta(|u|^2/2)dx\right|&\leq\int_{\partial B_{r_j}(0)}|u||\partial_\nu u|ds\\ &\leq M\left(2\pi r_j\int_{\partial B_{r_j}(0)}|\nabla u|^2ds\right)^{\frac12}=o(1),\quad when\ n=2;\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{B_{r_j}(0)}\Delta(|u|^2/2)dx\right|\leq\int_{\partial B_{r_j}(0)}|u||\partial_\nu u|ds\\ &\leq\left(r_j\int_{\partial B_{r_j}(0)}|u|^{2^*}ds\right)^{\frac{1}{2^*}}\left(r_j\int_{\partial B_{r_j}(0)}|\nabla u|^2ds\right)^{\frac12}|\partial B_{r_j}(0)|^{1-\frac12-\frac{1}{2^*}}r_j^{-\frac12-\frac{1}{2^*}}\\ &=o(1),\quad when\ n\geq 3.\end{aligned}$$

Thus, by (3.1), we have

$$\int_{\mathbb{R}^n}|u|^2(\beta^2-|u|^2)dx=\|\nabla u\|^2_{L^2(\mathbb{R}^n)}<\infty. \tag{3.2}$$

Set $S_*=\{x\in\mathbb{R}^n;\epsilon_0\leq|u|\leq m/2\}$ for some fixed small constant $\epsilon_0\in(0,m/3)$. By (1.4) and (1.6),

$$\int_{\mathbb{R}^n}|u|^2(\beta^2-|u|^2)dx\geq\int_{S_*}|u|^2(\beta^2-|u|^2)dx\geq\frac{3m^2}{4}\epsilon_0^2|S_*|.$$

This and (3.2) imply $|S_*|<\infty$. Thus, there exists suitably large $R_0>0$ such that $S_*\subset B_{R_0}(0)$. Since u is continuous and $\mathbb{R}^n\setminus B_{R_0}(0)$ is connected, either $|u|\leq\epsilon_0$ or $|u|\geq m/2$ holds true on $\mathbb{R}^n\setminus B_{R_0}(0)$.

When $|u|\leq\epsilon_0$ on $\mathbb{R}^n\setminus B_{R_0}(0)$, it is led to $\beta^2-|u|^2\geq m^2-\epsilon_0^2$ on $\mathbb{R}^n\setminus B_{R_0}(0)$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^n}|u|^2dx&=\int_{B_{R_0}(0)}|u|^2dx+\int_{\mathbb{R}^n\setminus B_{R_0}(0)}|u|^2dx\\ &\leq|B_{R_0}|M^2+\frac{1}{m^2-\epsilon_0^2}\int_{\mathbb{R}^n\setminus B_{R_0}(0)}|u|^2(\beta^2-|u|^2)dx<\infty,\end{aligned}$$

by virtue of (3.2). Namely, $u\in L^2(\mathbb{R}^n,\mathbb{R}^k)$.

When $|u|\geq m/2$ on $\mathbb{R}^n\setminus B_{R_0}(0)$, by the same argument above, we can see

$$\int_{\mathbb{R}^n}(\beta^2-|u|^2)dx<\infty$$

in view of (1.6). This furthermore implies $\beta^2 - |u|^2 \in L^2(\mathbb{R}^n)$ by (1.4) and (1.6). Therefore, the C^1-functional

$$E(U) = \int_{\mathbb{R}^n} |\nabla U|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} (\beta^2 - |U|^2)^2 dx$$

makes sense for $U \in B_\delta(u)$ for small $\delta > 0$. In addition, the classical solution u of (1.3) is a critical point of $E(U)$. Thus, the Frechet derivative of $E(U)$ at u is zero. This leads to

$$\left[\frac{d}{d\lambda} E(u(\lambda^{-1}x))\right]_{\lambda=1} = 0. \tag{3.3}$$

On the other hand, for $\lambda \neq 0$,

$$E(u(\lambda^{-1}x)) = \lambda^{n-2}\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx + \frac{\lambda^n}{2}\int_{\mathbb{R}^n} (\beta^2(\lambda x) - |u(x)|^2)^2 dx.$$

Therefore, from (3.3) we get

$$(n-2)\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{n}{2}\int_{\mathbb{R}^n} (\beta^2 - |u|^2)^2 dx + \int_{\mathbb{R}^n} (\beta^2 - |u|^2)(x \cdot \nabla \beta^2) dx = 0. \tag{3.4}$$

When $n = 2$, it follows that

$$\int_{\mathbb{R}^2} (\beta^2 - |u|^2)^2 dx = -\int_{\mathbb{R}^2} (\beta^2 - |u|^2)(x \cdot \nabla \beta^2) dx \leq 0,$$

which implies $|u| \equiv \beta$ a.e. in $\mathbb{R}^2$.

When $n \geq 3$,

$$(n\beta^2 + (n-4)|u|^2) > 0. \tag{3.5}$$

By (3.2), (3.4) and (3.5), there holds

$$\frac{1}{2}\int_{\mathbb{R}^n} (\beta^2 - |u|^2)(n\beta^2 + (n-4)|u|^2) dx = -\int_{\mathbb{R}^n} (\beta^2 - |u|^2)(x \cdot \nabla \beta^2) dx \leq 0.$$

This result leads to $|u| \equiv \beta$ a.e. in $\mathbb{R}^n$.

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