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A Fixed Point Problem Under a Finite Number of Equality Constraints on b-Banach spaces

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Abstract. In this manuscript, we investigate a fixed point problem under a finite number of equality constraints involving a well-known Ćirić type mappings in the context of b-metric space. We obtain sufficient conditions for the existence of solutions of such problems. We also express some immediate consequences of our main results.

1. Introduction

A fixed point problem, $f(x) = x$, can be reformulated as in the form $F(x) = 0$. It is usually expected to find a unique solution of this equation. On the other hand, the equations, abstracted from the real world problems, need not to have a unique solution. In particular, it is very well known that in several cases, the nonlinear integral or differential equations (that corresponding to a fixed point problem) have periodic solution or more than one solutions. Thus, non-unique fixed point problem, is very natural and has a wide application potential.

In 1974, Ćirić [18] proved a fixed point theorem for certain operators that do not need to possess a unique fixed point.

Theorem 1.1. [Non-unique fixed point theorem of Ćirić [18]] Let T be a self-map on a metric space (X, d). Suppose also that

(i) T is called orbitally continuous, that is,

$$\lim_{i\to\infty} T^{n_i}x = z \text{ implies } \lim_{i\to\infty} TT^{n_i}x = Tz \text{ for each } x \in X,$$

(ii) (X, d) is called orbitally complete, that is, each Cauchy sequence of type $\{T^{n_i}x\}_{i\in\mathbb{N}}$ converges in (X, d),

(iii) there is $k \in [0, 1)$ such that

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\} \le kd(x, y), \tag{1}$$

for all $x, y \in X$,

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Then, for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Here, the mapping T that satisfies (1) is called Ćirić-type-contraction. This remarkable result of Ćirić [18] initiated the trend of research on non-unique fixed points, and several authors have focused on this topic, see e.g. [2, 3, 16, 17, 19–24].

Very recently, Rakočivić and Samet [25], considered a solution for a finite number of equality constraints involving in the setting of metric space as an application of this trend. More precisely, they consider a system

$$\begin{cases} Tx = x \\ \varphi_i(x) = 0_E, \ i = 1, 2, ...r, \end{cases} \tag{2}$$

where $T, \varphi_i : E \to E, \ i = 1, 2, ..., r$ be a finite number of mappings defined on a Banach space $(E, \|\cdot\|)$ with a cone P, and 0_E is the zero vector of E, and T is mapping satisfying a Ćirić-type-contraction.

The main aim of this paper is to reconsider the system (2) in the setting of b-metric space and b-normed space. It is clear that our results improve and extend the results of [25].

2. Preliminaries

In this section we recall and recollect the necessary basic definitions and fundamental results to construct the our problem in the framework of b-metric. Throughout the paper, we let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We first express the definition of a b-metric space that was given by Bakhtin [4], Czerwik [12] and was announced earlier as quasi-metric by many authors, such as, Berinde [5–7, 26, 27].

Definition 2.1. *(Bakhtin [4], Czerwik [12]) Let X be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied:*

1. $d(x, y) = 0$ *if and only if* $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$,

for all $x, y, z \in X$. A pair (X, d) is called a b-metric space.

Example 2.2. *([5]) For $0 < p < 1$, the set*

$$l^p(\mathbb{R}) := \{(x_n) \subset \mathbb{R} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\}$$

together with the function

$$d(x, y) := \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}, \text{ for each } x, y \in l^p(\mathbb{R})$$

is a complete b-metric space with constant $s = 2^{1/p} > 1$.

Example 2.3. *([5]) For $0 < p < 1$, the set*

$$L^p[a, b] := \{x : [a, b] \to \mathbb{R} : \int_a^b |x(t)|^p dt < \infty\},$$

together with the function

$$d(x, y) := \left(\int_a^b |x(t) - y(t)|^p dt\right)^{\frac{1}{p}}, \text{ for each } x, y \in L^p[a, b],$$

is a complete b-metric space. Notice that in this case $s = 2^{1/p} > 1$.

Example 2.4. *([9]) Let E be a Banach space and P a normal cone in E with int(P) ≠ ∅. Denote by "≤" the partially order relation generated by P. If X is a nonempty set, then a mapping $d : X \times X \to E$ is called a cone metric on X if the usual axioms of the metric take place with respect to "≤". The cone P is called normal if there is a number $K \geq 1$ such that, for all $x, y \in E$, the following implication holds:*

$$0 \leq x \leq y \Longrightarrow \|x\| \leq K\|y\|.$$

If the cone P is normal with the coefficient of normality $K \geq 1$, then the functional

$$\hat{d} : X \times X \to \mathbb{R}_+,\ \hat{d}(x, y) := \|d(x, y)\|$$

is a b-metric on X with constant $s := K$.

For more details and examples on *b*-metric spaces, see e.g. [4, 5, 10–13].

Definition 2.5. *(Bakhtin [4], Czerwik [12]) Let X be a vector space over a field $\mathbb{K}$ (either $\mathbb{C}$ or $\mathbb{R}$) and let $s \geq 1$ be a given real number. A functional $\|x\| : X \to [0, \infty)$ is said to be a b-norm if the following conditions are satisfied:*

(N_1) $\|x\| = 0$ *if and only if* $x = 0$,
(N_2) $\|cx\| = |c|\|x\|$, *for all* $c \in \mathbb{F}$ *and for all* $x \in X$
(N_3) $\|x + y\| \leq s(\|x\| + \|y\|)$, *for all* $x, y \in X$.

for all $x, y, z \in X$. A pair (X, d) is called a b-metric space.

In the context of a linear space X, the pair $(X, \|\cdot\|)$ is called a *b*-normed space with constant $s \geq 1$ if the third axiom of the norm has the following form

Notice that in Example 2.2, the functional

$$\|x\|_p := (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$$

is a b-norm on $l^p(\mathbb{R})$ with constant $s = 2^{\frac{1}{p}}$ and the pair $(l^p(\mathbb{R}), \|\cdot\|_p)$ is a *b*-Banach space. Something similar takes place for Example 2.3.

If the *b*-metric generated by the *b*-norm $\|\cdot\|$ (i.e., $d(x, y) := \|x - y\|$) is complete, then the space $(X, \|\cdot\|)$ is called a *b*-Banach space.

For *b*-metric spaces, the notions of convergent sequence, Cauchy sequence, completeness are similar to those given for usual metric spaces. Moreover, a *b*-metric generates (in a similar way to the case of usual metric spaces) a topology τ on X (see [14]), but there are also major differences with the classical case of metric spaces. For example, the open ball $B(x_0; r) := \{x \in X : d(x_0, x) < r\}$ in a *b*-metric space (X, d) is not necessarily an open set, while the closed ball $\tilde{B}(x_0; r) := \{x \in X : d(x_0, x) \leq r\}$ is not necessarily closed, in the usual sense. Moreover, a *b*-metric is not necessarily continuous, which induces several problems in different approaches.

The following Lemma is used in the proof of our main results.

Lemma 2.6. *Let $(E, \|\cdot\|)$ be a b-Banach space and let $\{x_k\}_{k=0}^n \subset E$. Then:*

$$\|x_n - x_0\| \leq s\|x_0 - x_1\| + ... + s^{n-1}\|x_{n-2} - x_{n-1}\| + s^{n-1}\|x_{n-1} - x_n\|.$$

Proof. Applying inequality (N_3) *n*-times we obtain the desired result. □

A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \to 0$, $n \to \infty$, for any $t \in [0, \infty)$. We denote by Φ, the class of the comparison functions $\varphi : [0, \infty) \to [0, \infty)$. For more details and examples , see e.g. [7, 27].

We denote with Ψ the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ. It is clear that if $\Phi \subset \Psi$ (see e.g. [15]) and hence, by Lemma 2.7 (3), for $\psi \in \Psi$ we have $\psi(t) < t$, for any $t > 0$.

We recall the following essential result.

Lemma 2.7. *(Berinde [7], Rus [27]) If $\varphi : [0, \infty) \to [0, \infty)$ is a comparison function, then:*

(1) each iterate φ^k of φ, $k \geq 1$, is also a comparison function;

(2) φ is continuous at 0;

(3) $\varphi(t) < t$, for any $t > 0$.

The notion of a (b)-comparison function was given by Berinde [6] in order to expand some related fixed point results into the class of b-metric space.

Definition 2.8. *(Berinde [6]) Let $s \geq 1$ be a real number. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a (b)-comparison function if the following conditions are fulfilled*

(1) φ is monotone increasing;

(2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

We denote by Ψ_b for the class of (b)-comparison function $\varphi : [0, \infty) \to [0, \infty)$. In a special case of $b = 1$, (b)-comparison function get a special name: (c)-comparison function that has been used effectively in several fixed point results.

The following lemma has an important role in the proof of our main result.

Lemma 2.9. *(Berinde [5]) If $\varphi : [0, \infty) \to [0, \infty)$ is a (b)-comparison function, then we have the following*

(1) the series $\sum_{k=0}^{\infty} s^k\varphi^k(t)$ converges for any $t \in \mathbb{R}_+$;

(2) the function $s_b : [0, \infty) \to [0, \infty)$ defined by $s_b(t) = \sum_{k=0}^{\infty} s^k\varphi^k(t),\ t \in [0, \infty)$, is increasing and continuous at 0.

In what follows we recollect the notion of cone: A nonempty closed convex set $P \subset E$ is said to be a cone if it satisfies the following conditions:

(P1) $\lambda \geq 0, x \in P \Rightarrow \lambda x \in P$.

(P2) $-xx \in P \Rightarrow x = 0_E$.

We note that any (b)-comparison function is a comparison function.

Thorough this paper we consider the b-Banach space $(E, \|\cdot\|)$ which is supposed to be partially ordered by a cone P. We define the partial order $\leq_p$ in E induced by the cone P by

$$(x, y) \in E \times E,\ x \leq_p y \Leftrightarrow y - x \in P.$$

Thorough this paper we consider the b-Banach space $(E, \|\cdot\|)$ which is supposed to be partially ordered by a cone P. We define the partial order $\leq_p$ in E induced by the cone P by

$$(x, y) \in E \times E,\ x \leq_p y \Leftrightarrow y - x \in P.$$

Definition 2.10. *Let $\varphi : E \to E$ be a given mapping. We say that φ is 0_E-level closed from the left, if and only if the set*

$$\mathcal{L}(\varphi_{\geq_p}) := \{x \in E : \varphi(x) \geq_p 0_E\}$$

is nonempty and closed. Moreover, we say that φ is 0_E-level closed from the right, if and only if the set

$$\mathcal{L}(\varphi_{\leq_p}) := \{x \in E : \varphi(x) \leq_p 0_E\}$$

is nonempty and closed.

Definition 2.11. *Let $T : E \to E$ be a given mapping. For a given $x \in E$, we denote by $O(x)$ the orbit of x, that is,*

$$O(x) = \{T^n x : n = 0, 1, 2, ...\}.$$

We say that T is orbitally continuous on $F \subset E$, if and only if T is continuous on $O(x)$, for every $x \in F$.

Definition 2.12. *Let $T, \varphi_i : E \to E$ $(i = 1, 2, ...r)$ be a finite number of mappings. We say that T is a Ćirić-type operator with respect to $\{\varphi_i\}_{i=1}^r$, if and only if there exists a function $\psi : [0, \infty) \to [0, \infty)$ with $\psi \in \Psi_b$, such that*

$$\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq \psi(\|x - y\|),$$

for every $(x, y) \in E \times E$ such that:

$$\varphi_i(x) \leq_p 0_E \text{ and } \varphi_i(y) \geq_p 0_E,\ i = 1, 2, ..., r.$$

Remark 2.13. *Let $T, \varphi_i : E \to E$ $(i = 1, 2, ...r)$ be a finite number of mappings. If T is a Ćirić-type contraction with respect to $\{\varphi_i\}_{i=1}^r$, then by symmetry, we have*

$$\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq \psi(\|x - y\|),$$

where $\psi \in \Psi_b$, for every $(x, y) \in E \times E$ such that:

$$\varphi_i(x) \geq_p 0_E \text{ and } \varphi_i(y) \leq_p 0_E,\ i = 1, 2, ..., r.$$

3. Main results

In the first part of this paragraph we consider the analog of the system (2) for the case $r = 1$ and $\varphi_1 = \varphi$ in the context of b-Banach space:

$$\begin{cases} Tx = x \\ \varphi(x) = 0_E; \end{cases} \tag{3}$$

where T, φ be self-mappings on a b-Banach space $(E, \|\cdot\|)$ with a cone P and 0_E is the zero vector of E, and T forms a Ćirić-type-contraction.

Theorem 3.1. *Let $(E, \|\cdot\|)$ be a b-Banach space and let $T, \varphi : E \to E$ be two given mappings. Suppose the following conditions are satisfied:*

(i) T is orbitally continuous on $\mathcal{L}(\varphi_{\leq_p})$,

(ii) T is a Ćirić-type contraction with respect to φ, i.e.

$$\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq \psi(\|x - y\|),$$

where $\psi \in \Psi_b$, for every $(x, y) \in E \times E$ such that $\varphi(x) \leq_p 0_E$ and $\varphi(y) \geq_p 0_E$, *(iii)* φ *is* 0_E *level closed from left,*

(iv) *there exists* $x_0 \in E$ *such that* $\varphi(x_0) \leq_p 0_E$*,*

(v) *for every* $x \in E$ *we have,*

$$\varphi(x) \leq_p 0_E \Rightarrow \varphi(Tx) \geq_p 0_E,$$

and

$$\varphi(x) \geq_p 0_E \Rightarrow \varphi(Tx) \leq_p 0_E.$$

Then $\{T^n(x_0)\}$ *converges to a solution of (3).*

Proof. On account of (iv), there exists $x_0 \in E$ such that $\varphi(x_0) \leq_p 0_E$. By employing (v), we find

$$\varphi(x_0) \leq_p 0_E \Rightarrow \varphi(Tx_0) \geq_p 0_E.$$

We construct an iterative sequence $\{x_n\}$ in E by

$$x_{n+1} = Tx_n, \; n \in \mathbb{N}_0.$$

Taking (v) into account, we observe $\varphi(x_1) = \varphi(Tx_0) \geq_p 0_E \Rightarrow \varphi(Tx_1) \leq_p 0_E$, and further, we have $\varphi(x_2) = \varphi(Tx_1) \leq_p 0_E \Rightarrow \varphi(Tx_2) \geq_p 0_E$.

Iteratively, we derive that

$$\varphi(x_{2n}) \leq_p 0_E, \text{ and } \varphi(x_{2n+1}) \geq_p 0_E, \text{ for all } n \in \mathbb{N}_0. \tag{4}$$

Regarding Remark 2.13, together with (ii) and (4), we find that

$$\min\{\|Tx_{n-1} - Tx_n\|, \|x_{n-1} - Tx_{n-1}\|, \|x_n - Tx_n\|\} - \min\{\|x_{n-1} - Tx_n\|, \|x_n - Tx_{n-1}\|\} \\ \leq \psi(\|x_{n-1} - x_n\|),$$

for all $n \in \mathbb{N}$, that is

$$\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|, \|x_n - x_{n+1}\|\} - \min\{\|x_{n-1} - x_{n+1}\|, \|x_n - x_n\|\} \\ \leq \psi(\|x_{n-1} - x_n\|),$$

for all $n \in \mathbb{N}$. Accordingly, we have

$$\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|\} \leq \psi(\|x_{n-1} - x_n\|),$$

for all $n \in \mathbb{N}$.

We examine the inequality above in three cases.

Case 1. If $x_{2N} = x_{2N+1}$, for some $N \in \mathbb{N}_0$, then we have

$$x_{2N} = Tx_{2N}.$$

Moreover, from (4), we find

$$\varphi(x_{2N}) \leq_p 0_E \text{ and } \varphi(x_{2N}) = \varphi(Tx_{2N}) = \varphi(x_{2N+1}) \geq 0_E.$$

Consequently, we have $\varphi(x_{2N}) \in P$ and $-\varphi(x_{2N}) \in P$. Regarding that P is a cone, we conclude that $\varphi(x_{2N}) = 0_E$. Thus, $x_{2N} \in E$ forms a solution for (3).

Case 2. If $x_{2N+1} = x_{2N+2}$, for some $N \in \mathbb{N}_0$, then, by verbatim with the Case 1 we conclude that x_{2N+1} is a solution to (3).

Case 3. Suppose that $x_n \neq x_{n+1}$, for every $n \in \mathbb{N}$. Now, to examine the inequality below

$$\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|\} \leq \psi(\|x_{n-1} - x_n\|)$$

we assume that $\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|\} = \|x_{n-1} - x_n\|$. It yields that

$$\|x_{n-1} - x_n\| \leq \psi(\|x_{n-1} - x_n\|),$$

which is a contradiction (see Lemma 2.7).

Consequently, we observe that $\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|\} = \|x_n - x_{n+1}\|$, and hence

$$\|x_n - x_{n+1}\| \leq \psi(\|x_{n-1} - x_n\|).$$

for every $n \in \mathbb{N}$.

Recursively, we derive that

$$\|x_n - x_{n+1}\| \leq \psi^n(\|x_0 - x_1\|), \text{ for all } n \in \mathbb{N}. \tag{5}$$

From (5) and using the triangular inequality, for all $p \geq 1$, we have:

$$\begin{aligned}
\|x_n - x_{n+p}\| &\leq s\|x_n - x_{n+1}\| + s^2\|x_{n+1} - x_{n+2}\| + ... + s^{p-2}\|x_{n+p-3} - x_{n+p-2}\| \\
&\quad + s^{p-1}\|x_{n+p-2} - x_{n+p-1}\| + s^{p-1}\|x_{n+p-1} - x_{n+p}\| \\
&\leq s\psi^n(\|x_0 - x_1)\| + s^2\psi^{n+1}(\|x_0 - x_1)\| + ... + s^{p-2}\psi^{n+p-3}(\|x_0 - x_1)\| \\
&\quad + s^{p-1}\psi^{n+p-2}(\|x_0 - x_1)\| + s^{p-1}\psi^{n+p-1}(\|x_0 - x_1)\| \\
&= \tfrac{1}{s^{n-1}}[s^n\psi^n(\|x_0 - x_1)\| + s^{n+1}\psi^{n+1}(\|x_0 - x_1)\| + ... + s^{n+p-2}\psi^{n+p-2}(\|x_0 - x_1)\| \\
&\quad + s^{n+p-1}\psi^{n+p-1}(\|x_0 - x_1)\|]
\end{aligned}$$

Denoting $S_n = \sum_{k=0}^{n} s^k\psi^k(\|x_0 - x_1)\|$, $n \geq 1$ we obtain:

$$\|x_n, x_{n+p}\| \leq \frac{1}{s^{n-1}}[S_{n+p-1} - S_{n-1}], \ n \geq 1, \ p \geq 1. \tag{6}$$

Using Lemma 2.9, we conclude that the series $\sum_{k=0}^{n} s^k\psi^k(\|x_0, x_1)\|$ is convergent. Thus there exists $S = \lim_{n\to\infty} S_n \in [0, \infty)$. Regarding $s \geq 1$ and by (6), we obtain that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in the b-Banach space $(E, \|\cdot\|)$.

Therefore, there exists some $x^* \in E$ such that

$$\lim_{n\to\infty} \|x_n - x^*\| = 0 \tag{7}$$

Since T is orbitally continuous on $\mathcal{L}(\varphi_{\leq_p})$, then T is continuous on $O(x_0)$. Therefore, by (7), we deduce that

$$\lim_{n\to\infty} \|x_{n+1} - Tx^*\| = 0 \tag{8}$$

By the uniqueness of the limit, we obtain

$$Tx^* = x^* \tag{9}$$

On the other hand, from (4), we have

$$x_{2n+1} \subset \mathcal{L}(\varphi_{\geq_p}), \ n = 0, 1, ...$$

Since φ is 0_E-level closed from the left, we deduce from (7) that $x^* \in \mathcal{L}(\varphi_{\geq_p})$, that is,

$$\varphi(x^*) \geq_p 0_E.$$

However, by (v) and (9) , we derive that

$$\varphi(x^*) = \varphi(Tx^*) \leq_p 0_E.$$

As a result, we have $\varphi(x^*) \in P$ and $-\varphi(x^*) \in P$. By recalling that P is a cone, we get

$$\varphi(x^*) = 0_E. \tag{10}$$

Finally, (9) and (10) imply that $x^* \in E$ is a solution to (11) . □

The next result holds if we replace the condition (iii), "φ is 0_E level closed from left" in Theorem 3.1 with the condition (iii)′, that is, "φ is 0_E level closed from right".

Theorem 3.2. *Let $(E, \|\cdot\|)$ be a b-Banach space and let $T, \varphi : E \to E$ be two given mappings. Suppose the following conditions are satisfied:*

(i) T is orbitally continuous on $\mathcal{L}(\varphi_{\leq_p})$,

(ii) T is a Ćirić-type contraction w.r.t φ, i.e.

$$\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq \psi(\|x - y\|),$$

where $\psi \in \Psi_b$, for every $(x, y) \in E \times E$ such that $\varphi(x) \leq_p 0_E$ and $\varphi(y) \geq_p 0_E$.

(iii)′ φ is 0_E level closed from right

(iv) There exists $x_0 \in E$ such that $\varphi(x_0) \leq_p 0_E$

(v) For every $x \in E$ we have:

$$\varphi(x) \leq_p 0_E \Rightarrow \varphi(Tx) \geq_p 0_E$$

and

$$\varphi(x) \geq 0_E \Rightarrow \varphi(Tx) \leq_p 0_E$$

Then $\{T^n(x_0)\}$ converges to a solution of (3).

Proof. To obtain the desired result just replace φ by $-\varphi$ in Theorem 3.1. □

Remark 3.3. *We can observe that Theorems 3.1 and 3.2 are still valid if we replace condition (i) by the following condition: (i′) T is orbitally continuous on $\mathcal{L}(\varphi_{\geq_p})$.*

In the second part of this paragraph we study the solvability of system (2) in the setting of b-Banach spaces. More precisely, we investigate a system

$$\begin{cases} Tx = x \\ \varphi_i(x) = 0_E, \ i = 1, 2, ...r, \end{cases} \tag{11}$$

where $T, \varphi_i : E \to E, \ i = 1, 2, ..., r$ be a finite number of mappings defined on a b-Banach space $(E, \|\cdot\|)$ with a cone P, and 0_E is the zero vector of E, and T is mapping satisfying a Ćirić-type-contraction. For $r = 1$, this system coincide with (3).

The following is the first result of this part.

Theorem 3.4. *Let $(E, \|\cdot\|)$ be a b-Banach space and let $T, \varphi_i : E \to E$ $(i = 1, 2, ..., r)$ be a finite number of mappings. Suppose the following conditions are satisfied:*

(i) *T is orbitally continuous on* $\cap_{i=1}^{r}\mathcal{L}(\varphi_{i\leq_p})$ *;*

(ii) *T is a Ćirić-type contraction w.r.t* $\{\varphi_i\}_{i=1}^{r}$*, i.e.*

$$\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq \psi(\|x - y\|),$$

where $\psi \in \Psi_b$*, for every* $(x, y) \in E \times E$ *such that* $\varphi_i(x) \leq_p 0_E$ *and* $\varphi_i(y) \geq_p 0_E, i = 1, 2, ..., r$

(iii) $\varphi_i,\ i = 1, 2, ..., r$ *is* 0_E *level closed from left*

(iv) *There exists* $x_0 \in E$ *such that* $\varphi_i(x_0) \leq_p 0_E,\ i = 1, 2, ..., r$

(v) *For every* $x \in E$ *we have:*

$$\varphi_i(x) \leq_p 0_E,\ i = 1, 2, ..., r \Rightarrow \varphi_i(Tx) \geq_p 0_E,\ i = 1, 2, ..., r$$

and

$$\varphi_i(x), \geq 0_E\ i = 1, 2, ..., r \Rightarrow \varphi_i(Tx) \leq_p 0_E,\ i = 1, 2, ..., r$$

Then $\{T^n(x_0)\}$ *converges to a solution of (11).*

Proof. Due to (iv) there exists $x_0 \in E$ such that $\varphi_i(x_0) \leq_p 0_E,\ i = 1, 2, ..., r$. Applying (v), we get that

$$\varphi_i(x_0) \leq_p 0_E \Rightarrow \varphi_i(Tx_0) \geq_p 0_E,\ i = 1, 2, ..., r$$

Consider the iterative sequence $\{x_n\}$ in E by

$$x_{n+1} = Tx_n, \qquad n \in \mathbb{N}_0.$$

By employing (v) again, we find

$$\varphi_i(x_1) \geq_p 0_E \Rightarrow \varphi_i(Tx_1) \leq_p 0_E,\ i = 1, 2, ..., r,$$

and hence

$$\varphi_i(x_2) \leq_p 0_E \Rightarrow \varphi_i(Tx_2) \geq_p 0_E,\ i = 1, 2, ..., r.$$

Recursively, we find

$$\varphi_i(x_{2n}) \leq_p 0_E, \text{ and } \varphi_i(x_{2n+1}) \geq_p 0_E, \qquad n \in \mathbb{N}_0, i = 1, 2, ..., r. \tag{12}$$

Regarding the symmetry property, discussed in Remark 2.13 together with (ii) and (12), we get

$$\min\{\|Tx_{n-1} - Tx_n\|, \|x_{n-1} - Tx_{n-1}\|, \|x_n - Tx_n\|\} - \min\{\|x_{n-1} - Tx_n\|, \|x_n - Tx_{n-1}\|\} \leq \psi(\|x_{n-1} - x_n\|),\ n \in \mathbb{N},$$

that is,

$$\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|, \|x_n - x_{n+1}\|\} - \min\{\|x_{n-1} - x_{n+1}\|, \|x_n - x_n\|\} \leq \psi(\|x_{n-1} - x_n\|),\ n \in \mathbb{N}.$$

Consequently, we find that

$$\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|, \|x_n - x_{n+1}\|\} \leq \psi(\|x_{n-1} - x_n\|),\ n \in \mathbb{N},$$

which is equivalent to

$$\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|\} \leq \psi(\|x_{n-1} - x_n\|),\ n \in \mathbb{N}.$$

For analyzing the inequality above, we shall consider three cases.

Case 1. If $x_{2N} = x_{2N+1}$, for some $N \in \mathbb{N}_0$, then $x_{2N} = Tx_{2N}$. Hence, from (12), we find

$$\varphi_i(x_{2N}) \leq_p 0_E, \quad \varphi_i(x_{2N}) = \varphi_i(Tx_{2N}) = \varphi_i(x_{2N+1}) \geq 0_E., \ i = 1, 2, ..., r.$$

Therefore, $\varphi_i(x_{2N}) \in P$ and $-\varphi_i(x_{2N}) \in P$, for every $i = 1, 2..., r$. Regarding that P is a cone, we conclude that $\varphi_i(x_{2N}) = 0_E$, $i = 1, 2, ..., r$. Hence, $x_{2N} \in E$ is a solution to (11).

Case 2. If $x_{2N+1} = x_{2N+2}$, for some $N \in \mathbb{N}_0$, by verbatim, we deduce that x_{2N+1} is a solution to (11).

Case 3. Suppose that $x_n \neq x_{n+1}$, for every $n = 1, 2, ...$ with

$$\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|\} - \leq \psi(\|x_{n-1} - x_n\|).$$

On account of function ψ we have

$$\|x_n - x_{n+1}\| \leq \psi(\|x_{n-1} - x_n\|).$$

Iteratively, we find that

$$\|x_n - x_{n+1}\| \leq \psi^n(\|x_0 - x_1\|), \text{ for all } n = 1, 2, ... \tag{13}$$

From (13) and using the triangular inequality, for all $p \geq 1$, we have:

$$\begin{aligned}
\|x_n - x_{n+p}\| &\leq s\|x_n - x_{n+1}\| + s^2\|x_{n+1} - x_{n+2}\| + ... + s^{p-2}\|x_{n+p-3} - x_{n+p-2}\| \\
&\qquad +s^{p-1}\|x_{n+p-2} - x_{n+p-1}\| + s^{p-1}\|x_{n+p-1} - x_{n+p}\| \\
&\leq s\psi^n(\|x_0 - x_1)\| + s^2\psi^{n+1}(\|x_0 - x_1)\| + ... + s^{p-2}\psi^{n+p-3}(\|x_0 - x_1)\| \\
&\qquad +s^{p-1}\psi^{n+p-2}(\|x_0 - x_1)\| + s^{p-1}\psi^{n+p-1}(\|x_0 - x_1)\| \\
&= \tfrac{1}{s^{n-1}}[s^n\psi^n(\|x_0 - x_1)\| + s^{n+1}\psi^{n+1}(\|x_0 - x_1)\| + ... + s^{n+p-2}\psi^{n+p-2}(\|x_0 - x_1)\| \\
&\qquad +s^{n+p-1}\psi^{n+p-1}(\|x_0 - x_1)\|]
\end{aligned}$$

By letting $S_n = \sum_{k=0}^{n} s^k\psi^k(\|x_0 - x_1)\|$, $n \geq 1$ we obtain:

$$\|x_n, x_{n+p}\| \leq \frac{1}{s^{n-1}}[S_{n+p-1} - S_{n-1}], \ n \geq 1, \ p \geq 1. \tag{14}$$

Using Lemma 2.9, we conclude that the series $\sum_{k=0}^{n} s^k\psi^k(\|x_0, x_1)\|$ is convergent. Thus there exists $S = \lim_{n\to\infty} S_n \in [0, \infty)$. Since $s \geq 1$ and taking (6) into account, we obtain that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in the b-Banach space $(E, \|\cdot\|)$. Thus, there exists some $x^* \in E$ such that

$$\lim_{n\to\infty} \|x_n - x^*\| = 0 \tag{15}$$

Since T is orbitally continuous on $\cap_{i=1}^{r}\mathcal{L}(\varphi_{i\leq_p})$, then T is continuous on $O(x_0)$. Therefore, by (15), we deduce that

$$\lim_{n\to\infty} \|x_{n+1} - Tx^*\| = 0 \tag{16}$$

By the uniqueness of the limit, we obtain

$$Tx^* = x^* \tag{17}$$

On the other hand, from (12), we have

$\{x_{2n+1}\} \subset \cap_{i=1}^{r} \mathcal{L}(\varphi_{i \geq_p}),\ n = 0, 1, ...$

Since φ is 0_E-level closed from the left, we deduce from (15) that $x^* \in \cap_{i=1}^{r} \mathcal{L}(\varphi_{i \geq_p})$, that is,

$\varphi_i(x^*) \geq_p 0_E\ i = 1, 2, ..., r.$

But by (v) and (17) , we obtain

$\varphi_i(x^*) = \varphi(Tx^*) \leq_p 0_E,\ i = 1, 2, ..., r.$

Hence, we have $\varphi_i(x^*) \in P$ and $-\varphi_i(x^*) \in P,\ i = 1, 2, ..., r.$. Since P is a cone, we get

$$\varphi(x^*) = 0_E,\ i = 1, 2, ..., r. \tag{18}$$

So we have that $x^* \in E$ is a solution to (11) . □

The next result holds if we replace the condition (iii) in the above theorem.

Theorem 3.5. *Let $(E, \|\cdot\|)$ be a b-Banach space and let $T, \varphi_i : E \to E$ $(i = 1, 2, ..., r)$ be a finite number of mappings. Suppose the following following conditions are satisfied:*

(i) T is orbitally continuous on $\cap_{i=1}^{r} \mathcal{L}(\varphi_{i \leq_p})$,

(ii) T is a Ćirić-type contraction w.r.t $\{\varphi_i\}_{i=1}^{r}$, i.e.

$$\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq \psi(\|x - y\|),$$

where $\psi \in \Psi_b$, for every $(x, y) \in E \times E$ such that $\varphi_i(x) \leq_p 0_E$ and $\varphi_i(y) \geq_p 0_E, i = 1, 2, ..., r,$

(iii)2 $\varphi_i,\ i = 1, 2, ..., r$ is 0_E level closed from right,

(iv) There exists $x_0 \in E$ such that $\varphi_i(x_0) \leq_p 0_E,\ i = 1, 2, ..., r,$

(v) For every $x \in E$ we have:

$$\varphi_i(x) \leq_p 0_E,\ (i = 1, 2, ..., r) \Rightarrow \varphi_i(Tx) \geq_p 0_E,\ i = 1, 2, ..., r$$

and

$$\varphi_i(x), \geq 0_E\ (i = 1, 2, ..., r) \Rightarrow \varphi_i(Tx) \leq_p 0_E,\ i = 1, 2, ..., r$$

Then $\{T^n(x_0)\}$ converges to a solution of (11).

Remark 3.6. *We can observe that Theorems 3.4 and 3.5 are still valid if we replace condition (i) by the following condition:*
(i') T is orbitally continuous on $\cap_{i=1}^{r} \mathcal{L}(\varphi_{i \geq_p})$

4. Consequences

We present first a fixed point result that can be deduced from Theorem 3.1.

Corollary 4.1. *Let $(E, \|\cdot\|)$ be a b-Banach space and let $T : E \to E$ be a given mappings. Suppose the following conditions are satisfied:*

(i) T is orbitally continuous on E ;

(ii) T is a Ćirić-type contraction, i.e. there exists a function $\psi \in \Psi_b$*, such that for every* $(x, y) \in E \times E$,

$$\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq \psi(\|x - y\|)$$

Then $\{T^n(x)\}$ *converges to a fixed point of T, for every* $x \in E$.

An interesting consequence can be obtained applying the main results from the previous paragraph. We consider the following common fixed point problem, where $T, F : E \to E$ are two given mappings.

$$\begin{cases} x = Tx \\ x = Fx \end{cases} \tag{19}$$

This system is equivalent to

$$\begin{cases} x = Tx \\ \varphi(x) = 0_E, \end{cases} \tag{20}$$

where $\varphi : E \to E$ is defined by $\varphi(x) = Fx - x,\ x \in E$
We define the sets:

$$H_1 := \{x \in E : Fx \leq_p x\}$$

and

$$H_1 := \{x \in E : Fx \leq_p x\}.$$

and we can state the following result.

Corollary 4.2. *Let* $(E, \|\cdot\|)$ *be a b-Banach space and let* $T, F : E \to E$ *be two given mappings. Suppose the following conditions are satisfied:*

(i) T is orbitally continuous on H_1 *;*

(ii) There exists a function $\psi : [0, \infty) \to [0, \infty),\ \psi \in \Psi_b$*, such that for every* $(x, y) \in E \times E$

$$Fx \leq_p 0_E,\ Fy \geq_p 0_E \Rightarrow \min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq \psi(\|x - y\|),$$

(iii) H_2 *is a closed subset of E*

(iv) There exists $x_0 \in E$ *such that* $Fx_0 \leq_p x_0$

(v) For every $x \in E$ *we have:*

$$Sx \leq_p x \Rightarrow F(Tx) \geq_p Tx$$

and

$$Sx \geq_p x \Rightarrow F(Tx) \leq_p Tx$$

Then the Picard sequence $\{T^n(x_0)\}$ *converges to a solution of (19).*

Proof. Applying Theorem 3.1 with the function φ defined by $\varphi(x) = Fx - x,\ x \in E$ the conclusion follows. □

5. Conclusions

Notice that we extend the results in [25] in two-folds. First, we expanded the abstract spaces from Banach to b-Banach. Secondly, by using b-comparison function, we consider more general form of Ćirić operators. Thus, even, if we let $s = 1$, we get more general results when we compare the corresponding results in [25].

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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