



Generalised Distances of Sequences II: *B*-Distances with Weight Sequences

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Abstract. In this paper, we investigate the weighted *B*-distances of infinite sequences. The general neighbourhood sequences were introduced for measuring distances in digital geometry (\mathbb{Z}^n), and the theory was recently extended for application to sequences. By assigning various weights to the elements of the sequences the concept is further generalized. An algorithm is presented which provides a shortest path between two sequences. Formula is also provided to calculate the weighted *B*-distance of any two sequences with a neighbourhood sequence *B* and a weight sequence. There are several neighbourhood sequences, which do not generate metrics. We prove a necessary and sufficient condition for a *B*-distance to define a generalized metric space above the sequences. Moreover, our results can be applied if the elements of the sequences used with various weights. In case of weight functions used *B*-distances we present also the metric conditions.

1. Introduction

Finite and infinite sequences are widely used in mathematical analysis, calculus and in various other fields, see, e.g., [6].

The Hamming-distance (*H*-distance) of two same-length sequences can be extended to infinite sequences over infinite alphabets (\mathbb{Z} , or \mathbb{R}), see [11]. Other possibilities are the supremum norm (sup-distance) (see [7]) and the inf-distance, but this latter is rarely used.

The so-called *L*-distances are widely used in functional analysis, measure theory and in statistical methods also [1, 2, 5]. In most cases for finite sequences (or with other terminology, for points in finite, let us say, *n* dimension), they are defined as $d(p, q) = \left(\sum_{i=1}^n |p(i) - q(i)|^l \right)^{\frac{1}{l}}$, depending on the value of *l*. For infinite sequences they are not pleasant, but using a weight-sequence we can use them [2]. (The used weight sequence is usually non-negative, monotonous decreasing and its sum is finite in this case.) There is also a possibility to use the weighted sup-distance.

In this paper, we mainly work with distances with integer values, they are based on neighbour relations among the sequences and various neighbourhood sequences (*B*).

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The theory of neighbourhood sequences (n-sequences, for short) comes from digital geometry, but they can also be applied for infinite sequences [11] and for (formal) languages [8]. In digital geometry, finite integer sequences are used according to the dimension of the used digital space. The neighbourhood relations are natural; there are various neighbours of a point (sequence). The theory is, then, extended to infinite dimension in [4, 9, 10]. By dropping the criteria of usage of integer values in a sequence, the theory is further extended from digital spaces to (infinite) sequences [11].

By the help of an n-sequence $B = (b(i))_{i=1}^{\infty}$ the B -distance of sequences p, q is defined as follows [11]. The length of a shortest path from p to q is taken, such that at the i -th step, we may move from a sequence to another if and only if they are $b(i)$ -neighbours.

In [9] we have presented an algorithm which provides a shortest path between digital points, both in case of finite and infinite dimensional spaces. It is easy to find an n-sequence B , even a periodic one, such that the B -distance does not provide (generalized) metrics on the set of sequences. In [9] we have proved a necessary and sufficient condition for distance functions based on n-sequences to define a generalized metric in \mathbb{Z}^{∞} . One of the purposes of this paper to generalize these concepts from \mathbb{Z}^{∞} to \mathbb{R}^{∞} .

In this paper, we generalize the concept of B -distances of sequences of [11] by the following reasons: Our previous theory works very well if the elements of the sequences play equal roles. However, in some cases, for instance, the tails have less important role than the first elements of the sequences. In these cases, the weighted sup-distance or generally, the weighted L -distances were used [2]. We show here that the theory of B -distances can also be extended. The structure of this paper is as follows. In the next section, we give our notation and we recall the basic concepts. In the third section, we present our new definitions by extending and generalizing the previous ones: we introduce the weighted B_c -distances by using non-negative weight-sequences c . In section four, an algorithm is presented to solve the shortest path problem between any two sequences. In section five, a formula is provided to compute weighted B -distances and in section six we describe some properties of weighted B -distances including a proof for necessary and sufficient conditions for the n-sequence B to define a (generalized) metric over the set (i.e., the space) of sequences using also a weight sequence. In the last section we summarize our results.

2. Preliminaries

In this section we recall some basic concepts and our notations (that is similar to [11]).

Throughout the paper \mathbb{R}^{∞} will denote the set of all sequences.

In this paper, similarly to [11], we use generalized types of convergence: k -convergences and k -sequences.

Definition 2.1. A sequence $p \in \mathbb{R}^{\infty}$ is k -convergent (for a fixed non-negative value k), if there exists $n \in \mathbb{N}$ such that for all i, j with $i > n$ and $j > n$ we have $|p(i) - p(j)| \leq k$.

Definition 2.2. A sequence $p \in \mathbb{R}^{\infty}$ is k -sequence (for a fixed positive value of k) if there exists a natural number n such that for all $i \in \mathbb{N}$ if $i > n$ then $|p(i)| < k$.

From here we will use the terms k -convergence and k -sequence with arbitrary non-negative integer values of k , however our definition works for all (not necessary integer) non-negative value of k .

Now, for sake of completeness, we recall some basic definitions about distance functions.

Definition 2.3. A function $d : \mathbb{R}^{\infty} \times \mathbb{R}^{\infty} \rightarrow \mathbb{R} \cup \{\infty\}$ is called a generalized metric on \mathbb{R}^{∞} , if it satisfies the following conditions:

a) $\forall p, q \in \mathbb{R}^{\infty}$: $d(p, q) \geq 0$, and $d(p, q) = 0$ if and only if $p = q$ (positive definiteness),

b) $\forall p, q \in \mathbb{R}^{\infty}$: $d(p, q) = d(q, p)$, (symmetry)

c) $\forall p, q, r \in \mathbb{R}^{\infty}$: $d(p, q) + d(q, r) \geq d(p, r)$ (triangle inequality).

Moreover, if for every possible pair of $p, q \in \mathbb{R}^{\infty}$ the distance $d(p, q)$ is finite, then it is a metric.

If instead of point a) we have only

a') $\forall p, q \in \mathbb{R}^\infty: d(p, q) \geq 0$, and $d(p, p) = 0$,
 then the function $d : \mathbb{R}^\infty \times \mathbb{R}^\infty \rightarrow \mathbb{R} \cup \{\infty\}$ is a semi-metric on \mathbb{R}^∞ .

For measuring distances of sequences, the supremum norm is used in [7]. The Hamming-distance could also be an option (mostly when the two sequences differ only in finitely many places). The H -distance is a discrete distance.

Definition 2.4. The sup-distance of the sequence p and q is given by

$$d(p, q; sup) = \sup(|p(i) - q(i)|).$$

The Hamming-distance of p and q is

$$d(p, q; H) = \sum_{p(i) \neq q(i)} 1.$$

The inf-distances of p and q is

$$d(p, q; inf) = \inf(|p(i) - q(i)|).$$

The discrete metric over the set of sequences is the following:

$$d(p, q; disc) = \begin{cases} 0, & \text{if } p = q \\ 1, & \text{if } p \neq q. \end{cases}$$

Both $d(p, q; sup)$ and $d(p, q; H)$ are generalized metrics over \mathbb{R}^∞ . The inf-distance is rarely used since only properties b) and a') of Definition 2.3 hold. The discrete metric is the simplest metric.

One of our most important investigations is introducing the neighbourhood relation among sequences.

Definition 2.5. Let p and q be two sequences in \mathbb{R}^∞ . Let k be a non-negative integer. The sequences p and q are k -neighbours, if the following two conditions hold:

- $|p(i) - q(i)| \leq 1$ for all $i \in \mathbb{N}$, and
- $\sum_{i \in \mathbb{N}, p(i) \neq q(i)} 1 \leq k$.

Definition 2.6. The infinite sequence $B = (b(i))_{i=1}^\infty$ ($b(i) \in \mathbb{N} \cup \{\infty\}$) is called a neighbourhood sequence (or shortly n -sequence.). If for some $l \in \mathbb{N}$, $b(i) = b(i + l)$ holds for every $i \in \mathbb{N}$, then B is called periodic (with period l).

For investigating distances of sequences, we will use their difference sequences in the following way.

Notation 2.7. Let p and q be two sequences. Put $w(i) = |p(i) - q(i)|$ for all i , and $w = (w(i))_{i=1}^\infty$. The sequence w is called the (absolute) difference of p and q .

The up-integer-difference-sequence (uids) u of p and q is defined by the top (i.e., ceiling) of the elements of their absolute difference as $u(i) = \lceil w(i) \rceil = \lceil |p(i) - q(i)| \rceil$, where $\lceil x \rceil$ is the upper integer part of the real number x , i.e. $\lceil x \rceil = \inf\{k \in \mathbb{Z}, k \geq x\}$.

Definition 2.8. Let p and q be two sequences and $B = (b(i))_{i=1}^\infty$ be an n -sequence. A finite sequence of sequences $\Pi(p, q; B)$ of the form $p = p_0, p_1, \dots, p_m = q$, where $p_{i-1}, p_i \in \mathbb{R}^\infty$ are $b(i)$ -neighbours for $1 \leq i \leq m$, is called a B -path from p to q . We write $m = |\Pi(p, q; B)|$ for the length of the path.

We should note the following about the existence of B -paths.

Remark 2.9. It is possible that there are no B -paths between two sequences. For example, if the set $\{|p(i) - q(i)| : i \in \mathbb{N}\}$ –, i.e. the set of elements of the difference sequence of them – is unbounded, then there are no neighbourhood sequence B , for which a B -path would exist between the sequences $p = (p(i))_{i=1}^\infty$ and $q = (q(i))_{i=1}^\infty$.

Now, we recall the B -distance of two sequences.

Definition 2.10. Let $p, q \in \mathbb{R}^\infty$ and B be an n -sequence. If there is no B -path between p and q , then we put $d(p, q; B) = \infty$. Otherwise, denote by $\Pi^*(p, q; B)$ a shortest path (i.e., a B -path with minimal length) from p to q , and set $d(p, q; B) = |\Pi^*(p, q; B)|$. We call $d(p, q; B)$ the B -distance of the sequences p and q .

It is evident that using the definition of B -distance above, it is positive definite (point a) of Definition 2.3) for any n -sequence B .

Definition 2.11. Let B_1 and B_2 be two neighbourhood sequences. We say that B_1 is faster than B_2 , if

$$d(p, q; B_1) \leq d(p, q; B_2) \quad \text{for all } p, q \in \mathbb{R}^\infty.$$

We denote this relation by $B_1 \supseteq^* B_2$.

Originally, the relation \supseteq^* was introduced by Das [3] in the two dimensional digital space, and by Fazekas *et al.* in [4] for higher dimensions. We will use it for infinite sequences (\mathbb{R}^∞).

For later use we need to introduce some further notations.

Definition 2.12. Let $m \in \mathbb{N}$ and $B = (b(i))_{i=1}^\infty$ an n -sequence. Put

$$b^{(m)}(i) = \min(b(i), m) \text{ and } B^{(m)} = (b^{(m)}(i))_{i=1}^\infty.$$

The sequence $B^{(m)}$ is called the m -limited sequence of B . Denote by $f_k(i)$ the i -th subsums of the k -limited sequence of B , i.e., put

$$f_k(i) = \begin{cases} \sum_{j=1}^i b^{(k)}(j), & \text{if } i \geq 1, \\ 0, & \text{if } i = 0. \end{cases}$$

Definition 2.13. Let $B = (b(i))_{i=1}^\infty$ an n -sequence. The sequence $B(j) = (b(i))_{i=j}^\infty$ is called the j -shifted n -sequence of B .

In the next section we generalize the concept of B -distances.

3. Definition of weighted B -distances

In this section, by introducing weight sequences, we generalize the notion of B -distances. Note that in our description, we do not use every restriction for the elements of the weight sequence that is usually used.

Definition 3.1. The sequence $c = (c(i))_{i=1}^\infty$ is called a weight sequence if $c(i) \in \mathbb{R} \cup \{\infty\}$, and $c(i) \geq 0$ for all $i \in \mathbb{N}$.

Observe that we can use non-descending sequences and/or we allow that the sum of the sequences go to infinity, moreover we allow the symbol ∞ in our weight-sequence. (Our restriction is only that each element is non-negative.)

The used L -distances (including the sup-distance) with weight-sequence are defined in the form below.

Definition 3.2. Let c be a weight-sequence which does not contain the symbol ∞ . The L -distance with c is:

$$d_c(p, q; L_j) = \left(\sum_{i=1}^\infty c(i) |p(i) - q(i)|^j \right)^{\frac{1}{j}}.$$

For special case $j = 1$: the formula

$$d_c(p, q; L_1) = \sum_{i=1}^\infty c(i) |p(i) - q(i)|$$

is used. In the case $j = \infty$ we get the weighted sup-distance as

$$d_c(p, q; L_\infty) = \sup_{i=1}^\infty (c(i) |p(i) - q(i)|).$$

One can ask the following question about our results presented in [11]: Why have we defined the neighbourhood relation and allowed to change a value in a step by at most 1? The answer is that it looks simple and natural, but here we refine the definition. We can change the elements of the sequence depending on their place in the sequence and on the used weight function, as we present it in the next definitions.

Definition 3.3. The sequence d is the inverse weight-sequence of c if for all $i \in \mathbb{N}$, $d(i) = \frac{1}{c(i)}$. Here we use $d(i) = 0$, when $c(i) = \infty$; and $d(i) = \infty$, when $c(i) = 0$.

We will use the elements of the weight-sequences as the weights of the members of the sequences at the same position, i.e., the more important places in the sequences have higher weight values. (The distance of sequences p and q is not less if we use lower weight-values in the weight-sequence.)

Definition 3.4. The sequences p and q are k_c -neighbours for $k \in \mathbb{N}$ with the weight-sequence c (let its inverse weight-sequence be d) if the following conditions hold:

- $|p(i) - q(i)| \leq d(i)$ for all $i \in \mathbb{N}$, and
- $\sum_{\forall i \in \mathbb{N}, p(i) \neq q(i)} 1 \leq k$.

We define the B_c -paths and B_c -distances of the sequences in the same way as they were presented in Definitions 2.8 and 2.10, respectively, using the neighbourhood relation of Definition 3.4.

Remark 3.5. There is an alternative way to define the neighbourhood relations with weight-sequences on \mathbb{R}^∞ . For a $k \in \mathbb{N}$ and a weight sequence c let the sequences p and q be k_c -neighbours if

- $c(i)|p(i) - q(i)| \leq 1$ for all i and,
- $\sum_{\forall i, c(i)(p(i)-q(i)) \neq 0} 1 \leq k$.

The difference (between the neighbourhood relations defined in Definition 3.4 and this alternative neighbourhood relations) occurs when the weight-sequence c contains the element 0. In the first case, the distance of any two distinct sequences is greater than 0, i.e., they are not 0-neighbours. With the second type of criteria the distance of the sequences is 0 if they differ only in such elements ($p(i) \neq q(i)$) for those $c(i) = 0$ (i.e., the position has 0 weight, and thus 0 importance). In that case our distance function is a semi-metric if the triangle inequality holds for it. (For the condition for n -sequences that satisfy the triangle inequality see Theorem 6.2.) We can get a (generalized) metric by grouping the sequences with distance 0 to the same class. In this way, we get (finite or infinite) sub-sequences of the original ones deleting the i -th elements from all elements of \mathbb{R}^∞ , c and of course d , for all i for which $c(i) = 0$. After deletion of these items the alternative definition coincides with Definition 3.4, because, in this case, for all remained indices i the condition $c(i) > 0$ holds. Thus, if $c(i) > 0$ for all i , then both definitions are the same and for any n -sequence B the properties a) and b) of Definition 2.3 hold.

For using the relation of Definition 3.4 we will define the weighted (absolute) difference sequence and the weighted uids of two sequences, using the weight-sequence c in analogous way as they are given in Notation 2.7.

Definition 3.6. Let p and q be two sequences and d be the inverse weight-sequence of a weight sequence c . Put

$$w'(i) = \begin{cases} 0, & \text{if } p(i) = q(i) \\ \min(1, |p(i) - q(i)|), & \text{if } d(i) = \infty \\ \infty, & \text{if } d(i) = 0 \text{ and } p(i) \neq q(i) \\ \frac{|p(i)-q(i)|}{d(i)}, & \text{otherwise;} \end{cases}$$

for all i , and let $w_c = (w'(i))_{i=1}^\infty$. The sequence w_c is called the weighted (absolute) difference of p and q with respect to the weight sequence c .

The weighted up-integer-difference-sequence (wuids) $u_c = (u'(i))_{i=1}^\infty$ of p and q is defined as the ceiling (top) of the elements of their weighted absolute difference, i.e., $u'(i) = \lceil w'(i) \rceil$.

It is obvious that the sequence u_c contains integers which give us the value of the number of steps we need to change the corresponding element of p to reach the corresponding element of q . Therefore, if there is an element $c(i) = \infty$ (it is equivalent to $d(i) = 0$) of the weight-sequence c and p and q differs at the i -th element ($p(i) \neq q(i)$), then $d_c(p, q; B) = \infty$ for any n -sequence B . The elements of the weight sequence mean how important are the values of the sequences at these places. In our terminology, the more important values play more important role in the distance. The weight $c(i) = \infty$ means that the i -th elements are so important that if they differs, then the sequences are in infinite distances from each-other.

4. Shortest weighted B -paths between sequences

Now we recall Algorithm 1 from [11] and we present its variation for the case of weighted n -sequences.

Algorithm 1

Input: An n -sequence $B = (b(i))_{i=1}^\infty$, a weight sequence c (and its inverse d) and $p, q \in \mathbb{R}^\infty$, such that $d_c(p, q; B) < \infty$.

- step 1. Let $w^{(0)}$ be the absolute difference of p and q as defined in Notation 2.7, and $w_c^{(0)} = (w'(i))_{i=1}^\infty$ be the weighted difference of p and q as in Definition 3.6, $t(i) = \text{sgn}(p(i) - q(i))$ $i \in \mathbb{N}$, and put $j = 0$ and $\Pi = (p)$.
- step 2. If $w^{(j)}(i) = 0$ for every i , then goto step 8, else set $j = j + 1$.
- step 3. Put $w^{(j)} = w^{(j-1)}$ and $w_c^{(j)} = w_c^{(j-1)}$.
- step 4. If $b(j)$ is finite, then select the largest $b(j)$ entries of $w_c^{(j)}$, and the respective values in $w^{(j)}$. If $b(j)$ is infinite, then select all the entries of $w^{(j)}$ and $w_c^{(j)}$.
- step 5. For each selected $w^{(j)}(i)$ if $w^{(j)}(i) \geq d(i)$, then let $w'^{(j)}(i) = w'^{(j-1)}(i) - 1$ and $w^{(j)}(i) = w^{(j-1)}(i) - d(i)$ else let $w^{(j)}(i) = w'^{(j)}(i) = 0$.
- step 6. Append to the path Π the sequence x_j defined by $x_j(i) = q(i) + w^{(j)}(i)t(i)$ for all i .
- step 7. Goto 2.
- step 8. Output Π as a minimal B_c -path between p and q , and j as the length of this path.

Using Algorithm 1 we get the answer for the shortest path problem using by n -sequences and weight-sequences together. It can be shown similarly to the proof of Theorem 1 in [11] with the definition of w_c , that this algorithm is also correct. (We have only new scales for the elements depending on the values of $c(i)$.) The algorithm works if the B_c -distance of sequences p and q is finite. The theorem below states a necessary and sufficient condition for B_c -distance to be finite, and, thus, allows to check this property before one applies the algorithm.

Theorem 4.1. *The B_c -distance of two arbitrary sequences p and q is finite for an n -sequence B including the symbol ∞ k times if and only if their weighted difference w_c is a k -sequence. (And their distance is infinite if the number of ∞ is k , and w_c is not k -sequence.)*

Proof. This theorem is a consequence of Theorem 2 in [11] using the re-scaled property of w_c . \square

5. Formula for B -distances with weight sequences

For calculating B_c -distances we have the following theorem. For the calculation we use the sequence $v_c = (v'(i))_{i=1}^\infty$, which has the same elements as the uids u_c , sorting by non-decreasing order (i.e. the multiset of elements of u_c is the same as for v_c and for all $i < j$ the condition $v'(i) \geq v'(j)$ holds), as the algorithm changes those values of u_c first, which have greatest values among them.

Theorem 5.1. Let $B^{(j)}$ the j -limited sequence of the n -sequence B and c is a weight-sequence. The weighted B_c -distance of $p, q \in \mathbb{R}^\infty$ is

$$d_c(p, q; B) = \max_{i \in \mathbb{N}} \{d_c^{(i)}(p, q)\},$$

where

$$d_c^{(i)}(p, q) = \max \left\{ h \mid \sum_{k=1}^i v'(k) > \sum_{k=1}^{h-1} b^{(i)}(k) \right\}.$$

Proof. It is obvious by proof of Proposition 5 of [11] and the definition of v_c . \square

The effect of using a weight-sequence is to ‘re-scale’ the difference of the sequences.

We consider the items of p and q by help of their difference sequence w and the given weight-sequence c .

Lemma 5.2. Let p and q be arbitrary sequences in \mathbb{R}^∞ , and let the n -sequence B_1 be faster than the n -sequence B_2 (i.e., $B_1 \sqsupseteq^* B_2$). Using a weight-sequence c we have the following statement. If $d_c(p, q; B_1) = \infty$, then $d_c(p, q; B_2) = \infty$.

Proof. It is evident. \square

We have the following properties using the weight-sequence c .

Proposition 5.3. For the weight sequence $c(i) = 0$ for all i , and the n -sequence $(1)_{i=1}^\infty$ we get the H -distance of the sequences p and q for all possible $p, q \in \mathbb{R}^\infty$:

$$d_{\underline{0}}(p, q; (1)_{i=1}^\infty) = d(p, q; H).$$

Using the weight-sequence $c = \underline{0} = (0)_{i=1}^\infty$ and an n -sequence B for which $b(1) = \infty$ the distance B_c is similar to the discrete metric.

$$d_{\underline{0}}(p, q; (\infty, \dots)) = d(p, q; disc).$$

Using weight-sequences it is possible that the B_c -distance of the sequences p and q is finite, however, their absolute difference sequence w is divergent. More precisely, we have the following theorem about the role of the weight-sequences.

Theorem 5.4. The B_c -distance of $p, q \in \mathbb{R}^\infty$ is finite if and only if the B -distance of the sequences $o = (0)_{i=1}^\infty$ and w_c – defined by Definition 3.6 – is finite.

Proof. It is obvious. \square

In the case above, the difference-sequence w is called a k_c -sequence with the weight sequence c and with a suitable value of k based on Theorem 4.1.

6. Metric properties of distances

Sometimes it seems to be strange to call a distance function as a ‘distance’ without satisfying the conditions to be a metric. Since for B -distances the metric conditions are not fulfilled automatically, in this section we analyze their metric properties.

Lemma 6.1. If B_1 is faster than B_2 , then

$$d_c(q, r; B_1) \leq d_c(q, r; B_2)$$

for any weight-sequence c .

Proof. The weight-sequence is nothing else, but a re-scaling, therefore using Algorithm 1 the statement follows. \square

Now we extend Theorem 3 from [11] to the case of weighted distances, and we prove this more general case. (Using weight sequence $c(i) = 1$ for all i , this proof can also be considered as the proof of Theorem 3 of [11].)

Theorem 6.2. *The weighted distance function based on an n -sequence B and weight-sequence c generates a generalized metric space on the set \mathbb{R}^∞ , if and only if $B(i)$ is faster than B for all $i \in \mathbb{N}$.*

Proof. First, we prove sufficiency. The validity of properties a) of Definition 2.3 is trivial; it can be seen, e.g., following Algorithm 1. Indeed, the distance $d_c(p, q; B)$ depends only on the weighted absolute-difference w_c of p and q , and on B . (As we noted after Theorem 5.1) As the definition of w and so w_c are symmetric in p and q , thus $d_c(p, q; B) = d_c(q, p; B)$ for arbitrary $p, q \in \mathbb{R}^\infty$ and for arbitrary n -sequence B and weight-sequence c . It is clear that the distance is zero if and only if the difference sequences (both w and w_c) of the sequences has only zero elements, i.e., if the sequences are the same. Otherwise, the distance is a positive integer or infinite. Therefore, all distances generated by an n -sequence satisfy these two properties independently of the weight-sequence c . Hence, it is enough to deal with the triangle inequality.

Now we prove that property c) is true if and only if $B(i)$ is faster than B for all $i \in \mathbb{N}$. Let $p, q, r \in \mathbb{R}^\infty$ be three sequences, such that their distances are finite with the weight-sequence c . Then, we can find a B_c -path Π between p and r which is a concatenation of a minimal B_c -path between p and q , and a minimal $(B(i))_c$ -path between q and r , where $i = d_c(p, q; B) + 1$, and $B(i)$ is the i -shifted sequence of B . Hence,

$$|\Pi| = d_c(p, q; B) + d_c(q, r; B(i)).$$

The assumption that $B(i)$ is faster than B means that

$$d_c(q, r; B(i)) \leq d_c(q, r; B)$$

(by Lemma 6.1). Thus,

$$|\Pi| \leq d_c(p, q; B) + d_c(q, r; B).$$

By the definition of the B_c -distance we have

$$d_c(p, r; B) \leq |\Pi|,$$

hence

$$d_c(p, r; B) \leq d_c(p, q; B) + d_c(q, r; B).$$

Now, suppose that not all the distances are finite between p, q and r . If $d_c(p, q; B) = \infty$ or $d_c(q, r; B) = \infty$ then c) is trivially valid. Assume, that $d_c(p, r; B) = \infty$, but $d_c(p, q; B) = s < \infty$. If there would be a $(B(s))_c$ -path between q and r , then there would also be a B_c -path between p and r . (We could concatenate a shortest B_c -path between p and q , with length s , and a $(B(s))_c$ -path between q and r .) As the shifted sequence $B(s)$ is faster than B , by Lemma 5.2 there is no B_c -path between q and r . So, $d_c(q, r; B) = \infty$, and c) is valid in this case, too.

Now, we prove necessity by an indirect method. Assume that for some $j \in \mathbb{N}$, $B(j)$ is not faster than B , but the B_c -distance has property c). In this case, by Definition 2.3 there exist two sequences $p, q \in \mathbb{R}^\infty$ and $k \in \mathbb{N}$ such that $d_c(p, q; B(j)) = k$, and $d_c(p, q; B) < k$. Define the elements of sequence $r \in \mathbb{R}^\infty$ in the following way:

$$r(i) = q(i) + (-1)^{sg(p(i)-q(i))} \cdot c(i) \cdot t_{l(i)},$$

where

$$sg(p(i) - q(i)) = \begin{cases} 1, & \text{if } p(i) \geq q(i), \\ 0, & \text{in other case.} \end{cases}$$

$l(i)$ is a permutation of \mathbb{N} such a way, that $w'(l(i)) \geq w'(l(n))$ for all $l(i) < l(n)$. (It is a non-decreasing ordering of w_c , as v_c was at Theorem 5.1.) The values $t_{l(i)}$ are the number of elements of B among the first j ones such that their values are at least l .

Actually, we use the $(n$ -th) greatest value of $t_{l(i)}$ (which is the number of items at least value 1 (n) among the first j elements of B) calculating the value of $r(i)$, for which i is the index of the $(n$ -th) greatest element of the sequence w_c .

By our algorithm, it is easy to see that $d_c(q, r; B) = j$ and q is an element of one of the shortest paths between p and r . Then,

$$d_c(p, r; B) = d_c(q, r; B) + d_c(p, q; B(j)) = j + k,$$

as a shortest B_c -path between p and r can be obtained as a concatenation of a shortest $(B(j))_c$ -path from p to q and a shortest B_c -path from q to r . Thus,

$$d_c(p, q; B) + d_c(q, r; B) < k + j = d_c(p, r; B).$$

But we assumed that $d(p, q; B)$ has property c). This is a contradiction, and the proof is complete. \square

Consequence 6.3. *If a B_c -distance is a generalized metric, and the symbol ∞ occurs in B , then it occurs infinitely many times.*

Theorem 6.4. *A B_c -distance is a metric if and only if $B(i)$ is faster than B for all i , ∞ is an element of B and the weight-sequence c contains only finitely many non-zero elements.*

Proof. By Theorem 6.2 we know that a B_c -distance is a generalized metric iff each $B(i)$ is faster than B . We need to prove about the finiteness. Assume that B contains the symbol ∞ and the weight function c has only finitely many non-zero elements. Let $k \in \mathbb{N}$ such that $b(k) = \infty$, and there are no $j \in \mathbb{N}$ for which $j < k$ and $b(j) = \infty$. Using Algorithm 1 after the k -th step there are only finitely many non-zero items in $w_c^{(k)}$ and in $w^{(k)}$. The sum of the ceiling of these values, i.e., $\left(\sum_{i=1}^{\infty} u^{(k)}(i)\right)$ is a natural number. It will be decreasing by a positive integer value in each of the steps. Therefore the B_c distance will be finite for every pair of sequences.

For proving the other direction, the distance $d_c\left((0)_{i=1}^{\infty}, (i)_{i=1}^{\infty}; B\right)$ is infinite if B does not contain the symbol ∞ (independently of the used weight-sequence c).

The B_c distance of the sequences p and q is infinite (independently of the used n -sequence B), if the sequence w_c of p and q is divergent. The sequence w_c is divergent if and only if its subsequence keeping only the values for which $p(i) \neq q(i)$ is divergent. But, those elements of w_c are given by

$$w'(i) = c(i)|p(i) - q(i)| = c(i)w(i).$$

Since p and q are arbitrary pair of sequences w can be arbitrary (we can restrict our analysis to w with only non-negative values). For surely non-divergent sequence $w_c = (c(i)w(i))_{i=1}^{\infty}$ independently of w the following condition is needed: there exist some $j \in \mathbb{N}$ such that $c(i) = 0$ for all $i > j$, which was to be proven. \square

To check the weight sequences and the n -sequences that the distance defined by them is a (generalized) metric or not one can use Proposition 7 from [11] together with the previous result, Theorem 6.4.

Consequence 6.5. *If a B -distance (or a B_c -distance) is a generalized metric, then the B^k -distances (or the B_c^k -distances) using limited n -sequences are generalized metrics (and not necessarily metrics) for all $k \in \mathbb{N}$.*

7. Conclusion

In paper [11], we have investigated B -distances of sequences, which depend on the elements of sequences with equal weight. These distance functions are positive definite, i.e., the distance is non-negative and it is zero only in case when the sequences are the same. We have defined k -convergent and k -sequences based on properties of their tails.

In the recent paper, using various weight-sequences instead of the constant $(1)_{i=1}^{\infty}$ sequence, we have investigated the weighted B_c -distances, in which we can use various weight-functions and n -sequences

to get wide variety of distances. Using them we can get finite distances between sequences which are divergent.

We have presented an algorithm which solve the shortest path problem between any two the sequences (having a finite distance). A formula to calculate a B_c -distance of any two sequences has also been derived. We have shown the connection between the existence of a finite path between the sequences p and q and the k -sequence property of their (weighted) absolute difference sequence. We have proven a necessary and sufficient condition for the n -sequence B to provide a generalized metric over the set of sequences and a necessary and sufficient condition for the weight-sequence and the n -sequence for defining metric distances.

Moreover, our definitions and results work with finite sequences also. Calculating with two finite sequences we need only the assumption that their length are the same (or we can use 0 as the other elements of the respective infinite sequences).

It is an interesting future work to study the spheres in these spaces for various radii r ($r \in \mathbb{N}$).

We should note here that B -distances were also defined for formal languages in [8], and other types of combination of weights and n -sequences were presented for the square grid in [12–14, 16] and for three-dimensional grids in [15, 17].

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