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Connections Between H_v-S-Act, GHS-Act and S-Act

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Abstract. The largest class of hyperstructures is the one which satisfies the weak properties and they are called H_v-structures. In this paper, the concept of H_v-S-act is introduced and some of their properties are investigated. The present paper establishes a possible connection between S-act, GHS-act and H_v-S-act. It is shown that the quotient of GHS-act with any equivalence relation is H_v-S-act. The main tool to study all hyperstructures is the fundamental relations. The study of fundamental relations in H_v-S-act reveals some interesting results. Specifically, these relations connect weak hyperactions with the corresponding classical actions.

1. Introduction

Algebraic hyperstructures are a natural extension of classical algebraic structures. Theory of hyperstructure is initiated in 1934 by the French Mathematician Marty [11]. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. This particular character of hyperstructure attracted mathematicians and researchers towards this direction. During last decades hyperstructures seem to have a variety of applications not only in other branches of mathematics but also in many other sciences including the social sciences. These applications range from biomathematics and hardonic physics to automata theory. Hyperstructure can now be widely applied in industry and production. A recent book contains a wealth of applications [2]. Via this book, Corsini and Leoreanu presented some of numerous applications of the algebraic hyperstructures. Different hyperstructures are extensively studied from the theoretical perspective such as in fuzzy set theory, rough set theory, optimization theory, cryptography, codes, analysis of computer programs, automata, formal language theory, combinatorics, artificial intelligence, probability, graphs and hypergraphs, geometry, lattices and binary relations, see [5], [6], [7], [8], [9] [10] and [21].

H_v-structures were introduced by Vougiouklis in Fourth AHA Congress. Vougiouklis defined the notion of an H_v-group [18]. H_v-structures satisfy the weak axioms, where the non-empty intersection replaces the equality. Since then many papers concerning various H_v-structures have appeared in literature, see [2]. Vougiouklis defined the concept of H_v-vector space which is a generalization of the concept of vector space in classical theory [18]. Davvaz introduced H_v-module of fractions of a hypermodule which is a generalization of the concept of module of fractions [3]. Davvaz surveyed the theory of H_v-structures [4]. The reader will find some principal notions and theorems about H_v-structures in book "Hyperstructures

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and their representations" [20]. Applications of H_v-structures in other sciences can be seen in [6], [7] and [20].

One of the very competent conception in many branches of mathematics as well as in computer science is the action of a semigroup or a monoid on a non-empty set. A representation of a semigroup S by transformation of a set defines an S-act. Sen et al. [13] and Shahbaz [14] have introduced the concept of hyperaction. Their approach of defining hyperaction lacks perfection. Shabir et al., modified this conception by introducing the notion of GHS-act [14].

In this paper we present the idea of weak hyperaction. This paper is arranged in the following manner. Section 2 is a collection of definitions of basic terms and theorems concerning hyperstructure and semigroup action. In Section 3, we introduce the action of H_v-monoid on a non-empty set and call it H_v-S-act. Furthermore, some basic properties of H_v-S-acts are investigated. Section 3 is devoted to the study of congruences and quotients of hyperactions. It is shown that the quotient of a GHS-act with an equivalence relation is H_v-S-act. The main tools in the theory of hyperstructures are the fundamental relations. In section 5, we study the fundamental relations in H_v-S-act which relates weak hyperactions with classical actions. In the end, some concluding remarks are given.

2. Preliminaries

In this section some basic concepts pertaining to hyperstructure and semigroup acts are given, which will be required in later sections.

Definition 1. *[2] Let S be a non-empty set and $P^*(S)$ be the set of all non-empty subsets of S. A n-hyperoperation on S is a map $f : S^n \longrightarrow P^*(S)$. The number n is called the arity of f. A set S, endowed with a family Γ of hyperoperations is called a hyperstructure or a multivalued algebra. If Γ is singleton that is $\Gamma = \{f\}$, where arity of f is 2, then the hyperstructure is called a hypergroupoid.*

Definition 2. *[2] If $\circ : S \times S \longrightarrow \mathcal{P}^*(S)$ is a hyperoperation or join operation, then the image of the pair (s, t) of $S \times S$ is denoted by $s \circ t$ and is called the hyperproduct of s and t.*

If S_1 and S_2 are non-empty subsets of $(S, \circ)$, then $S_1 \circ S_2 = \bigcup_{\substack{s \in S_1 \\ s' \in S_2}} s \circ s'$.

In the pursuit, we state some basic notions related to hypergroupoids.

Definition 3. *[2] A hypergroupoid $(S, \circ)$ is called a semihypergroup if for all $s_1, s_2, s_3 \in S$, $(s_1 \circ s_2) \circ s_3 = s_1 \circ (s_2 \circ s_3)$.*

Definition 4. *[16] A hypergroupoid $(S, \circ)$ is called an H_v-semigroup if*

$$(s_1 \circ s_2) \circ s_3 \cap s_1 \circ (s_2 \circ s_3) \neq \emptyset \text{ for all } s_1, s_2, s_3 \in S.$$

An H_v-semigroup is called an H_v-group if

$$s \circ S = S \circ s = S \text{ for all } s \in S.$$

Definition 5. *[16] An element e in a semihypergroup (H_v-semigroup) $(S, \circ)$ is called an identity element if $s \in e \circ s = s \circ e$ ($s \in e \circ s \cap s \circ e$) for all $s \in S$. A hypermonoid (H_v- monoid) is the semihypergroup (H_v-semigroup) with an identity element.*

Definition 6. *[16] An element 0 in a semihypergroup (H_v-semigroup) $(S, \circ)$ is called a zero element if $0 \in 0 \circ s = s \circ 0$ ($0 \in s \circ 0 \cap 0 \circ s$) for all $s \in S$.*

Definition 7. *[2] A semihypergroup (H_v-semigroup) $(S, \circ)$ is commutative if $s \circ t = t \circ s$ ($s \circ t \cap t \circ s \neq \emptyset$) for all $s, t \in S$.*

Definition 8. *[9] A non-empty subset T of a semihypergroup* $(S, \circ)$ *is called a subsemihypergroup of* $(S, \circ)$ *if* $T \circ T \subseteq T$.

The idea of representing an object by some other object which is better known at least in some respects is quite familiar in mathematics. Representation of semigroups (monoids) by transformations of sets give rise to the notion of action of semigroups (monoids).

Definition 9. *[11] Let* $(S, \cdot)$ *be a monoid and A be a non-empty set. A right action of S on A is a function* $\xi : A \times S \longrightarrow A$ *(usually denoted by* $\xi(a, s) \longmapsto as$*) such that*

(i) $a(st) = (as)t$,

(ii) $ae = a$, for all $a \in A$ and $s, t \in S$.

Definition 10. *[15] Let* $(S, \circ)$ *be a hypermonoid with identity element e and A be a non-empty set. A generalized hyperaction of S on A is a function* $*$ *defined as*

$$\begin{aligned} * \quad &: \quad A \times S \longrightarrow \mathcal{P}^*(A) \\ (a, s) \quad &\longmapsto \quad a * s \in \mathcal{P}^*(A) \end{aligned}$$

where $\mathcal{P}^*(A)$ *is the family of all non-empty subsets of A. A non-empty set A endowed with hyperaction* $*$ *is called right GHS-act if for all* $a \in A$ *and* $s, t \in S$

(i) $a * (s \circ t) = (a * s) * t$,

(ii) $a \in a * e$.

Example 1. *[15] Let A be a non-empty set and* $\mathcal{T}(A)$ *be the set of all transformations from A to A. Define* $\circ : \mathcal{T}(A) \times \mathcal{T}(A) \longrightarrow \mathcal{P}^*(\mathcal{T}(A))$ *by* $f \circ g = \{f, g, fg\}$ *for all* $f, g \in \mathcal{T}(A)$*, where fg represents the composition of two maps. Then* $(\mathcal{T}(A), \circ)$ *is a hypermonoid. Now define* $* : \mathcal{T}(A) \times A \longrightarrow \mathcal{P}^*(A)$ *by* $f * a = \{a, f(a)\}$*. Then* ${}_{\mathcal{T}(A)}A$ *is a left GH*$\mathcal{T}(A)$*-act. Indeed, for* $f, g \in \mathcal{T}(A)$ *and* $a \in A$, $g * (f * a) = \{a, f(a), g(a), g(f(a))\} = (g \circ f) * a$.

3. On Weak Hyperaction

In this section, we define the hyperaction of an H_v-monoid on a non-empty set and call it H_v-S-act. The notion of an H_v-S-act is a generalization of GHS-act in hyperstructure as well as S-act notion in classical theory.

Definition 11. *Let* $(S, \circ)$ *be an* H_v*-monoid and A be a non-empty set. A weak hyperaction of S on A is a function*

$$\begin{aligned} *_{\mathcal{H}} : \quad & A \times S \longrightarrow \mathcal{P}^*(A) \\ & (a, s) \longmapsto a *_{\mathcal{H}} s \in \mathcal{P}^*(A) \end{aligned}$$

where $\mathcal{P}^*(A)$ *is the family of all non-empty subsets of A. A non-empty set A endowed with weak hyperaction* $*_{\mathcal{H}}$ *is called right* H_v*-S-act or right* H_v*-act over S if for all* $a \in A$ *and* $s, t \in S$

(i) $a *_{\mathcal{H}} (s \circ t) \cap (a *_{\mathcal{H}} s) *_{\mathcal{H}} t \neq \emptyset$,

(ii) $a \in a *_{\mathcal{H}} e$.

We write $(A_S, *_{\mathcal{H}})$ to indicate that A is a right H_v-S-act. Analogously, one can define a left H_v-S-act, written as $({}_SA, *_{\mathcal{H}})$.

In order to understand the concept, consider the following examples.

Example 2. *Consider the H_v-monoid $(S, \circ)$, where $S = \{e, s, t, q\}$ and $\circ$ is defined in Table 1.*

$\circ$	e	s	t	q
e	e	s	t	q
s	s	s	$\{s,t\}$	$\{s,t\}$
t	t	q	q	q
q	q	$\{s,t\}$	$\{s,t\}$	$\{s,t\}$

Table 1

Let $A = \{a, b, c, d\}$ and weak hyperaction $*_{\mathcal{H}}$ of S on A is presented in Table 2.

$*_{\mathcal{H}}$	e	s	t	q
a	a	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	b	b	b	b
c	c	d	d	d
d	d	$\{c,d\}$	$\{c,d\}$	$\{c,d\}$

Table 2

Then $(A_S, *_{\mathcal{H}})$ is a right H_v-S-act over H_v-monoid $(S, \circ)$.

Example 3. *Consider the classical differential ring of real valued functions $C^1(\mathbb{R})$ with the usual differentiation. For any $f, g \in C^1(\mathbb{R})$, define a hyperoperation on the ring $C^1(\mathbb{R})$ by*

$$f \circ g = \{f, g, fg\}$$

where fg is defined as $(fg)(x) = f(x)\,g(x)$ for all $x \in \mathbb{R}$. Then for $f, g, h \in C^1(\mathbb{R})$

$$(f \circ g) \circ h = \{f, g, h, fg, fh, gh, (fg)h\} = f \circ (g \circ h)$$

and also I(identity function)$\in C^1(\mathbb{R})$ and $I \in I \circ f = f \circ I$. Therefore, $\left(C^1(\mathbb{R}), \circ\right)$ is a hypermonoid. Define $*_{\mathcal{H}} : \mathbb{R} \times C^1(\mathbb{R}) \longrightarrow \mathcal{P}^*(\mathbb{R})$ (described as $(a, f) \longmapsto a *_{\mathcal{H}} f$) by

$$a *_{\mathcal{H}} f = \{a, f(a), f'(a)\}.$$

Here $a *_{\mathcal{H}} I = \{a, 1\}$, for all $a \in \mathbb{R}$. Also

$$\begin{aligned}
(a *_{\mathcal{H}} f) *_{\mathcal{H}} g &= \{a, f(a), f'(a)\} *_{\mathcal{H}} g \\
&= \{a, f(a), g(a), f'(a), g'(a), g(f(a)), g(f'(a)), g'(f(a)), g'(f'(a))\}. \\
a *_{\mathcal{H}} (f \circ g) &= a *_{\mathcal{H}} \{f, g, fg\} \\
&= \left\{a, f(a), g(a), (fg)(a), f'(a), g'(a), (fg)'(a)\right\}.
\end{aligned}$$

As, $(a *_{\mathcal{H}} f) *_{\mathcal{H}} g \cap a *_{\mathcal{H}} (f \circ g) \neq \emptyset$, therefore $\mathbb{R}$ is an H_v-$C^1(\mathbb{R})$-act.

Remark 1. *As every hypermonoid is an H_v-monoid, we can compare generalized hyperaction and weak hyperaction of a hypermonid on a non-empty set. For a hypermonoid $(S, \circ)$, every right GHS-act is an H_v-S-act but the converse is not true in general. Consider the hypermonoid $(S, \circ)$, where $S = \{e, s, t, q\}$ and $\circ$ is defined in Table 3.*

$\circ$	e	s	t	q
e	e	s	t	q
s	s	s	$\{s,t\}$	s
t	$\{s,t\}$	s	t	s
q	q	s	s	q

Table 3

Let $A = \{a, b\}$ and hyperaction $*_{\mathcal{H}}$ of S on A is presented in Table 4.

$*_{\mathcal{H}}$	e	s	t	q
a	a	a	b	A
b	A	a	A	A

Table 4

Then $(A_S, *_{\mathcal{H}})$ is a right H_v-S-act over hypermonoid $(S, \circ)$ which is not a GHS-act because $a *_{\mathcal{H}} (s \circ t) \neq (a *_{\mathcal{H}} s) *_{\mathcal{H}} t$.

All properties of H_v-S-acts are also true for subsets. Therefore, we have the following result.

Proposition 1. *Let A be a non-empty set, $(S, \circ)$ be an H_v-monoid and $*_{\mathcal{H}}$ be a weak hyperaction of S on A. Then $(A_S, *_{\mathcal{H}})$ is an H_v-S-act if and only if for all $A' \in \mathcal{P}^*(A)$ and $S_1, S_2 \in \mathcal{P}^*(S)$ the following conditions hold:*

(i) $A' *_{\mathcal{H}} (S_1 \circ S_2) \cap (A' *_{\mathcal{H}} S_1) *_{\mathcal{H}} S_2 \neq \emptyset$,

(ii) $A' \subseteq A' *_{\mathcal{H}} e$.

Proof. Suppose $(A_S, *_{\mathcal{H}})$ is an H_v-S-act. Then for $A' \in \mathcal{P}^*(A)$ and $S_1, S_2 \in \mathcal{P}^*(S)$, we have

$$\begin{aligned} A' *_{\mathcal{H}} (S_1 \circ S_2) &= \bigcup_{\substack{a \in A' \\ s_1, s_2 \in S}} a *_{\mathcal{H}} (s_1 \circ s_2) \\ (A' *_{\mathcal{H}} S_1) *_{\mathcal{H}} S_2 &= \bigcup_{\substack{a \in A' \\ s_1, s_2 \in S}} (a *_{\mathcal{H}} s_1) *_{\mathcal{H}} s_2. \end{aligned}$$

As $(A_S, *_{\mathcal{H}})$ is an H_v-S-act, therefore $A' *_{\mathcal{H}} (S_1 \circ S_2) \cap (A' *_{\mathcal{H}} S_1) *_{\mathcal{H}} S_2 \neq \emptyset$. Also $A' \subseteq A' *_{\mathcal{H}} e$ for $A' \in \mathcal{P}^*(A)$. Converse is obvious. □

Remark 2. *If $(S, \circ)$ is a commutative H_v-monoid, then every left H_v-S-act can be considered as a right H_v-S-act. Indeed, if $({}_S A, *_{\mathcal{H}})$ is a left H_v-S-act, we may define a right multiplication by elements of S as:*

$$a \circledast s = s *_{\mathcal{H}} a \text{ for } a \in A, s \in S.$$

Then $a \in a \circledast e = e *_{\mathcal{H}} a$ for all $a \in A$ and $(a \circledast s_1) \circledast s_2 \cap a \circledast (s_1 \circ s_2) \neq \emptyset$ for all $s_1, s_2 \in S$ and $a \in A$.

Proposition 2. *Let $(A_S, *_{\mathcal{H}})$ and $(B_S, *'_{\mathcal{H}})$ be two right H_v-acts over an H_v-monoid $(S, \circ)$. Then $A \times B$ can induce an H_v-S-act.*

Proof. Define the weak hyperaction $\circledast$ of S on Cartesian product $A \times B$ by

$$(a, b) \circledast s = (a *_{\mathcal{H}} s) \times \left(b *'_{\mathcal{H}} s\right) \text{ for } (a, b) \in A \times B \text{ and } s \in S.$$

Then for all $(a, b) \in A \times B$ and $s, t \in S$, we have

$$\begin{aligned} ((a, b) \circledast s) \circledast t &= \bigcup_{(a', b') \in (a,b) \circledast s} (a', b') \circledast t \\ &= \bigcup_{\substack{a' \in a *_{\mathcal{H}} s \\ b' \in b *_{\mathcal{H}} s}} (a' *_{\mathcal{H}} t) \times \left(b' *'_{\mathcal{H}} t\right) \\ &= ((a *_{\mathcal{H}} s) *_{\mathcal{H}} t) \times \left(\left(b *'_{\mathcal{H}} s\right) *_{\mathcal{H}} t\right). \end{aligned}$$

$$(a, b) \circledast (s \circ t) = (a *_{\mathcal{H}} (s \circ t)) \times \left(b *'_{\mathcal{H}} (s \circ t)\right).$$

As $(a *_{\mathcal{H}} s) *_{\mathcal{H}} t \cap a *_{\mathcal{H}} (s \circ t) \neq \emptyset$ and $\left(b *'_{\mathcal{H}} s\right) *_{\mathcal{H}} t \cap b *_{\mathcal{H}} (s \circ t) \neq \emptyset$, we have $((a, b) \circledast s) \circledast t \cap (a, b) \circledast (s \circ t) \neq \emptyset$. Also $a \in a *_{\mathcal{H}} e$ and $b \in b *'_{\mathcal{H}} e$ imply that $(a, b) \in (a *_{\mathcal{H}} e) \times (b *_{\mathcal{H}} e) = (a, b) \circledast e$. Hence $((A \times B)_S, \circledast)$ is an H_v-S-act. □

Sen et al. (2011) defined the Cartesian product of two hypermonoids. In a similar way, we can define Cartesian product of two H_v-monoids.

Let $(S, \circ)$ and $(T, \circ')$ be two H_v-monoids with identities e and e', respectively. Then their Cartesian product $S \times T$ can induce an H_v-monoid with respect to the hyperoperation $\otimes$ defined as:

$$(s_1, t_1) \otimes (s_2, t_2) = (s_1 \circ s_2) \times (t_1 \circ' t_2) = \{(s, t) | s \in s_1 \circ s_2, t \in t_1 \circ' t_2\}.$$

Identity element of $S \times T$ is (e, e'). The H_v-monoid $S \times T$ is called the direct product of S and T, written as $(S \times T, \otimes)$.

Proposition 3. *Let $(A_S, *_{\mathcal{H}})$ and $\left(B_T, *'_{\mathcal{H}}\right)$ be an H_v-S-act and an H_v-T-act, respectively. Then $A \times B$ can induce an H_v-$(S \times T)$-act.*

Proof. Define $\circledast : (A \times B) \times (S \times T) \longrightarrow \mathcal{P}^* (A \times B)$ by

$$(a, b) \circledast (s, t) = (a *_{\mathcal{H}} s) \times \left(b *'_{\mathcal{H}} t\right) \text{ for all } (a, b) \in (A \times B) \text{ and } (s, t) \in S \times T.$$

Then for all $(a, b) \in A \times B$ and $(s_1, t_1), (s_2, t_2) \in S \times T$, we have

$$\begin{aligned} ((a, b) \circledast (s_1, t_1)) \circledast (s_2, t_2) &= \bigcup_{\substack{a' \in a *_{\mathcal{H}} s_1 \\ b' \in b *'_{\mathcal{H}} t_1}} (a', b') \circledast (s_2, t_2) \\ &= \bigcup_{\substack{a' \in a *_{\mathcal{H}} s_1 \\ b' \in b *'_{\mathcal{H}} t_1}} (a' *_{\mathcal{H}} s_2) \times \left(b' *'_{\mathcal{H}} t_2\right) \\ &= ((a *_{\mathcal{H}} s_1) *_{\mathcal{H}} s_2) \times \left(\left(b *'_{\mathcal{H}} t_1\right) *'_{\mathcal{H}} t_2\right). \end{aligned}$$

$$\begin{aligned} (a, b) \circledast ((s_1, t_1) \otimes (s_2, t_2)) &= (a, b) \circledast ((s_1 \circ s_2) \times (s_2 \circ' t_2)) \\ &= (a *_{\mathcal{H}} (s_1 \circ s_2)) \times \left(b *'_{\mathcal{H}} (t_1 \circ' t_2)\right) \end{aligned}$$

As $(A_S, *_{\mathcal{H}})$ is an H_v-S-act and $\left(B_T, *'_{\mathcal{H}}\right)$ is an H_v-T-act, we have $(((a, b) \circledast (s_1, t_1)) \circledast (s_2, t_2)) \cap (a, b) \circledast ((s_1, t_1) \otimes (s_2, t_2)) \neq \emptyset$. Also, $a \in a *_{\mathcal{H}} e$ and $b \in b *'_{\mathcal{H}} e'$ imply that $(a, b) \in (a *_{\mathcal{H}} e) \times \left(b *'_{\mathcal{H}} e'\right) = (a, b) \circledast (e, e')$. Hence, $A \times B$ is an H_v-$(S \times T)$-act. □

Definition 12. *Let $(X_S, *)$ be an H_v-S-act. An element θ of X is called an absorbing element of X if $\theta \in \theta * s$ for all $s \in S$.*

Note that an H_v-S-act may have several absorbing elements, it may also have no absorbing element. In order to understand the concept, consider the following example in which every element is an absorbing element.

Example 4. *Let $(S, \circ)$ be an H_v-monoid, where $S = \{e, p, q, s, t, v\}$ and $\circ$ is defined in Table .*

$\circ$	e	p	q	s	t	v
e	e	$\{e, p\}$	$\{e, q\}$	$\{e, s\}$	$\{e, t\}$	$\{e, v\}$
p	$\{e, p\}$	p	$\{s, t\}$	t	p	$\{s, v\}$
q	$\{e, q\}$	$\{s, v\}$	q	$\{p, t\}$	$\{e, t\}$	v
s	$\{e, s\}$	s	$\{t, v\}$	s	t	v
t	$\{e, t\}$	$\{q, s\}$	t	$\{p, q\}$	t	v
v	$\{e, v\}$	$\{s, t\}$	$\{s, t\}$	$\{s, t\}$	$\{s, t\}$	v

Table 5

Let $X = \{x, y, z\}$ and the weak hyperaction $*$ of S on X is exhibited in Table .

$*$	e	p	q	s	t	v
x	x	$\{x, y\}$	$\{x, z\}$	x	$\{x, y, z\}$	$\{x, z\}$
y	y	$\{x, y\}$	$\{x, y\}$	$\{x, y\}$	$\{x, y\}$	$\{x, z\}$
z	z	$\{x, y, z\}$	z	z	z	$\{x, y, z\}$

Table 6

Then $(x_s, *)$ is an H_v-S-act and $X \in X * s$ for all $x \in x$ and $s \in S$.

Proposition 4. *Let $(A_S, *_{\mathcal{H}})$ and $(B_S, *'_{\mathcal{H}})$ be two H_v-S-acts. If θ and θ' are absorbing elements of A and B, respectively, then (θ, θ') is an absorbing element of $A \times B$, which is an H_v-S-act with weak hyperaction $\circledast$.*

Proof. As θ and θ' are absorbing elements of A and B, we have $\theta \in \theta *_{\mathcal{H}} s$ and $\theta' \in \theta' *_{\mathcal{H}} s$ for all $s \in S$ which implies $(\theta, \theta') \in (\theta *_{\mathcal{H}} s) \times (\theta' *_{\mathcal{H}} s) = (\theta, \theta') \circledast s$ for all $s \in S$. Hence (θ, θ') is an absorbing element of $A \times B$. □

Proposition 5. *Let $(A_S, *_{\mathcal{H}})$ and $\left(B_T, *'_{\mathcal{H}}\right)$ be an H_v-S-act and an H_v-T-act, $*_{\mathcal{H}}$. If θ and θ' are absorbing elements of A and B, respectively, then (θ, θ') is an absorbing element of $A \times B$, which is an H_v-$(S \times T)$-act with weak hyperaction $\circledast$.*

Proof. Directly follows from Proposition 3. □

Definition 13. *Let $(A_S, *_{\mathcal{H}})$ be a right H_v-S-act. A subset $A' \neq \emptyset$ of A is called an H_v-S-subact of A if $A' *_{\mathcal{H}} S \subseteq A'$, that is, $a' *_{\mathcal{H}} s \subseteq A'$ for all $a' \in A'$ and $s \in S$.*

Proposition 6. *Let $(A_S, *_{\mathcal{H}})$ be an H_v-S-act and A_1 and A_2 be any two H_v-S-subacts of A. Then $A_1 \cap A_2$ is also an H_v-S-subact of A if $A_1 \cap A_2$ is non-empty.*

Proof. The proof is straightforward. □

By the definition of an H_v-S-act and the product of H_v-S-acts we have the next proposition.

Proposition 7. *Let $(A_S, *_{\mathcal{H}})$ and $(B_T, *'_{\mathcal{H}})$ be two H_v-S-acts and A' and B' be H_v-S-subacts of A and B, respectively. Then $A' \times B'$ is an H_v-S-subact of $A \times B$, which is an H_v-S-act with weak hyperaction $\circledast$.*

Definition 14. *Let $(X_S, *)$ and $(Y_S, *')$ be two H_v-S -acts. A mapping $f : X \to Y$ is called*
(i) *weak S-homomorphism, if $f(X * s) \cap (f(X) *' s) \neq \emptyset$ for all $x \in X, s \in S$.*
(ii) *inclusion S-homomorphism, if $f(X * s) \subseteq (f(x) *' s)$ for all $x \in X, s \in S$.*
(iii) *strong S-homomorphism, if $f(x * s) = f(X) *'_{\mathcal{H}} s$ for all $x \in X, s \in S$.*

Obviously, every s-S-homomorphism is i-S-homomorphism and every i-S-homomorphism is w-S-homomorphism. But the converse is not true in general. An w-S-homomorphism (resp. i-S-homomorphism, s-S-homomorphism) $f : A \to B$ is called w-S-isomorphism (resp. i-S-isomorphism, s-S-isomorphism) if f is bijective and in this situation it is denoted by $A_S \sim B_S$ (resp. $A_S \simeq B_S$, $A_S \cong B_S$).

Proposition 8. *Let $f : (A_S, *_{\mathcal{H}}) \longrightarrow (B_S, *'_{\mathcal{H}})$ be an i-S-homomorphism of H_v-S-acts. If θ is an absorbing element of A_S, then $f(\theta)$ is an absorbing element of B_S.*

Proof. As θ is an absorbing element, we have $\theta \in \theta *_{\mathcal{H}} s$ for all $s \in S$. Then

$$f(\theta) \in f(\theta *_{\mathcal{H}} s) \subseteq f(\theta) *'_{\mathcal{H}} s \text{ for all } s \in S.$$

Therefore, $f(\theta)$ is an absorbing element of $(B_S, *'_{\mathcal{H}})$. □

Proposition 9. *Let* $f : (A_S, *_{\mathcal{H}}) \longrightarrow \left(B_S, *'_{\mathcal{H}}\right)$ *be an s-S-homomorphism of* H_v*-S-acts. Then the followings are satisfied.*

(i) If A' *is an* H_v*-S-subact of* A*, then* $f(A')$ *is an* H_v*-S-subact of* B*.*

(ii) If f *is surjective and* B' *is an* H_v*-S-subact of* B*, then* $f^{-1}(B')$ *is an* H_v*-S-subact of* A*.*

Definition 15. *Let* $(X_S, *)$ *be a right* H_v*-S-act. An element* $s \in S$ *acts on* X *weakly injective if*

$$x * s = x' * s \Longrightarrow x = x' \text{ for all } x, x' \in X.$$

And $s \in S$ *acts on* x *strongly injective if*

$$x * s \cap x' * s \neq \emptyset \Longrightarrow x = x' \text{ for all } x, x' \in X.$$

Remark 3. *Clearly, an element which acts strongly injective also acts weakly injective but the converse is not true in general. Let* $(S, \circ)$ *be an* H_v*-monoid, where* $S = \{e, s, t, q\}$ *and* $\circ$ *is defined in Table .*

$\circ$	e	s	t	q
e	e	$\{a, t\}$	t	$\{e, q\}$
s	$\{s, t\}$	$\{s, t\}$	$\{s, t\}$	$\{s, t\}$
t	t	t	t	t
q	$\{e, q\}$	$\{s, q\}$	$\{e, q\}$	q

Table 7

Let $x = \{x_1, y_1, x_2, y_2, x_3, y_3\}$ *and weak hyperaction* $*$ *of* S *on* X *is defined in Table .*

$*_{\mathcal{H}}$	e	s	t	q
x_1	x_1	y_1	y_1	$\{x_1, y_1\}$
y_1	y_1	y_1	y_1	y_1
x_2	$\{x_2, x_3\}$	$\{y_2, y_3\}$	$\{y_2, y_3\}$	$\{x_2, x_3, y_3\}$
y_2	y_2	y_2	y_2	y_2
x_3	$\{y_1, x_3\}$	$\{y_1, x_3\}$	$\{y_1, x_3\}$	$\{y_1, x_3\}$
y_3	y_3	y_3	y_3	y_3

Txble 8

Then $(x_S, *)$ *is an* H_v*-S-act. The elements* e *and* q *of* S *acts on* x *weakly injective but not strongly injective because* $(y_1 * e) \cap (x_3 * e) \neq \emptyset$ *and* $(x_1 * q) \cap (y_2 * q) \neq \emptyset$*.*

Proposition 10. *Let* $(A_S, *_{\mathcal{H}})$ *and* $(B_S, *'_{\mathcal{H}})$ *be two* H_v*-S-acts. If* $s \in S$ *acts strongly injective on* A *and* B*, then* s *also acts strongly injective on* $A \times B$*, which is an* H_v*-S-act with weak hyperaction* $\circledast$*.*

Proof. If $(a, b) \circledast s \cap (a', b') \circledast s \neq \emptyset$ for all $(a, b), (a', b') \in A \times B$, then $(a *_{\mathcal{H}} s) \times (b *'_{\mathcal{H}} s) \cap (a' *_{\mathcal{H}} s) \times (b' *'_{\mathcal{H}} s) \neq \emptyset$ which implies $a *_{\mathcal{H}} s \cap a' *_{\mathcal{H}} s \neq \emptyset$ and $b *'_{\mathcal{H}} s \cap b' *'_{\mathcal{H}} s \neq \emptyset$. As $s \in S$ acts strongly injective on A and B, we have $a = a'$ and $b = b'$. Therefore, s acts strongly injective on $A \times B$. □

Proposition 11. *Let* $(A_S, *_{\mathcal{H}})$ *be an* H_v*-S-act and* $\left(B_T, *'_{\mathcal{H}}\right)$ *be an* H_v*-T-act. If* s *and* t *acts strongly injective on* A *and* B*, respectively, then* (s, t) *acts strongly injective on* $A \times B$*, which is an* H_v*-*$S \times T$*-act* $A \times B$ *with weak hyperaction* $\circledast$*.*

Proof. The proof is straightforward. □

4. Relationship Between H_v-S-Act and GHS-Act

This section is devoted to the study of congruence and quotients of hyperaction which relates H_v-S-act and GHS-act. Throughout this section, unless otherwise stated, $(S, \circ)$ is a hypermonoid with identity element e.

Definition 16. *[21] An equivalence relation σ on a right GHS-act $(A_S, *)$ is called a congruence relation if for every $a, b \in A$ and $s \in S$*

$$a\sigma b \Longrightarrow [a * s]_\sigma = [b * s]_\sigma$$

where, for $B \subseteq A$, $[B]_\sigma = \{[b]_\sigma : b \in B\}$ and $[b]_\sigma$ is the equivalence class of b with respect to σ.

Note that for every $A_1, A_2 \subseteq A$, $[A_1]_\sigma = [A_2]_\sigma$ if and only if for every $a_1 \in A_1$ there exists $a_2 \in A_2$ such that $a_1\sigma a_2$ and for every $a_2 \in A_2$ there exists $a_1 \in A_1$ such that $a_1\sigma a_2$.

The set of all equivalence relations on A_S is denoted by $\mathrm{Eq}(A_S)$ and the set of all congruences on A_S is denoted by $\mathrm{Con}(A_S)$.

Define hyperaction $\boxtimes$ of S on $A/\sigma = \{[a]_\sigma : a \in A\}$ by

$$[a]_\sigma \boxtimes s = \bigcup_{x\in[a]_\sigma} [x * s]_\sigma \text{ for all } a \in A \text{ and } s \in S.$$

Firstly, we prove that $\boxtimes$ is well-defined. Suppose that $[a]_\sigma = [b]_\sigma$ imply that $a\sigma b$. Let $[y]_\sigma \in [a]_\sigma \boxtimes s = \bigcup_{x\in[a]_\sigma} [x*s]_\sigma$. So

$$[y]_\sigma \in [x * s]_\sigma \text{ for some } x \in [a]_\sigma.$$

As σ is an equivalence relation, we have $x\sigma b$ which imply that $[y]_\sigma \in [b]_\sigma \boxtimes s$. Similarly, $[b]_\sigma \boxtimes s \subseteq [a]_\sigma \boxtimes s$. Therefore, $\boxtimes$ is well defined. Also, $(A/\sigma, \boxtimes)$ is an H_v-S-act. Indeed, $[a * s]_\sigma \subseteq [a]_\sigma \boxtimes s$ for $a \in A$ and $s \in S$. So

$$\begin{aligned} [(a * s) * t]_\sigma &\subseteq ([a]_\sigma \boxtimes s) \boxtimes t, \\ [a * (s \circ t)]_\sigma &\subseteq [a]_\sigma \boxtimes (s \circ t) \text{ for } s, t \in S. \end{aligned}$$

Thus, $(([a]_\sigma \boxtimes s) \boxtimes t) \cap ([a]_\sigma \boxtimes (s \circ t)) \neq \emptyset$ and $[a]_\sigma \subseteq [a]_\sigma \boxtimes e$ which implies that $(A/\sigma, \boxtimes)$ is an H_v-S-act.

Notice if σ is a congruence on $(A_S, *)$, then

$$[a]_\sigma \boxtimes s = [a * s]_\sigma \text{ for all } s \in S.$$

If σ is a congruence relation, then $(A/\sigma, \boxtimes)$ is a GHS-act.

The above arguments have been summarized in the following theorem.

Theorem 1. *Let $(A_S, *)$ be a right GHS-act. Then*

(i) $(A/\sigma, \boxtimes)$ is an H_v-S-act if $\sigma \in \mathrm{Eq}(A_S)$.

(ii) $(A/\sigma, \boxtimes)$ is a GHS-act if $\sigma \in \mathrm{Con}(A_S)$.

The above theorem establishes a link between GHS-act and H_v-S-act. If $(A_S, *)$ is a right H_v-S-act and $\sigma \in \mathrm{Con}(A_S)$, then we have the following result.

Lemma 1. *Let $(A_S, *)$ be a right H_v-S-act and $\sigma \in Con(A_S)$. Then $(A/\sigma, \boxtimes)$ is an H_v-S-act.*

Theorem 2. *Let $(X_S, *)$ be a right GHS-act and $\sigma \in Eq(A_S)$. Then we have the following:*

(i) The natural map $\pi : X \to X/\sigma$ given by $\pi(x) = [x]_\sigma$ for $x \in X$ is an i-S-homomorphism.

(ii) The natural map $\pi : X \to X/\sigma$ is an s-S-homomorphism if and only if $\sigma \in \mathrm{Con}(X_S)$.

Proof. (i) For $a \in A$ and $s \in S$, $\pi(a * s) = [a * s]_\sigma \subseteq \bigcup_{x \in [a]_\sigma} [x * s]_\sigma = [a]_\sigma \boxtimes s = \pi(a) \boxtimes s$.

(ii) For $\sigma \in$Con(A_S), $a \in A$ and $s \in S$ $\pi(a * s) = [a * s]_\sigma = [a]_\sigma \boxtimes s = \pi(a) \boxtimes s$. Thus π is strong S-homomorphism.

Conversely, suppose that π is a strong s-S-homomorphism, $a, b \in A$, $a\sigma b$ and $s \in S$. Then $[a * s]_\sigma \subseteq \bigcup_{x \in [b]_\sigma} [x * s]_\sigma = [b]_\sigma \boxtimes s = \pi(b) \boxtimes s = \pi(b * s) = [b * s]_\sigma$. Similarly, $[b * s]_\sigma \subseteq [a * s]_\sigma$ and hence $\sigma \in$Con(A_S). □

Theorem 3. *Let $(X_S, *)$ and $(X_S, *')$ be two GHS-acts and $f : (X_S, *) \longrightarrow (X_S, *')$ be w-S-homomorphism, then $\sigma = \{(x, x') : f(x) = f(x')\}$ is an equivalence relation on X. If f is an s-S-homomorphism, then σ is a congruence on X.*

Proof. The proof is straightforward. □

Theorem 4. *Let $(X_S, *)$ and $(Y_S, *')$ be two GHS-acts, $f : (X, *) \longrightarrow (Y, *')$ be a w-S-homomorphism and $\sigma = \{(x, x') : f(x) = f(x')\}$. Then there exists a unique w-S-homomorphism $\alpha : X/\sigma \to Y$ defined by $\alpha([x]_\sigma) = f(x)$ for all $x \in X$ such that $\alpha \circ \pi = f$.*

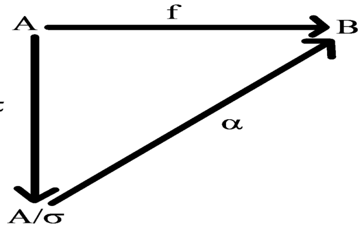

Figure 1

Proof. Let $[a]_\sigma = [b]_\sigma$ in A/σ. Then, $a\sigma b$ implies $f(a) = f(b)$. Therefore, $\alpha([a]_\sigma) = f(a)$ is well-defined. Also, α is w-S-homomorphism. Indeed, for $[a]_\sigma \in A/\sigma$ and $s \in S$

$$\begin{aligned} \alpha([a]_\sigma \boxtimes s) &= \alpha(\bigcup_{x \in [a]_\sigma} [x * s]_\sigma) \\ &= \bigcup_{x \in [a]_\sigma} \alpha([x * s]_\sigma) = \bigcup_{x \in [a]_\sigma} f(x * s). \end{aligned}$$

And

$$\alpha([a]_\sigma) *' s = f(a) *' s.$$

So $\alpha([a]_\sigma \boxtimes s) \cap \alpha([a]_\sigma) *' s \neq \emptyset$. □

Corollary 1. *If $f : (A_S, *) \longrightarrow (B_S, *')$ be a w-S-epimorphism, then $A/\sigma \sim B$.*

The above result remains valid if $(A_S, *)$ and $(B_S, *')$ are H_v-S-acts and f is an s-S-homomorphism.

Theorem 5. *Let $(X_S, *)$ be a right GHS-act and $\rho \in Eq(X_S)$ and $\sigma \in Con(X_S)$ such that $\rho \subseteq \sigma$. Then $\sigma/\rho = \{([x]_\rho, [x']_\rho) \in X/\rho \times X/\rho : (x, x') \in \sigma\}$ is a congruence relation on X/ρ and $(X/\rho)/(\sigma/\rho) \doteq X/\sigma$.*

Proof. From Theorem 1 $(A/\rho, \boxtimes)$ and $(A/\sigma, \boxtimes')$ are H_v-S-acts. Define $\alpha : A/\rho \longrightarrow A/\sigma$ by $\alpha([a]_\rho) = [a]_\sigma$. Firstly, we show that the map α is s-S-homomorphism. Let $a \in A$ and $s \in S$. Then

$$\begin{aligned} \alpha([a]_\rho \boxtimes s) &= \alpha(\bigcup_{x \in [a]\rho} [x * s]_\rho) \\ &= \bigcup_{x \in [a]_\rho} \alpha\left([x * s]_\rho\right) \\ &= \bigcup_{x \in [a]_\rho} [x * s]_\sigma \\ \alpha([a]_\rho) \boxtimes' s &= [a]_\sigma \boxtimes' s \\ &= \bigcup_{x \in [a]\sigma} [x * s]_\sigma. \end{aligned}$$

But σ is a congruence on A_S. So we have $\alpha([a]_\rho \boxtimes s) = [a * s]_\sigma = [a]_\sigma \boxtimes' s = \alpha([a]_\rho) \boxtimes' s$ which implies α is an s-S-homomorphism. Obviously, α is a bijection. Now it remains to prove that

$$\sigma/\rho = \{([a]_\rho, [b]_\rho) \in A/\rho \times A/\rho : \alpha([a]_\rho) = \alpha([b]_\rho)\}.$$

Let $([a]_\rho, [b]_\rho) \in A/\rho \times A/\rho$ such that $\alpha([a]_\rho) = \alpha([b]_\rho) \Longleftrightarrow [a]_\sigma = [b]_\sigma \Longleftrightarrow a\sigma b \Longleftrightarrow ([a]_\rho, [b]_\rho) \in \sigma/\rho$. Thus σ/ρ is a congruence by Theorem 3. Hence by Corollary 1 $(A/\rho)/(\sigma/\rho) \sim A/\sigma$. □

5. Actions Obtained from H_v-S-Acts

The main tools in the theory of an H_v-structures are fundamental relations. These relations were introduced and first studied by Vougiouklis [17]. In this section, we studied the fundamental relations for H_v-S-act. This establishes a link between weak hyperactions and the corresponding classical actions.

Let $(S, \circ)$ be an H_v-monoid and $\mathcal{V}$ be the set of all expressions consisting of finite hyperoperations of elements of S. Define a binary relation β on S by

$$s\beta t \iff \text{there exists } v \in \mathcal{V} \text{ such that } \{s, t\} \subset v$$

and denote by $\bar{\beta}$ the transitive closure of the relation β [2].

Proposition 12. *[2] Let $(S, \circ)$ be an H_v-monoid. Then $\bar{\beta}$ is the smallest equivalence relation such that $S/\bar{\beta}$ is a monoid.*

The relation $\bar{\beta}$ is the fundamental equivalence relation on S and $S/\bar{\beta}$ is the fundamental monoid. Following the similar technique.

Let $(A_S, *_{\mathcal{H}})$ be an H_v-S-act and $\mathcal{U}$ denote the set of all finite hyperactions of elements of S on A.

Define the relation γ on A as follows:

$$a\gamma b \iff \{a, b\} \subset u \text{ for some } u \in \mathcal{U}.$$

Clearly, γ is reflexive and symmetric but not a transitive relation. Let us denote $\bar{\gamma}$ the transitive closure of relation γ. The relation $\bar{\gamma}$ is an equivalence relation and $[a]_{\bar{\gamma}}$ is an equivalence class of the element a.

We can rewrite the definition of $\bar{\gamma}$ on A as follows:

$$\begin{aligned} a\bar{\gamma}b &\iff \exists\, a_1, a_2, ..., a_{n+1} \in A \text{ with } a = a_1, b = a_{n+1} \text{ and} \\ \exists\, u_1, u_2, u_3, ..., u_n &\in \mathcal{U} \text{ such that } \{a_i, a_{i+1}\} \subset u_i \text{ for } i = 1, 2, ..., n. \end{aligned}$$

Theorem 6. *Let $(A_S, *_{\mathcal{H}})$ be an H_v-S-act. Then $\bar{\gamma}$ is the smallest equivalence relation defined on A such that $A/\bar{\gamma}$ is an $S/\bar{\beta}$-act.*

Proof. First we prove that the quotient set $A/\bar{\gamma}$ is $S/\bar{\beta}$-act. On $A/\bar{\gamma}$, the operation $\circledast$ using $\bar{\beta}$ classes in S is defined as follows:

$$\begin{aligned} [x]_{\bar{\sigma}} \circledast [s]_\beta &= \{[c]_{\bar{\sigma}} : c \in [a]_{\bar{\sigma}} * [s]_{\bar{\beta}}\} \\ \text{for } [x]_{\bar{\sigma}} &\in X/\bar{\sigma} \text{ and } [s]_{\bar{\beta}} \in S/\bar{\beta}. \end{aligned}$$

Firstly, we prove that $[a]_{\bar{\gamma}} \circledast [s]_{\bar{\beta}}$ is a singleton. For this, let $a' \in [a]_{\bar{\gamma}}$ and $s' \in [s]_{\bar{\beta}}$. We have

$$\begin{aligned} a'\bar{\gamma}a &\implies \exists\, a_1, a_2, ..., a_{n+1} \in A \text{ with } a' = a_1, a = a_{n+1} \text{ and} \\ \exists\, u_1, u_2, u_3, ..., u_n &\in \mathcal{U} \text{ such that } \{a_i, a_{i+1}\} \subset u_i, \text{ for } i = 1, 2, ..., n. \end{aligned}$$

And

$$\begin{aligned} s'\bar{\beta}s &\implies \exists\, s_1, s_2, ..., s_{m+1} \in S \text{ with } s' = s_1, s = s_{m+1} \text{ and} \\ \exists\, v_1, v_2, v_3, ..., v_m &\in \mathcal{V} \text{ such that } \{s_j, s_{j+1}\} \subset v_j, \text{ for } j = 1, 2, ..., m. \end{aligned}$$

From these we obtain

$$\begin{aligned}\{a_i, a_{i+1}\} *_{\mathcal{H}} s_1 &\subset u_i *_{\mathcal{H}} v_1,\ i = 1, 2, ..., n-1\\ a_{n+1} *_{\mathcal{H}} \{s_j, s_{j+1}\} &\subset u_n *_{\mathcal{H}} v_j,\ j = 1, 2, ..., m.\end{aligned}$$

Here the sets

$$u_i *_{\mathcal{H}} v_1 = t_i,\ i = 1, 2, ..., n-1 \text{ and } u_n *_{\mathcal{H}} v_j = t_{n-1+j},\ j = 1, 2, ..., m$$

are elements of $\mathcal{U}$. Now, pick up elements $z_1, z_2, ..., z_{n+m}$ such that

$$z_i \in a_i *_{\mathcal{H}} s_1,\ i = 1, 2, ..., n \text{ and } z_{n+j} \in a_{n+1} *_{\mathcal{H}} s_{j+1},\ j = 1, 2, ..., m.$$

Using the above relation, we have

$$\{z_k, z_{k+1}\} \subset t_k,\ k = 1, 2, ..., m+n-1.$$

Thus, every element $z_1 \in a_1 *_{\mathcal{H}} s_1 = a' *_{\mathcal{H}} s'$ is $\bar{\gamma}$ equivalent to every element $z_{m+n} \in a *_{\mathcal{H}} s$. Thus $[a]_{\bar{\gamma}} \circledast [s]_{\bar{\beta}}$ is singleton. So, we can write

$$[a]_{\bar{\gamma}} \circledast [s]_{\bar{\beta}} = [c]_{\bar{\gamma}} \text{ for all } c \in [a]_{\bar{\gamma}} *_{\mathcal{H}} [s]_{\bar{\beta}}.$$

Obviously $A/\bar{\gamma}$ is an $S/\bar{\beta}$-act.

Let σ be any other equivalence relation on A such that A/σ is an $S/\bar{\beta}$-act. Then $[a]_\sigma \circledast [s]_{\bar{\beta}}$ are singletons, that is

$$[a]_\sigma \circledast [s]_{\bar{\beta}} = [c]_\sigma \text{ for all } c \in [a]_\sigma *_{\mathcal{H}} [s]_{\bar{\beta}}.$$

Thus, we can write for $a \in A$, $s \in S$ and $A' \subset [a]_\sigma$, $S' \subset [s]_{\bar{\beta}}$

$$[a]_\sigma \circledast [s]_{\bar{\beta}} = [A' *_{\mathcal{H}} S']_\sigma = [a *_{\mathcal{H}} s]_\sigma.$$

Let $a\gamma a'$. Then $\{a, a'\} \subset u$ for some $u \in \mathcal{U}$. Take $u = x *_{\mathcal{H}} s$ for some $x \in A$ and $s \in S$. Then, using relation σ, $x *_{\mathcal{H}} s$ is a subset of one class, say $[u_i]_\sigma$, for some i, so $u = x *_{\mathcal{H}} s \subset [u_i]_\sigma \Longrightarrow [a]_\sigma = [a']_\sigma \Longrightarrow a\sigma a'$and as σ is transitive, we have

$$a\bar{\gamma}a' \Longrightarrow a\sigma a'.$$

Therefore, $\bar{\gamma}$ is the smallest equivalence relation such that quotient is an $S/\bar{\beta}$-act. □

Remark 4. *From Theorem 6, we conclude that $\bar{\gamma}$ is the smallest equivalence relation such that $A/\bar{\gamma}$ is an $S/\bar{\beta}$-act. The relation $\bar{\gamma}$ is a fundamental relation on A and the quotient is said to be a fundamental $S/\bar{\beta}$-act.*

Theorem 7. *Let $(A_S, *_{\mathcal{H}})$ and $(B_S, *'_{\mathcal{H}})$ be two H_v-S-acts, $f : A \longrightarrow B$ be an s-S-homomorphism and $\bar{\gamma}_1, \bar{\gamma}_2$ and $\bar{\beta}$ be the fundamental relations on A, B and S, respectively. Then the map $\bar{f} : (A/\bar{\gamma}_1, \boxtimes) \longrightarrow (B/\bar{\gamma}_2, \boxtimes')$ defined by $\bar{f}([a]_{\bar{\gamma_1}}) = [f(a)]_{\bar{\gamma_2}}$ is an $S/\bar{\beta}$-homomorphism of $S/\bar{\beta}$-acts.*

Proof. Clearly, $A/\bar{\gamma}_1$ and $B/\bar{\gamma}_2$ are $S/\bar{\beta}$-acts. First we show that $\bar{f}$ is well-defined. Suppose that

$$[a]_{\bar{\gamma_1}} = [b]_{\bar{\gamma_1}}.$$

Then $a\bar{\gamma}_1 b \Longrightarrow \exists\, a_1, a_2, ..., a_{n+1} \in A$ with $a' = a_1$, $a = a_{n+1}$ and $\exists\, u_1, u_2, u_3, ..., u_n \in \mathcal{U}_A$ such that $\{a_i, a_{i+1}\} \subset u_i$, for $i = 1, 2, ..., n$. Since f is an s-S-homomorphism and $u_i \in \mathcal{U}_A$, we get $f(u_i) \in \mathcal{U}_B$. Therefore $f(a)\bar{\gamma}_2 f(b)$ which implies $[f(a)]_{\bar{\gamma_2}} = [f(b)]_{\bar{\gamma_2}}$, and so $\bar{f}([a]_{\bar{\gamma_1}}) = \bar{f}([b]_{\bar{\gamma_1}})$. Thus $\bar{f}$ is well-defined. Now,

$$\begin{aligned}\bar{f}([a]_{\bar{\gamma_1}} \boxtimes [s]_{\bar{\beta}}) &= \bar{f}([a *_{\mathcal{H}} s]_{\bar{\gamma_1}})\\ &= [f(a *_{\mathcal{H}} s)]_{\bar{\gamma_2}}\\ &= [f(a) *'_{\mathcal{H}} s]_{\bar{\gamma_2}}\\ &= [f(a)]_{\bar{\gamma_2}} *'_{\mathcal{H}} [s]_{\bar{\beta}}\\ &= \bar{f}([a]_{\bar{\gamma_1}}) \boxtimes' [s]_{\bar{\beta}}.\end{aligned}$$

□

Theorem 8. *Let $(A, *_{\mathcal{H}})$ and $(B, *'_{\mathcal{H}})$ be H_v-S-acts, $f : A \longrightarrow B$ be a s-S-homomorphism and $\bar{\gamma}_1, \bar{\gamma}_2$ and $\bar{\beta}$ be the fundamental relations on A, B and S, respectively. Then the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g_X & & \downarrow g_Y \\ X/\bar{\sigma}_1 & \xrightarrow{\bar{f}} & Y/\bar{\sigma}_2 \end{array}$$

Figure 2.

is commutative, where g_A, g_B are the natural projections of $(A, *_{\mathcal{H}})$ and $(B, *'_{\mathcal{H}})$, respectively.

6. Conclusion

The class of hyperstructures called H_v-structures has been studied from numerous aspects as well as in association with many other topics of mathematics. Here, in this paper, we introduced the concept of H_v-S-act and investigated some basic properties. A link between H_v-S-act, GHS-act and S-act (action notion in classical theory) have been established.

In future, we will focus on application of H_v-S-act in biology, chemistry, physics and social sciences mainly the use of H_v-S-act in questionnaire. We will also characterized H_v-S-act in term of primeness.

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