



## Results on L-Functions of Certain Differential Polynomials Sharing One Finite Value

Xiao-Min Li<sup>a</sup>, Fang Liu<sup>a</sup>, Hong-Xun Yi<sup>b</sup>

<sup>a</sup>Department of Mathematics, Ocean University of China, Qingdao, Shandong 266100, P. R. China

<sup>b</sup>Department of Mathematics, Shandong University, Jinan, Shandong 250199, P. R. China

**Abstract.** We study a uniqueness question of meromorphic functions whose certain nonlinear differential polynomials share a finite nonzero value. The results in this paper extend the corresponding results from Steuding [14, p.152], Li[9] and Fang [1]. The studied question is concerning a question posed by Fang in 2009.

### 1. Introduction and main results

In this paper, by L-functions we always mean L-functions that are Dirichlet series with the Riemann zeta function  $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$  as the prototype and are important objects in number theory. The Selberg class  $S$  of L-functions is the set of all Dirichlet series  $L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$  of a complex variable  $s = \sigma + it$  with  $a(1) = 1$ , satisfying the following axioms (cf.[13, 14]):

- (i) Ramanujan hypothesis:  $a(n) \ll n^{\varepsilon}$  for every  $\varepsilon > 0$ .
- (ii) Analytic continuation: There is a nonnegative integer  $k$  such that  $(s-1)^k L(s)$  is an entire function of finite order.
- (iii) Functional equation:  $L$  satisfies a functional equation of type  $\Lambda_L(s) = \overline{\omega \Lambda_L(1-\bar{s})}$ , where  $\Lambda_L(s) = L(s) Q^s \prod_{j=1}^K \Gamma(\lambda_j s + \nu_j)$  with positive real numbers  $Q$ ,  $\lambda_j$  and complex numbers  $\nu_j$ ,  $\omega$  with  $\text{Re} \nu_j \geq 0$  and  $|\omega| = 1$ .
- (iv) Euler product hypothesis:  $L(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$  with suitable coefficients  $b(p^k)$  satisfying  $b(p^k) \ll p^{k\theta}$  for some  $\theta < 1/2$ , where the product is taken over all prime numbers  $p$ .

In the last few years, value distribution of L-functions has been studied extensively, which can be found, for example in Steuding [11]. Value distribution of L-functions concerns the distribution of zeros of an

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*Email addresses:* lixiaomin@ouc.edu.cn (Xiao-Min Li), liufang2115@163.com (Fang Liu), hxyi@sdu.edu.cn (Hong-Xun Yi)

L-function  $L$  and, more generally, the  $c$ -points of  $L$ , i. e., the roots of the equation  $L(s) = c$ , or the points in the pre-image  $L^{-1} = \{s \in \mathbb{C} : L(s) = c\}$ , here and throughout the paper,  $s$  denotes the complex variables in the complex plane  $\mathbb{C}$  and  $c$  denotes a value in the extended complex plane  $\mathbb{C} \cup \{\infty\}$ . L-functions can be analytically continued as meromorphic functions in  $\mathbb{C}$ . Two meromorphic functions  $f$  and  $g$  in the complex plane are said to share a value  $c \in \mathbb{C} \cup \{\infty\}$  IM (ignoring multiplicities) if  $f^{-1}(c) = g^{-1}(c)$  as two sets in  $\mathbb{C}$ . Moreover,  $f$  and  $g$  are said to share a value  $c$  CM (counting multiplicities) if they share the value  $c$  and if the roots of the equations  $f(s) = c$  and  $g(s) = c$  have the same multiplicities. In terms of sharing values, two nonconstant meromorphic functions in the complex plane must be identically equal if they share five values IM, and one must be a Möbius transformation of the other if they share four values CM. The numbers “five” and “four” are the best possible, as shown by Nevanlinna (cf.[3, 12, 16, 17]), which are famous theorems due to Nevanlinna and often referred to as Nevanlinna’s uniqueness theorems.

Throughout this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. To prove the main results in the present paper, we will apply Nevanlinna’s theory and adopt the standard notations of the Nevanlinna’s theory. We assume that the readers are familiar with the standard notations which are used in the Nevanlinna’s theory such as the characteristic function  $T(r, f)$ , the proximity function  $m(r, f)$ , the counting function  $N(r, f)$  and the reduced counting function  $\bar{N}(r, f)$  that are explained in [3, 6, 16, 17]. Here  $f$  is a meromorphic function. It will be convenient to let  $E$  denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. In addition, we will use the lower order  $\mu(f)$  and the order  $\rho(f)$  of a meromorphic function  $f$ , which can be found, for example in [3, 6, 16, 17], and are in turn defined as follows:

$$\mu(f) = \liminf_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r}, \quad \rho(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r}.$$

For a nonconstant meromorphic function  $h$ , we denote by  $S(r, h)$  any quantity satisfying  $S(r, h) = o(T(r, h))$ , as  $r \rightarrow \infty$  and  $r \notin E$ . We say that a meromorphic function  $a$  is a small function with respect to  $h$ , if  $T(r, a) = S(r, h)$  (cf.[16]). We also need the following two definitions:

**Definition 1.1.** ([7, Definition 1]). Let  $p$  be a positive integer and  $a \in \mathbb{C} \cup \{\infty\}$ . Next we denote by  $N_p\left(r, \frac{1}{f-a}\right)$  the counting function of those  $a$ -points of  $f$  (counted with proper multiplicities) whose multiplicities are not greater than  $p$ , and denote by  $N_{(p)}\left(r, \frac{1}{f-a}\right)$  the counting function of those  $a$ -points of  $f$  (counted with proper multiplicities) whose multiplicities are not less than  $p$ . We denote by  $\bar{N}_p\left(r, \frac{1}{f-a}\right)$  and  $\bar{N}_{(p)}\left(r, \frac{1}{f-a}\right)$  the reduced forms of  $N_p\left(r, \frac{1}{f-a}\right)$  and  $N_{(p)}\left(r, \frac{1}{f-a}\right)$  respectively. Here  $N_p\left(r, \frac{1}{f-\infty}\right)$ ,  $\bar{N}_p\left(r, \frac{1}{f-\infty}\right)$ ,  $N_{(p)}\left(r, \frac{1}{f-\infty}\right)$  and  $\bar{N}_{(p)}\left(r, \frac{1}{f-\infty}\right)$  mean  $N_p(r, f)$ ,  $\bar{N}_p(r, f)$ ,  $N_{(p)}(r, f)$  and  $\bar{N}_{(p)}(r, f)$  respectively.

**Definition 1.2.** Let  $a$  be an any value in the extended complex plane and let  $k$  be an arbitrary nonnegative integer. We define

$$\Theta(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}, \quad \delta_k(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_k\left(r, \frac{1}{f-a}\right)}{T(r, f)},$$

where

$$N_k\left(r, \frac{1}{f-a}\right) = \bar{N}\left(r, \frac{1}{f-a}\right) + \bar{N}_{(2)}\left(r, \frac{1}{f-a}\right) + \cdots + \bar{N}_{(k)}\left(r, \frac{1}{f-a}\right).$$

**Remark 1.3.** By Definition 1.2 we have

$$0 \leq \delta_k(a, f) \leq \delta_{k-1}(a, f) \leq \delta_1(a, f) \leq \Theta(a, f) \leq 1.$$

We first recall the following result due to Steuding [11], which actually holds without the Euler product hypothesis:

**Theorem 1.4.** ([14, p.152]). If two L-functions  $L_1$  and  $L_2$  with  $a(1) = 1$  share a complex value  $c \neq \infty$  CM, then  $L_1 = L_2$ .

**Remark 1.5.** Recently Hu and Li pointed out that Theorem 1.4 is false when  $c = 1$ . A counter example was given by Hu and Li, see [5].

In 2010, Li [9] introduced the following question posed by Chung-Chun Yang:

**Question 1.6.** ([9]). If  $f$  is a meromorphic function in  $\mathbb{C}$  that shares three distinct values  $a, b$  CM and  $c$  IM with the Riemann zeta function  $\zeta$ , where  $c \notin \{a, b, 0, \infty\}$ , is  $f$  equal to  $\zeta$ ?

Li [9] also proved the following result to deal with Question 1.6:

**Theorem 1.7.** ([9]). Let  $a$  and  $b$  be two distinct finite values, and let  $f$  be a meromorphic function in the complex plane such that  $f$  has finitely many poles in the complex plane. If  $f$  and a nonconstant L-function  $L$  share  $a$  CM and  $b$  IM, then  $L = f$ .

**Remark 1.8.** In 2012, Gao and Li completely solved Question 1.6, see [2].

Concerning the value distribution of nonlinear differential polynomials of meromorphic functions, we recall the following result proved by Fang in 2002:

**Theorem 1.9.** ([1, Theorem 2]). Let  $f$  and  $g$  be two nonconstant entire functions, and let  $n, k$  be two positive integers satisfying  $n \geq 2k + 8$ . If  $(f^n(f-1))^{(k)}$  and  $(g^n(g-1))^{(k)}$  share 1 CM, then  $f = g$ .

Regarding Theorem 1.9, one may ask, what can be said about the relationship between two meromorphic functions  $f$  and  $g$ , if  $(f^n(f-1))^{(k)}$  and  $(g^n(g-1))^{(k)}$  share 1 CM (IM), where  $n$  and  $k$  are positive integers? which was also posed by Professor M. L. Fang in 2009. By now this question is still open. In this paper, we will prove the following result by considering the nonlinear differential polynomials of L-functions, we will prove the following result, which deals with the special case of this question:

**Theorem 1.10.** Let  $f$  be a nonconstant meromorphic function, let  $L$  be an L-function, and let  $n$  and  $k$  be two positive integers with  $n > 3k + 9$  and  $k \geq 2$ . If  $(f^n(f-1))^{(k)}$  and  $(L^n(L-1))^{(k)}$  share 1 CM, then  $f = L$ .

In the same manner as in the proof of Theorem 1.10 in Section 3 of this paper, we can get the following result by Lemma 2.3 in Section 3 of this paper:

**Theorem 1.11.** Let  $f$  be a nonconstant meromorphic function, let  $L$  be an L-function, and let  $n$  and  $k$  be two positive integers satisfying  $n > 7k + 17$  and  $k \geq 2$ . If  $(f^n(f-1))^{(k)}$  and  $(L^n(L-1))^{(k)}$  share 1 IM, then  $f = L$ .

## 2. Preliminaries

In this section, we will give the following lemmas that play an important role in proving the main results in this paper. First of all, we introduce the following lemma from [11]:

**Lemma 2.1.** (Valiron-Mokhonoko, [11]). Let  $f$  be a nonconstant meromorphic function, and let

$$F = \frac{\sum_{k=0}^p a_k f^k}{\sum_{j=0}^q b_j f^j}$$

be an irreducible rational function in  $f$  with constant coefficients  $\{a_k\}$  and  $\{b_j\}$ , where  $a_p \neq 0$  and  $b_q \neq 0$ . Then  $T(r, F) = dT(r, f) + O(1)$ , where  $d = \max\{p, q\}$ .

The following two lemmas are from Li-Yi[10]:

**Lemma 2.2.** ([10, Lemma 2.5]). Let  $F$  and  $G$  be two nonconstant meromorphic functions such that  $F^{(k)} - P$  and  $G^{(k)} - P$  share 0 CM, where  $k \geq 1$  is a positive integer,  $P \neq 0$  is a polynomial. If

$$(k + 2)\Theta(\infty, F) + 2\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + \delta_{k+1}(0, F) + \delta_{k+1}(0, G) > k + 7$$

and

$$(k + 2)\Theta(\infty, G) + 2\Theta(\infty, F) + \Theta(0, G) + \Theta(0, F) + \delta_{k+1}(0, G) + \delta_{k+1}(0, F) > k + 7,$$

then either  $F^{(k)}G^{(k)} = P^2$  or  $F = G$ .

**Lemma 2.3.** ([10, Lemma 2.4]). Let  $F$  and  $G$  be two nonconstant meromorphic functions such that  $F^{(k)} - P$  and  $G^{(k)} - P$  share 0 IM, where  $k \geq 1$  is a positive integer,  $P \neq 0$  is a polynomial. If

$$(2k + 3)\Theta(\infty, F) + (2k + 4)\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + 2\delta_{k+1}(0, F) + 3\delta_{k+1}(0, G) > 4k + 13$$

and

$$(2k + 3)\Theta(\infty, G) + (2k + 4)\Theta(\infty, F) + \Theta(0, G) + \Theta(0, F) + 2\delta_{k+1}(0, G) + 3\delta_{k+1}(0, F) > 4k + 13,$$

then either  $F^{(k)}G^{(k)} = P^2$  or  $F = G$ .

**Lemma 2.4.** ([8, Theorem 1.2]). Suppose that  $f$  is a meromorphic of finite order in the plane, and that  $f^{(k)}$  has finitely many zeros for some  $k \geq 2$ . Then  $f$  has finitely many poles in the complex plane.

**Lemma 2.5.** ([18, Lemma 6]). Let  $f_1$  and  $f_2$  be two nonconstant meromorphic functions such that

$$\bar{N}(r, f_j) + \bar{N}\left(r, \frac{1}{f_j}\right) = S(r)$$

for  $1 \leq j \leq 2$ . Then, either  $\bar{N}_0(r, 1; f_1, f_2) = S(r)$  or that there exist two integers  $p$  and  $q$  satisfying  $|p| + |q| > 0$ , such that  $f_1^p f_2^q = 1$ , where  $\bar{N}_0(r, 1; f_1, f_2)$  denotes the reduced counting function of the common 1-points of  $f_1$  and  $f_2$  in  $|z| < r$ ,  $T(r) = T(r, f_1) + T(r, f_2)$  and  $S(r) = o(T(r))$ , as  $r \notin E$  and  $r \rightarrow \infty$ . Here  $E \subset (0, +\infty)$  is a subset of finite linear measure.

**Lemma 2.6.** ([4]). Let  $f$  be a transcendental meromorphic function in the complex plane. Then, for each  $K > 1$ , there exists a set  $M(K) \subset (0, +\infty)$  of the lower logarithmic density at most  $d(K) = 1 - (2e^{K-1} - 1)^{-1} > 0$ , that is

$$\underline{\log \text{dens}} M(K) = \liminf_{r \rightarrow \infty} \frac{1}{\log r} \int_{M(K) \cap [1, r]} \frac{dt}{t} \leq d(K),$$

such that, for every positive integer  $k$ , we have

$$\limsup_{\substack{r \rightarrow \infty \\ r \notin M(K)}} \frac{T(r, f)}{T(r, f^{(k)})} \leq 3eK.$$

**Lemma 2.7.** ([18]). Let  $s > 0$  and  $t$  be relatively prime integers, and let  $c$  be a finite complex number such that  $c^s = 1$ , then there exists one and only one common zero of  $\omega^s - 1$  and  $\omega^t - c$ .

3. Proof of the main results

**Proof of Theorem 1.1.** First of all, we denote by  $d$  the degree of  $L$ . Then  $d = 2 \sum_{j=1}^K \lambda_j > 0$  (cf.[14, p.113]), where  $K$  and  $\lambda_j$  are respectively the positive integer and the positive real number in the functional equation of the axiom (iii) of the definition of L-function. Therefore, by Steuding [14, p.150] we have

$$T(r, L) = \frac{d}{\pi} r \log r + O(r). \tag{1}$$

Next we set

$$F_1 = f^n(f - 1), \quad G_1 = L^n(L - 1). \tag{2}$$

Now we let

$$\Delta_1 = (k + 2)\Theta(\infty, F_1) + 2\Theta(\infty, G_1) + \Theta(0, F_1) + \Theta(0, G_1) + \delta_{k+1}(0, F_1) + \delta_{k+1}(0, G_1) \tag{3}$$

and

$$\Delta_2 = (k + 2)\Theta(\infty, G_1) + 2\Theta(\infty, F_1) + \Theta(0, G_1) + \Theta(0, F_1) + \delta_{k+1}(0, G_1) + \delta_{k+1}(0, F_1). \tag{4}$$

By Lemma 2.1 we have

$$\Theta(\infty, F_1) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, F_1)}{T(r, F_1)} = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{(n + 1)T(r, f) + O(1)} \geq 1 - \frac{1}{n + 1}, \tag{5}$$

$$\begin{aligned} \delta_{k+1}(0, F_1) &= 1 - \limsup_{r \rightarrow \infty} \frac{N_{k+1}\left(r, \frac{1}{F_1}\right)}{T(r, F_1)} \\ &\geq 1 - \limsup_{r \rightarrow \infty} \frac{(k + 1)\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right)}{(n + 1)T(r, f) + O(1)} \\ &\geq 1 - \frac{k + 2}{n + 1} \end{aligned} \tag{6}$$

and

$$\Theta(0, F_1) \geq 1 - \frac{2}{n + 1}, \quad \Theta(0, G_1) \geq 1 - \frac{2}{n + 1}, \quad \delta_{k+1}(0, G_1) \geq 1 - \frac{k + 2}{n + 1}. \tag{7}$$

By noting that an L-function has at most one pole  $z = 1$  in the complex plane, we have by (1) that

$$\Theta(\infty, G_1) = 1. \tag{8}$$

By (3), (5)-(8) we have

$$\Delta_1 \geq k + 8 - \frac{3k + 10}{n + 1}, \quad \Delta_2 \geq k + 8 - \frac{2k + 10}{n + 1}. \tag{9}$$

By (9) and the assumption  $n > 3k + 9$ , we have  $\Delta_1 > k + 7$  and  $\Delta_2 > k + 7$ . This together with (3), (4), Lemma 2.2 and the assumption that  $F_1^{(k)}$  and  $G_1^{(k)}$  share 1 CM gives  $F_1^{(k)}G_1^{(k)} = 1$  or  $F_1 = G_1$ . We consider the following two cases:

**Case 1.** Suppose that  $F_1^{(k)}G_1^{(k)} = 1$ . Then, by (2) we have

$$(f^n(f - 1))^{(k)}(L^n(L - 1))^{(k)} = 1. \tag{10}$$

On the other hand, by (1) and (10), Lemma 2.1, a result from Whittaker [15, p.82] and the definition of the order of a meromorphic function we have

$$\begin{aligned} \rho(f) &= \rho(f^n(f-1)) = \rho((f^n(f-1))^{(k)}) = \rho((L^n(L-1))^{(k)}) = \rho(L^n(L-1)) \\ &= \rho(L) = 1. \end{aligned} \tag{11}$$

By (11) we can see that  $f$  is a transcendental meromorphic function. Since an L-function at most has one pole  $z = 1$  in the complex plane, we deduce by (10) that  $(f^n(f-1))^{(k)}$  at most has one zero  $z = 1$  in the complex plane. Combining this with (11), Lemma 2.4 and the assumption  $k \geq 2$ , we have that  $f^n(f-1)$ , and so  $f$  has at most finitely many poles in the complex plane. This together with (10) implies that  $(L^n(L-1))^{(k)}$  has at most finitely many zeros in the complex plane. Therefore, by (2) we have

$$\overline{N}(r, F_1^{(k)}) + \overline{N}\left(r, \frac{1}{F_1^{(k)}}\right) \leq O(\log r) \tag{12}$$

and

$$\overline{N}(r, G_1^{(k)}) + \overline{N}\left(r, \frac{1}{G_1^{(k)}}\right) \leq O(\log r). \tag{13}$$

We now set

$$f_1 = \frac{F_1^{(k)}}{G_1^{(k)}}, \quad f_2 = \frac{F_1^{(k)} - 1}{G_1^{(k)} - 1}. \tag{14}$$

By (14) and the assumption that  $f$  and  $L$  are transcendental meromorphic functions, we have  $f_1 \not\equiv 0$  and  $f_2 \not\equiv 0$ . Suppose that one of  $f_1$  and  $f_2$  is a nonzero constant. Then, by (14) we see that  $F_1^{(k)}$  and  $G_1^{(k)}$  share  $\infty$  CM. Combining this with  $F_1^{(k)}G_1^{(k)} = 1$  we deduce that  $\infty$  is a Picard exceptional value of  $f$  and  $L$ . Next we suppose that  $f_1$  and  $f_2$  are nonconstant meromorphic functions. We set

$$F_2 = F_1^{(k)}, \quad G_2 = G_1^{(k)}. \tag{15}$$

Then, by (14) and (15) we have

$$F_2 = \frac{f_1(1-f_2)}{f_1-f_2}, \quad G_2 = \frac{1-f_2}{f_1-f_2}. \tag{16}$$

By (16) we can find that there exists a subset  $I \subset (0, +\infty)$  with infinite linear measure such that  $S(r) = o(T(r))$  and

$$T(r, F_2) \leq 2(T(r, f_1) + T(r, f_2)) + S(r) \leq 8T(r, F_2) + S(r) \tag{17}$$

or

$$T(r, G_2) \leq 2(T(r, f_1) + T(r, f_2)) + S(r) \leq 8T(r, G_2) + S(r), \tag{18}$$

as  $r \in I$  and  $r \rightarrow \infty$ , where  $T(r) = T(r, f_1) + T(r, f_2)$ . Without loss of generality, we suppose that (17) holds. Then we have  $S(r) = S(r, F_2)$ , as  $r \in I$  and  $r \rightarrow \infty$ . By  $F_2G_2 = 1$  we see that  $F_2$  and  $G_2$  share 1 and  $-1$  CM. By noting that  $F_2$  and  $G_2$  are transcendental meromorphic functions such that  $F_2$  and  $G_2$  share 1 CM, we deduce by (12)-(14) that

$$\overline{N}\left(r, \frac{1}{f_j}\right) + \overline{N}(r, f_j) = o(T(r)), \quad j = 1, 2, \tag{19}$$

as  $r \in I$  and  $r \rightarrow \infty$ . By noting that  $F_2$  and  $G_2$  share  $-1$  CM, we deduce by (14), (15) and the second fundamental theorem that

$$\begin{aligned} T(r, F_2) &\leq \bar{N}(r, F_2) + \bar{N}\left(r, \frac{1}{F_2}\right) + \bar{N}\left(r, \frac{1}{F_2 + 1}\right) + o(T(r, F_2)) \\ &\leq \bar{N}\left(r, \frac{1}{F_2 + 1}\right) + O(\log r) + o(T(r, F_2)) \\ &\leq \bar{N}_0(r, 1; f_1, f_2) + o(T(r, F_2)), \end{aligned} \tag{20}$$

as  $r \in I$  and  $r \rightarrow \infty$ . By (17) and (20) we have

$$T(r, f_1) + T(r, f_2) \leq \bar{N}_0(r, 1; f_1, f_2) + o(T(r)), \tag{21}$$

By (12)-(15), (19), (21) and Lemma 2.5 we find that there exist two relatively prime integers  $s$  and  $t$  satisfying  $|s| + |t| > 0$ , such that  $f_1^s f_2^t = 1$ . Combining this with (14) and (15), we have

$$\left(\frac{F_2}{G_2}\right)^s \left(\frac{F_2 - 1}{G_2 - 1}\right)^t = 1. \tag{22}$$

We discuss this as follows:

Suppose that  $st < 0$ , say  $s > 0$  and  $t < 0$ , say  $t = -t_1$ , where  $t_1$  is some positive integer. Then, (22) can be rewritten as

$$\left(\frac{F_2}{G_2}\right)^s = \left(\frac{F_2 - 1}{G_2 - 1}\right)^{t_1}. \tag{23}$$

Let  $z_1 \in \mathbb{C}$  be a pole of  $F_2$  of multiplicity  $p_1 \geq 1$ . Then, by  $F_2 G_2 = 1$  we can see that  $z_1$  be a zero of  $G_2$  of multiplicity  $p_1$ . Therefore, by (23) we deduce that  $2s = t_1 = -t$ . Combining this with the assumption that  $s$  and  $t$  are two relatively prime integers, we have  $s = 1$  and  $t = -t_1 = -2$ . Therefore, (3.23) can be rewritten as  $F_2(G_2 - 1)^2 = (F_2 - 1)^2 G_2$ , this equivalent to the obtained result  $F_2 G_2 = 1$ . Next we can deduce a contradiction by using the other method. Indeed, by (11), (13), the right equality of (2) and the fact that  $L$ , and so  $(L^n(L - 1))^{(k)}$  has at most one pole  $z = 1$  in the complex plane, we deduce

$$(L^n(z)(L(z) - 1))^{(k)} = \frac{P_1(z)}{(z - 1)^{p_2}} e^{A_1 z + B_1}, \tag{24}$$

where  $P_1$  is a nonzero polynomial,  $p_2 \geq 0$  is an integer,  $A_1 \neq 0$  and  $B_1$  are constants. By (24), Hayman[3, p.7], Lemmas 2.1 and 2.6 we deduce that there exists a subset  $I \subset (0, +\infty)$  with logarithmic measure  $\log \text{meas } I = \int_I \frac{dt}{t} = \infty$  such that for some given sufficiently large positive number  $K > 1$ , we have

$$\begin{aligned} (n + 1)T(r, L) &= T(r, (L^n(L - 1))) \\ &\leq 3eKT(r, (L^n(L - 1))^{(k)}) = \frac{3eK|A_1|r}{\pi}(1 + o(1)) + O(\log r), \end{aligned} \tag{25}$$

as  $r \in I$  and  $r \rightarrow \infty$ . By (1) and (25) we have a contradiction.

Suppose that  $st = 0$ , say  $s = 0$  and  $t \neq 0$ . Then, by (22) we can see that  $F_2$  and  $G_2$  share  $\infty$  CM. This together with (2), (15) and the assumption  $F_2 G_2 = 1$  implies that  $\infty$  is a Picard exceptional value of  $f$  and  $L$ .

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By (2), (13) and the assumption  $n > 3k + 9$  we deduce that  $L$  has at most finitely many zeros in the complex plane. This together with the obtained result that  $\infty$  is a Picard exceptional value of  $f$  and  $L$  gives

$$L(z) = P_3(z)e^{A_2 z + B_2}, \tag{26}$$

where  $P_3$  is a nonzero polynomial,  $A_2 \neq 0$  and  $B_2$  are constants. By (26) and Hayman[3, p.7] we have

$$T(r, L(z)) = T(r, P_3(z)e^{A_2z+B_2}) = \frac{|A_2|r}{\pi}(1 + o(1)) + O(\log r), \tag{27}$$

which contradicts (1).

**Case 2.** Suppose that  $F_1 = G_1$ . Then, by (2) we have

$$f^n(f - 1) = L^n(L - 1). \tag{28}$$

Set

$$H = \frac{L}{f}. \tag{29}$$

By (28) and (29) we deduce

$$(H^{n+1} - 1)f = H^n - 1. \tag{30}$$

We consider the following two subcases:

**Subcase 2.1** Suppose that  $H$  is a nonconstant meromorphic function. Then, by (30) we have

$$f = \frac{1 - H^n}{1 - H^{n+1}}. \tag{31}$$

By noting that  $n$  and  $n + 1$  are two relatively prime positive integers, we know by Lemma 2.7 that  $\omega = 1$  is the only one common zero of  $\omega^n - 1$  and  $\omega^{n+1} - 1$ . Therefore, (31) can be rewritten as

$$f = \frac{1 + H + \dots + H^{n-1}}{1 + H + \dots + H^n}, \tag{32}$$

By (32) and Lemma 2.1 we have

$$T(r, f) = T\left(r, \frac{1 + H + \dots + H^{n-1}}{1 + H + \dots + H^n}\right) = nT(r, H) + O(1). \tag{33}$$

By (11), (28), (32) and the second fundamental theorem we have

$$\bar{N}(r, L) = \bar{N}(r, f) = \sum_{j=1}^n \bar{N}\left(r, \frac{1}{H - \lambda_j}\right) + o(T(r, H)) \geq (n - 2)T(r, H), \tag{34}$$

as  $r \rightarrow \infty$ . Here  $\lambda_1, \lambda_2, \dots, \lambda_n$  are  $n$  distinct finite complex numbers satisfying  $\lambda_j \neq 1$  and  $\lambda_j^{n+1} = 1$  for  $1 \leq j \leq n$ . By noting that  $L$  is a transcendental meromorphic function such that  $L$  has at most one pole  $z = 1$  in the complex plane, we deduce by (34) that there exists some small positive number  $\varepsilon_0$  satisfying  $0 < \varepsilon_0 < 1$ , such that

$$(n - 2 - \varepsilon_0)T(r, H) \leq \bar{N}(r, L) = \log r + O(1). \tag{35}$$

By (35) and the assumption  $n > 3k + 9$  and  $k \geq 2$  we deduce that  $H$  is a nonconstant rational function such that

$$T(r, H) \geq \log r + O(1). \tag{36}$$

By (35) and (36) we can get a contradiction.



**Subcase 2.2** Suppose that  $H$  is a constant. If  $H^{n+1} \neq 1$ . By (30) we get (31), which contradicts the assumption that  $f$  is a nonconstant meromorphic function. Therefore,  $H^{n+1} = 1$ , and so it follows by (3.30) that  $H^{n+1} - 1 = H^n - 1 = 0$ , which implies that  $H = 1$ . Combining this with (29), we get the conclusion of Theorem 1.10. This completes the proof of Theorem 1.10.

**Proof of Theorem 1.2.** First of all, we denote by  $d$  the degree of  $L$ . Then  $d = 2 \sum_{j=1}^K \lambda_j > 0$  (cf.[14, p.113]), where  $K$  and  $\lambda_j$  are respectively the positive integer and the positive real number in the functional equation of the axiom (iii) of the definition of L-function. Therefore, by Steuding [14, p.150] we have (1). Now we let (2), and let

$$\begin{aligned} \Delta_3 = & (2k+3)\Theta(\infty, F_1) + (2k+4)\Theta(\infty, G_1) + \Theta(0, F_1) + \Theta(0, G_1) \\ & + 2\delta_{k+1}(0, F_1) + 3\delta_{k+1}(0, G_1) \end{aligned} \quad (37)$$

and

$$\begin{aligned} \Delta_4 = & (2k+3)\Theta(\infty, G_1) + (2k+4)\Theta(\infty, F_1) + \Theta(0, G_1) + \Theta(0, F_1) \\ & + 2\delta_{k+1}(0, G_1) + 3\delta_{k+1}(0, F_1). \end{aligned} \quad (38)$$

In the same manner as in the proof of Theorem 1.10 we have 5-8. By (5)-(8), (37) and (38) we have

$$\Delta_3 \geq 4k + 14 - \frac{7k + 18}{n + 1}, \quad \Delta_4 \geq 4k + 14 - \frac{7k + 17}{n + 1}. \quad (39)$$

By (38)-(39) and the assumption  $n > 7k + 17$  we deduce  $\Delta_3 > 4k + 13$  and  $\Delta_4 > 4k + 13$ . This together with Lemma 2.3 gives  $F_1^{(k)}G_1^{(k)} = 1$  or  $F_1 = G_1$ . We consider the following two cases:

**Case 1.** Suppose that  $F_1^{(k)}G_1^{(k)} = 1$ . Then, in the same manner as in Case 1 of the proof of Theorem 1.10 we have a contradiction.

**Case 2.** Suppose that  $F_1 = G_1$ . Then, in the same manner as in Case 2 of the proof of Theorem 1.1 we get the conclusion of Theorem 1.11. This completely proves Theorem 1.11.

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## References

- [1] M. L.Fang, Uniqueness and value sharing of entire functions, *Computers & Mathematics with Applications* 44(2002), no. 5-6, 823–831.
- [2] R. Gao, B. Q. Li, An answer to a question on value distribution of the Riemann zeta-function, *International Journal of Mathematics* 23(2012), no. 4, 17–18.
- [3] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
- [4] W. K. Hayman, J. Miles, On the growth of a meromorphic function and its derivatives, *Complex Variables Theory & Application An International Journal* 12 (1989), no. 1, 245–260.
- [5] P. C. Hu, B. Q. Li, A simple proof and strengthening of a uniqueness theorem for L-functions, *Canadian mathematical bulletin* 59 (2016), no. 1, 119–122.
- [6] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin/New York, 1993.
- [7] I. Lahiri, Weighted sharing of three values and uniqueness of meromorphic functions, *Kodai Mathematical Journal* 24(2001), no.3, 421–435.
- [8] J. K. Langley, The second derivative of a meromorphic function of finite order, *Bulletin of the London Mathematical Society* 35(2003), no.1, 97–108.
- [9] B. Q. Li, A result on value distribution of L-functions, *Proceedings of the American Mathematical Society* 138(2010), no. 6 2071–2077.
- [10] X. M. Li, H. X. Yi, Uniqueness of meromorphic functions whose certain nonlinear differential polynomials share a polynomial, *Computers & Mathematics with Applications* 62(2011), no.2, 539–550.

- [11] A.Z. Mokhonko, On the Nevanlinna characteristics of some meromorphic functions, in: *Theory of Functions, Functional Analysis and Their Applications*, vol.14, Izd-vo Khar'kovsk. Un-ta, 1971, pp. 83–87.
- [12] R. Nevanlinna, Einige Eindeutigkeitssätze in der Theorie der meromorphen Funktionen, *Acta Mathematica* **48**(1926), no. 3-4, 367–391.
- [13] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, *Proceedings of the Amalfi Conference on Analytic Number Theory* (Maiori, 1989), E. Bombieri et al. (eds.), *Collected papers*, Vol. II, Springer-Verlag, 1991, pp. 47-63.
- [14] J. Steuding, *Value distribution of L-functions*, *Lecture Notes in Mathematics*, Vol. 1877, Springer-Verlag, Berlin, 2007.
- [15] J. M. Whittaker, The order of the derivative of a meromorphic function, *Journal of the London Mathematical Society* **s1-11**(1936), no. 4, Page 320.
- [16] C. C. Yang, H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.
- [17] L. Yang, *Value Distribution Theory*, Springer-Verlag, Berlin Heidelberg, 1993.
- [18] Q. C. Zhang, Meromorphic functions sharing three values, *Indian Journal of Pure and Applied Mathematics* **30**(1999), no.7 667–682.